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Proton and Ion Linear Accelerators

4. Focusing of Intense Beams

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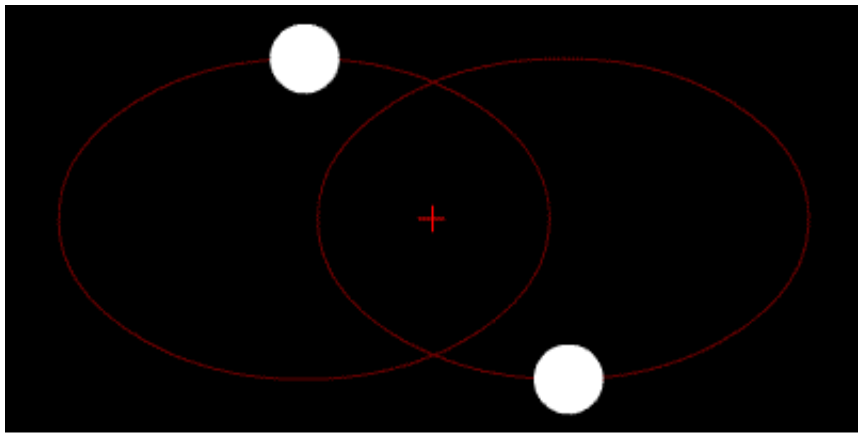
Self-Consistent Particle Dynamics

Example of self-consistent dynamics: two - body problem



In classical mechanics, the two-body problem is to determine the motion of two point particles that interact only with each other according to the gravitational law:

$$F = G\frac{m_1 m_2}{r^2}$$

Self-Consistent Approach to N-Particle Dynamics

Self-consistent approach: solution to the equations of motion of the particles, together with the equations for the electromagnetic field that they create. Evolution of charged particles interacting through long-range (Coulomb) forces is determined by *Vlasov's equation*

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \vec{x}}\frac{d\vec{x}}{dt} + \frac{\partial f}{\partial \vec{P}}\frac{d\vec{P}}{dt} = 0$$

Solution of self-consistent problem: the phase space density, as a constant of motion, can be expressed as a function of other constants of motion I_1, I_2, ….

$$f = f\,(I_1, I_2, \ldots.)$$

This equation automatically obeys Vlasov's equation

$$\frac{df}{dt} = \frac{\partial f}{\partial I_1}\frac{dI_1}{dt} + \frac{\partial f}{\partial I_2}\frac{dI_2}{dt} + \ldots. \quad = 0$$

because of vanishing derivatives, $dI_i/dt = 0$. Distribution function determined in this way is then substituted to Maxwell's equation to find self-consistent field created by the beam together with the external electromagnetic field.

Maxwell's Equations

Field created by the beam is described by Maxwell's equations:

$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$
$\nabla \cdot \mathbf{B} = 0$
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$

space charge density:

$$\rho = q \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f \, dP_x \, dP_y \, dP_z$$

beam current density:

$$\vec{j} = q \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \vec{v} f \, dP_x \, dP_y \, dP_z$$

Field Equations

Instead of electric field $\vec{E}$ and magnetic field $\vec{B}$, it is common to use vector potential $\vec{A}$ and scalar potential U:

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - gradU$$

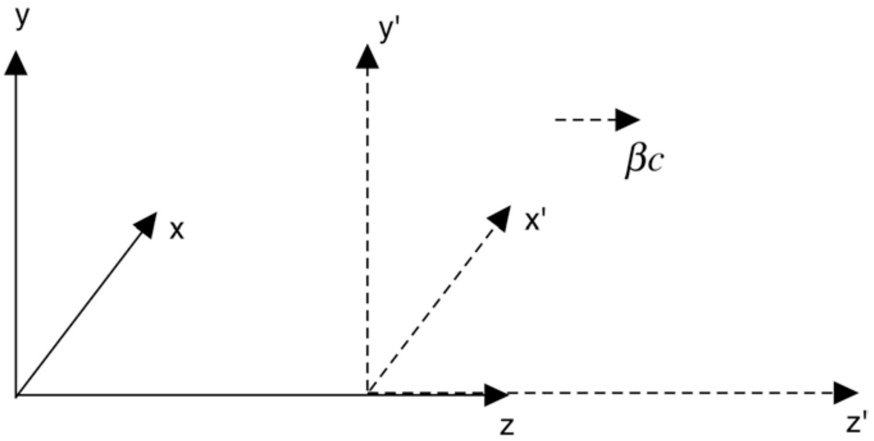
$$\vec{B} = rot\vec{A}$$

The field of the beam is described by the equations

$$\Delta U_b - \frac{1}{c^2}\frac{\partial^2 U_b}{\partial t^2} = -\frac{\rho}{\varepsilon_o}$$

$$\Delta \vec{A_b} - \frac{1}{c^2}\frac{\partial^2 \vec{A_b}}{\partial t^2} = -\mu_o \vec{j}$$

Lorentz Transformations



Laboratory and moving systems of coordinates

Consider system of coordinates, which moves with the average beam velocity β. We will denote all values in this frame by prime symbol. Potentials $U', \vec{A}'$ are connected with that in laboratory system, $U, \vec{A}$, by Lorentz transformation

$$A_z = \gamma(A_z' + \frac{\beta}{c} U')$$

$$U = \gamma\,(U' + \beta c A_z')$$

$$A_x = A_x', \quad A_y = A_y'$$

Self-Consistent Field Equation

In the moving system of coordinates, particles are static, therefore, vector potential of the beam equals to zero, $\vec{A}_b^{\,'} = 0$. According to Lorentz transformations, components of vector potential of the beam are converted into laboratory system of coordinates as follow

$$A_{xb} = 0\,,\quad A_{yb} = 0\,,\quad A_{zb} = \beta \frac{U_b}{c}$$

In a particle beam, the vector potential and the scalar potential are related via the expression $\vec{A}_b = \vec{v}_z \;/c^2 U_b$, therefore, it is sufficient to only solve the equation for the scalar potential. Equation for scalar potential of moving bunched beam is

$$\frac{\partial^2 U_b}{\partial x^2} + \frac{\partial^2 U_b}{\partial y^2} + \frac{\partial^2 U_b}{\gamma^2 \partial \zeta^2} = - \frac{1}{\varepsilon_o} \rho\,(x, y, \zeta)$$

Self-Consistent Field Equation

The unknown distribution function of the beam is then found by substituting equation for distribution function into the field equation and solving it. For beam transport, equation for unknown space charge potential is

$$\Delta U_b = - \frac{q}{\varepsilon_o} \int_{-\infty}^{\infty} f(I_1, I_2, \ldots)\, d\vec{P}$$

Equation for unknown potential of the beam together with Vlasov's equation for beam distribution function

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \vec{x}} \frac{d\vec{x}}{dt} + \frac{\partial f}{\partial \vec{P}} \frac{d\vec{P}}{dt} = 0$$

constitute *self-consistent system of equations* describing beam evolution in the field created by the beam itself

Applicability of Vlasov's Equation to Particle Dynamics

Vlasov's equation describes behavior of interactive particles in self field.

Charged particles within the beam interact between themselves:
(i) interaction of large number of particles resulted in smoothed collective charge density and current density distribution
(ii) individual particle - particle collisions, when particles approach to each other at the distance, much smaller than the average distance between particles.

First type of interaction results in generation of smoothed electromagnetic field, which, being added to the field of external sources, act at the beam as an external field.
The second type of interaction has a meaning of particle collisions resulting in appearance of additional fluctuating electromagnetic fields.

Applicability of Vlasov's Equation to Particle Dynamics (cont.)

Using Vlasov's eqauiton, we *formally* expand it to dynamics of interacting charged particles, assuming that the total electromagnetic filed of the structure $(U, \vec{A})$

$$U = U_{ext} + U_b$$

$$\vec{A} = \vec{A}_{ext} + \vec{A}_b$$

$U_{ext}, \vec{A}_{ext}$, external field

$U_b, \vec{A}_b$ field created by the beam.

and *neglecting* individual particle-particle interactions. Vlasov's equation treats collisionless plasma, where individual particle-particle interactions are negligible in comparison with the collective space charge field.

Applicability of Vlasov's Equation to Particle Dynamics (cont.)

Quantative treatment of validity of collisionless approximation dynamics to particle dynamics: n - particle density within the beam, $\bar{r}$ - the average distance between particles:

$$n\bar{r}^3 = 1 \quad \text{, or} \quad \bar{r} = n^{-1/3}$$

Individual particle-particle collisions are neglected, when kinetic energy of thermal particle motion within the beam is much larger than potential energy of Coulomb particle-particle interaction:

$$\frac{mv_t^2}{2} >> \frac{q^2}{4\pi\varepsilon_o \bar{r}}$$

v_t is the root-mean square velocity of chaotic particle motion within the beam:

$$\frac{mv_t^2}{2} = \frac{kT}{2}$$

T is the "temperature" of chaotic particle motion
$k = 8.617342 \times 10^{-5}$ eV K^{-1} = $1.3806504 \times 10^{-23}$ J K^{-1} is the Boltsman's constant.

Number of Particles in Debye Sphere

Radius of Debye shielding in plasma: $\lambda_D = \sqrt{\dfrac{\varepsilon_o kT}{q^2 n}}$

Combining all equation one gets: $\bar{r} << \sqrt{2\pi}\ \lambda_D$ or $(2\pi)^{3/2} n\lambda_D^3 >> 1$

Volume of Debye sphere is $V = (4/3)\pi\lambda_D^3$ and number of particles within Debye sphere is $N_D = (4/3)n\pi\lambda_D^3$.

Condition $(2\pi)^{3/2} n\lambda_D^3 >> 1$ can be rewritten as

$$\boxed{N_D = \frac{4}{3}\pi n\lambda_D^3 >> 1}$$

<u>Individual particle-particle collisions can be neglected if number of particles within Debye sphere is much larger than unity $N_D >> 1$ (or average distance between particles is much smaller than λ_D).</u>

Particle density within uniformly charged cylindrical beam of radius R, with current I, propagating with longitudinal velocity βc, is

$$n = \frac{I}{\pi q \beta c R^2}$$

Hamiltonian of Particle Motion in Quadrupole Focusing Channel

Hamiltonian of charged particle

$$H = c\sqrt{m^2c^2 + (P_x - qA_x)^2 + (P_y - qA_y)^2 + (P_z - qA_z)^2} + qU$$

Vector potential $\vec{A} = \vec{A}_{magn} + \vec{A}_b$

is a combination of that of magnetic lenses, $\vec{A}_{magn}$, and of that of the beam, $\vec{A}_b$,

Scalar potential $U = U_{el} + U_b$

is a combination of the scalar potential of the electrostatic focusing field, U_{el}, and of the space charge potential of the beam, U_b.

Hamiltonian of Particle Motion in Quadrupole Focusing Channel (cont.)

Vector - potential of an ideal magnetic quadrupole lens with gradient G inside the lens is given by

$$A_{z\,magn} = \frac{G}{2}(x^2 - y^2)$$

Electrostatic quadrupole with gradient G_{el}, creates the field with electrostatic potential

$$U_{el} = -\frac{G_{el}}{2}(x^2 - y^2)$$

Transversal components of mechanical momentum are equal to that of canonical momentum, $p_x = P_x$, $p_y = P_y$, and Hamiltonian can be written as:

$$K = c\sqrt{m^2c^2 + p_x^2 + p_y^2 + (P_z - q\,A_z)^2} + qU$$

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Hamiltonian of Particle Motion in Quadrupole Focusing Channel (cont.)

In the moving system of coordinates, particles are static, therefore, vector potential of the beam equals to zero, $\vec{A}_b' = 0$. According to Lorentz transformations, components of vector potential of the beam are converted into laboratory system of coordinates as follow

$$A_{xb} = 0\ , \qquad A_{yb} = 0\ , \qquad A_{zb} = \beta\frac{U_b}{c}$$

Total vector-potential of the structure is therefore

$$A_z = A_{z\ magn} + \frac{\beta}{c}U_b$$

Kinetic energy of the beam is typically much larger than the potential energy of focusing elements and than the potential energy of the beam. Therefore, $P_z >> qA_z$, and we can substitute canonical momentum by the mechanical momentum:

$$(P_z - qA_z)^2 \approx P_z^2 - 2\, P_z\, q\, A_z \approx p_z^2 - 2\, p_z\, q\, A_z$$

It corresponds to the case when longitudinal particle motion is not affected by the transverse motion, which is typical for beam transport.

Hamiltonian of Particle Motion in Quadrupole Focusing Channel (cont.)

Hamiltonian can be rewritten as

$$K = mc^2 \sqrt{(1 + \frac{p_z^2}{m^2c^2}) + \frac{p_x^2 + p_y^2}{m^2c^2} - \frac{2\,q\,p_z A_z}{m^2c^2}} + qU_{el} + qU_b$$

The term in brackets is close to square of reduced particle energy:

$$1 + \frac{p_z^2}{m^2c^2} \approx \gamma^2$$

Taking that term out of square root gives for Hamiltonian:

$$K = mc^2\gamma \sqrt{1 + \frac{p_x^2 + p_y^2}{(\gamma m\,c)^2} - \frac{2\,q\,p_z A_z}{(\gamma m\,c)^2}} + qU_{el} + qU_b$$

Hamiltonian of Particle Motion in Quadrupole Focusing Channel (cont.)

After expansion of small terms $\sqrt{1+x} \approx 1 + x/2$, the Hamiltonian becomes:

$$K = mc^2\gamma + \frac{p_x^2 + p_y^2}{2mc\gamma} - \frac{2\,q\,p_z\,(A_{z\ magn} + \frac{\beta}{c}U_b)}{2mc\gamma} + qU_{el} + qU_b$$

Removing the constant $mc^2\gamma$ results in the general form of Hamiltonian in a focusing channel:

$$H = \frac{p_x^2 + p_y^2}{2\,m\,\gamma} + q(U_{el} - \beta c A_{z\ magn}) + q\,\frac{U_b}{\gamma^2}$$

Both U_{el} and $A_{z\ magn}$ can be a combination of that of multipole lenses of an arbitrary order.

Kapcckinsky-Vladimirsky (KV) Beam Envelope Equations

Consider now dynamics of the beam in focusing quadrupole channel including space charge forces of the beam. All particles move with the same longitudinal velocity βc, and the longitudinal space charge forces are equal to zero. Hamiltonian of particle motion in qudrupole channel with space charge is given by

$$H = \frac{p_x^2 + p_y^2}{2m\gamma} + q\beta c G(z)\frac{x^2 - y^2}{2} + q\frac{U_b}{\gamma^2} \quad . \qquad (2.96)$$

Assume that transverse space charge forces are linear functions of coordinates. Correctness of this assumption will be checked later. Linear equation of motion are

$$\frac{d^2 x}{dz^2} + k_x^{'}(z)x = 0 \;, \qquad (2.97)$$

$$\frac{d^2 y}{d z^2} + k_y^{'}(z)\, y = 0 \;, \qquad (2.98)$$

where $k_x^{'}(z)$, $k_y^{'}(z)$ are modified focusing strengths including space charge. Equations of motion (2.97), (2.98) are linear, therefore, invariant of Courant-Snyder, is valid in both planes *(x, x'), (y, y')* for space charge regime as well.

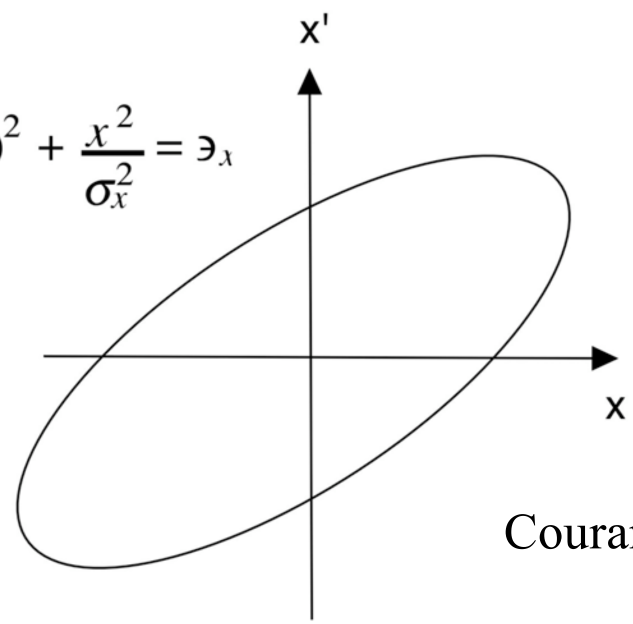
Courant-Snyder Invariants

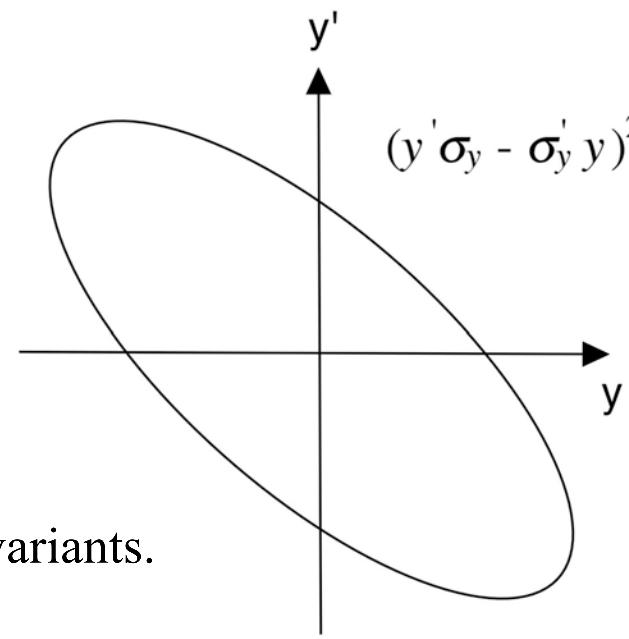
Self-consistent solution can be obtained when distribution function is expressed as a function of integrals of motion. Due to equations of motion in linear field are uncoupled, Courant-Snyder invariants are conserved at every phase plane:

$$(x'\sigma_x - \sigma_x' x)^2 + \frac{x^2}{\sigma_x^2} = \ni_x , \qquad (2.99)$$

$$(y'\sigma_y - \sigma_y' y)^2 + \frac{y^2}{\sigma_y^2} = \ni_y . \qquad (2.100)$$

$$(x'\sigma_x - \sigma_x' x)^2 + \frac{x^2}{\sigma_x^2} = \ni_x$$





$$(y'\sigma_y - \sigma_y' y)^2 + \frac{y^2}{\sigma_y^2} = \ni_y$$

Courant-Snyder invariants.

KV Distribution Function

Values of $\ni_x$, $\ni_y$ are areas of ellipses at phase planes (beam emittances), which are the constants of motion during beam transport. Let us express beam distribution function as a function of values $\ni_x$, $\ni_y$:

$$f = f_o\, \delta\,(\ni_x + \ni_y - F_o) \qquad (2.101)$$

where f_o, F_o, ν are constants defined below and $\delta(\xi)$ is the Dirac delta -function:

$$\delta(\xi) = \begin{cases} \infty, & \xi = 0 \\ 0, & \xi \neq 0 \end{cases}, \qquad (2.102)$$

$$\int_a^b f(\xi)\delta(\xi - X)\, d\xi = \begin{cases} 0, & X<a, \quad X>b, \\ 1/2 f(X), & X = a \ \ or \ \ X = b, \\ f(X), & a<X<b \end{cases} \qquad (2.103)$$

In the selected distribution, Eq. (2.101), particles are placed at the surface of four-dimensional ellipsoid:

$$F(x, x', y, y') = (x'\sigma_x - \sigma_x' x)^2 + \frac{x^2}{\sigma_x^2} + (y'\sigma_y - \sigma_y' y)^2 + \frac{y^2}{\sigma_y^2} - F_o = 0 . \qquad (2.104)$$

Boundary of (x - y) Projection of KV Beam Distribution

Let us find *boundary of projection* of the surface $F(x,x',y,y')=0$ on the plane (x, y). Boundary of projection of the four-dimensional surface $F(x,x',y,y')=0$ on arbitrary two-dimensional plane is obtained by equating to zero the partial derivatives of function $F(x,x',y,y')$ over the rest of variables:

$$\frac{\partial F(x,\ x',\ y,\ y')}{\partial x'}=0, \qquad \frac{\partial F(x,\ x',\ y,\ y')}{\partial y'}=0, \qquad (2.105)$$

and substitution of the solutions of equations (2.105) into equation $F(x,x',y,y')=0$. Actually, for every fixed value of x, the point at the boundary of projection corresponds to maximum possible value of y:

$$\frac{\partial y}{\partial x'}=0 \quad , \qquad \frac{\partial y}{\partial y'}=0, \qquad (2.106)$$

or, according to differentiation of implicit functions,

$$\frac{\partial y}{\partial x'}=-\frac{\dfrac{\partial F}{\partial x'}}{\dfrac{\partial F}{\partial y}}, \qquad \frac{\partial y}{\partial y'}=-\frac{\dfrac{\partial F}{\partial y'}}{\dfrac{\partial F}{\partial y}}, \qquad (2.107)$$

which coincides with Eq. (2.105).

Boundary of x-y Projection of KV Beam Distribution (cont.)

Partial derivatives over variables x', y' in equation of four-dimensional ellipsoid are:

$$\frac{\partial F}{\partial x'} = 2\,(x'\sigma_x - \sigma_x' x)\sigma_x = 0$$

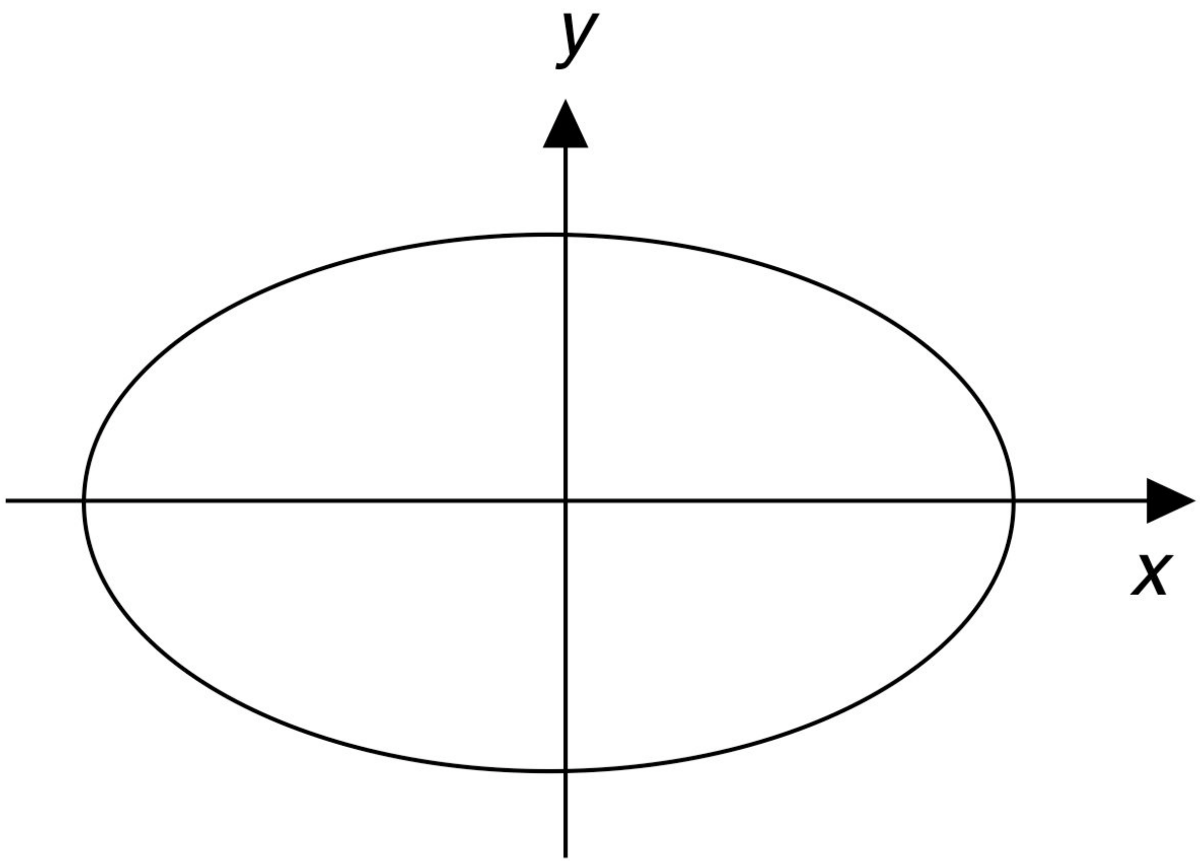
$$\frac{\partial F}{\partial y'} = 2\,(y'\sigma_y - \sigma_y' y)\sigma_y = 0$$

Substitution of solution of equations $\partial F/\partial x' = 0$, $\partial F/\partial y' = 0$ into equation $F(x, x', y, y') = 0$ gives the expression for the boundary of particle projection on plane (x, y):

$$\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} = F_o$$

Therefore, particles of KV beam distribution are surrounded by ellipse with semi-axes $R_x = \sigma_x\sqrt{F_o}$, $R_y = \sigma_y\sqrt{F_o}$ and the area of ellipse $S = \pi\,\sigma_x\,\sigma_y F_o$.

Boundary of (x - y) Projection of KV Beam Distribution (cont.)



Boundary of projection of KV beam on *(x,y)*.

Space Charge Density of KV Beam

Space charge density of the beam is an integral of distribution function over the rest variables x', y':

$$\rho(x,y) = f_o \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta\{(x'\sigma_x - \sigma_x' x)^2 + \frac{x^2}{\sigma_x^2} + (y'\sigma_y - \sigma_y' y)^2 + \frac{y^2}{\sigma_y^2} - F_o\} dx' dy' . \qquad (2.111)$$

To find particle density, Eq.(2.111), let us make substitution of the new variables, α, Ω, for old variables, x', y', according to transformation:

$$(x'\sigma_x - \sigma_x' x) = \alpha \cos\Omega , \qquad (2.112)$$

$$(y'\sigma_y - \sigma_y' y) = \alpha \sin\Omega . \qquad (2.113)$$

Inverse transformation is

$$x' = \frac{1}{\sigma_x}(\alpha \cos\Omega + x\sigma_x') , \qquad (2.114)$$

$$y' = \frac{1}{\sigma_y}(\alpha \sin\Omega + y\sigma_y') . \qquad (2.115)$$

Phase-space element is transformed according to: $dx' dy' = \begin{vmatrix} \frac{\partial x'}{\partial \alpha} & \frac{\partial x'}{\partial \Omega} \\ \frac{\partial y'}{\partial \alpha} & \frac{\partial y'}{\partial \Omega} \end{vmatrix} d\alpha\, d\Omega = \frac{\alpha\, d\alpha\, d\Omega}{\sigma_x \sigma_y}$ (2.116)

Space Charge Density of KV Beam (cont.)

With introduced transformation, Eqs. (2.112), (2.113), the space charge density of the beam is

$$\rho(x,\ y) = \frac{f_o}{\sigma_x \sigma_y} \int_o^{2\pi} \int_o^{\infty} \delta\left(\alpha^2 + \frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - F_o\right) \alpha\, d\alpha\, d\Omega \quad = \frac{\pi f_o}{\sigma_x \sigma_y} \int_o^{\infty} \delta\left(\alpha^2 + \frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - F_o\right) d\alpha^2$$

Let us use one more transformation: $\alpha^2 = u,$ (2.118)

$$\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - F_o = -u_o \quad . \qquad (2.119)$$

With new transformation, space charge density is $\rho(x,\ y) = \frac{\pi f_o}{\sigma_x\ \sigma_y} \int_o^{\infty} \delta(u - u_o)\, du \quad .$ (2.120)

As far as the value of u_o is always positive inside the ellipse, Eq. (2.110), the integral over delta function in Eq. (2.120) is equal to unity and space charge density is equal to constant:

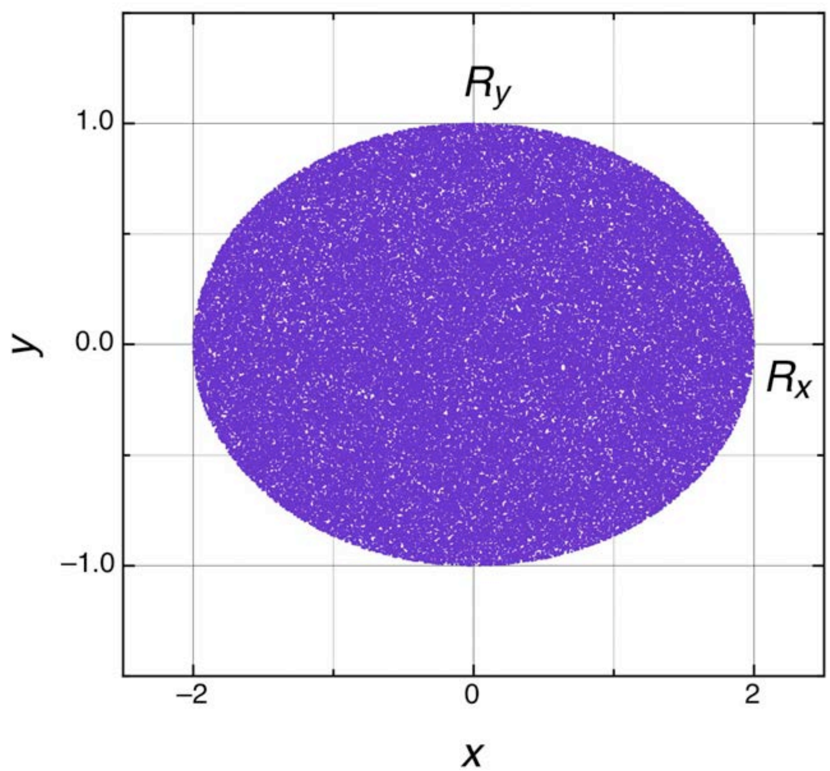
$$\rho(x,\ y) = \frac{\pi f_o}{\sigma_x\ \sigma_y} = \rho_o . \qquad (2.121)$$

KV distribution gives projection on plane (x, y) as uniformly populated ellipse, Eq. (2.110).

Space Charge Density of KV Beam (cont.)

Space charge density of elliptical beam with current I, semi-axis R_x, R_y, and longitudinal velocity β is

$$\rho_o = \frac{I}{\pi\beta c R_x R_y} \tag{2.122}$$



Projection of KV beam on *(x,y)*.

Boundary of KV Beam Distribution at *x-x'*

Consider particle distribution at phase plane *(x, x')*. Follow the method described above and put the following derivatives over variables *y*, *y'* to zero

$$\frac{\partial F(x, x', y, y')}{\partial y} = 0, \qquad \frac{\partial F(x, x', y, y')}{\partial y'} = 0 . \tag{2.123}$$

Substitution of the solution of Eqs. (2.123) into Eq. (2.101) gives us the boundary of particle distribution at phase plane (*x*, *x'*):

$$(x' \sigma_x - \sigma_x' x)^2 + \frac{x^2}{\sigma_x^2} = F_o , \tag{2.124}$$

which is also the ellipse. To find an area of ellipse, let us change the variables:

$$\begin{cases} \dfrac{x}{\sigma_x} = r_x \cos\theta \\ x \sigma_x' - x' \sigma_x = r_x \sin\theta \end{cases} . \tag{2.125}$$

Transformation, Eq. (2.125), in explicit form is

$$\begin{cases} x = r_x \sigma_x \cos\theta \\ x' = r_x \sigma_x' \cos\theta - \dfrac{r_x}{\sigma_x} \sin\theta \end{cases} . \tag{2.126}$$

Boundary of KV Beam Distribution at *x-x'* (cont.)

Phase space element is transformed analogously to Eq. (2.116) as

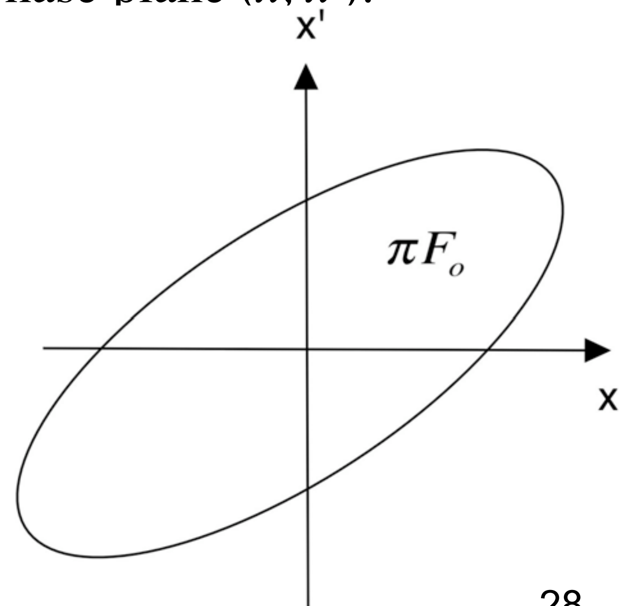
$$dx\, dx' = r_x\, dr_x\, d\theta \ . \tag{2.127}$$

With the new variables, equation for the ellipse boundary, Eq. (2.124), is $r_x^2 = F_o$. Area of the ellipse, occupied by the particles, is:

$$S = \int_o^{2\pi} \int_o^{F_o} r_x\, dr_x\, d\theta = \pi F_o \ . \tag{2.128}$$

Therefore, parameter $F_o = \ni_x$ is equal to beam emittance at phase plane (x, x').

Boundary of KV beam projection on *(x,x')*.



KV Beam Distribution at *x-x'* (cont.)

Distribution of particles at phase plane, $\rho_x(x, x')$, is obtained via integration of distribution function, Eq. (2.101), over remaining variables y, y':

$$\rho(x,x') = f_o \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta\{(x'\sigma_x - \sigma'_x x)^2 + \frac{x^2}{\sigma_x^2} + (y'\sigma_y - \sigma'_y y)^2 + \frac{y^2}{\sigma_y^2} - F_o\} dy\, dy' . \qquad (2.129)$$

Let us make transformation from variables y, y' to new variables T, ψ in Eq. (2.129):

$$(y'\sigma_y - \sigma'_y y)^2 = T\cos\psi , \qquad \frac{y^2}{\sigma_y^2} = T\sin\psi \qquad (2.130)$$

Phase space element $dy\, dy'$ is transformed analogously to (2.116):

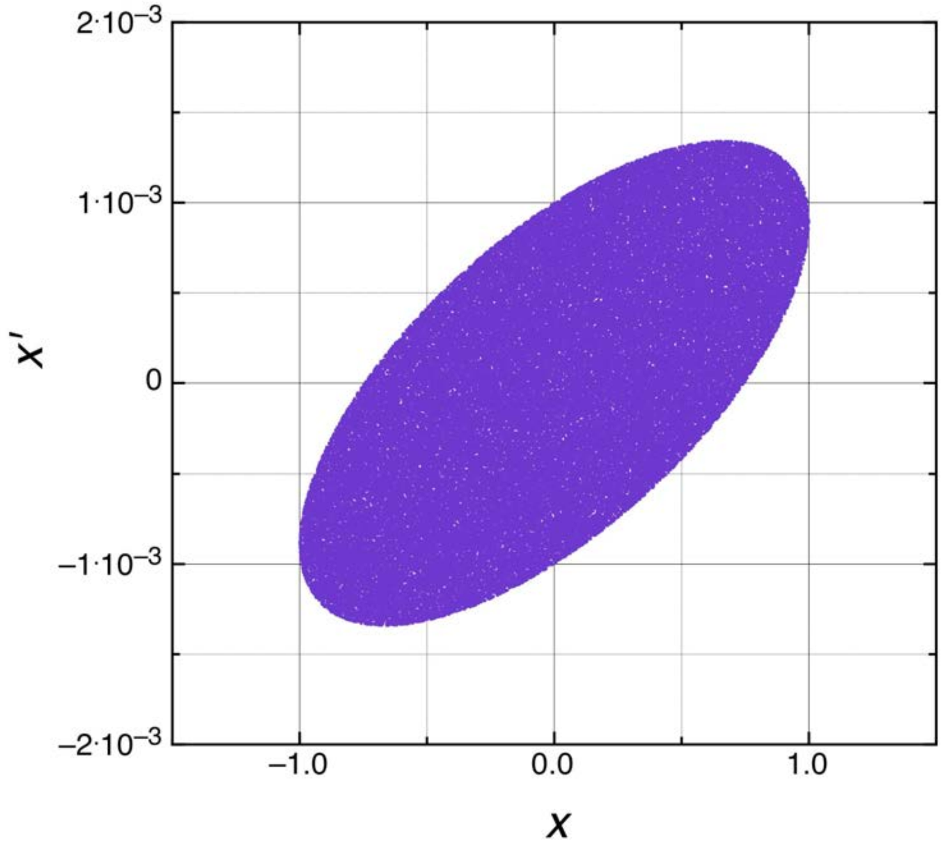
$$dy\, dy' = T\, dT\, d\psi . \qquad (2.132)$$

Integration of Eq. (2.129) gives distribution in phase plane $\rho_x(x, x') = \rho_x(r_x^2)$:

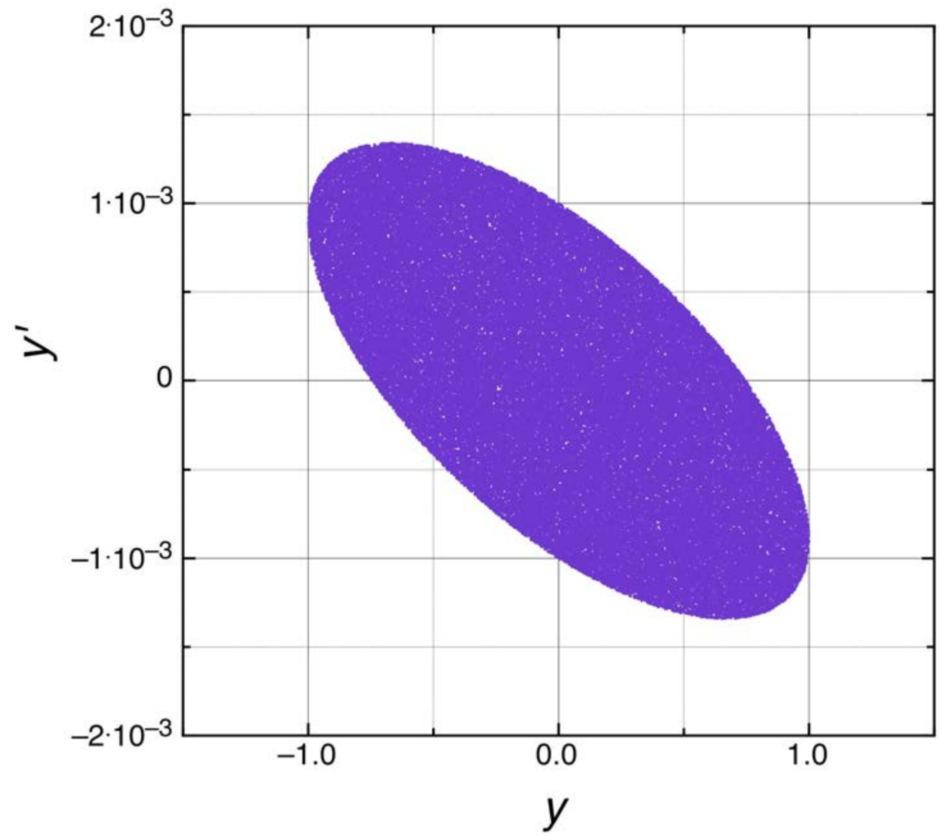
$$\rho_x(r_x^2) = \pi f_o \int_o^{\infty}\int_o^{2\pi} \delta(r_x^2 + T^2 - F_o)\, T\, dT\, d\psi = \pi f_o . \qquad (2.133)$$

Integral, Eq. (2.133), is evaluated in the same way as that in Eq. (2.117). Therefore, distribution of particles at phase plane (x, x') is uniform inside the ellipse, Eq. (2.124).

KV Beam Distribution on *x-x’* , *y-y’*



Projection of KV beam on (x-x’)



Projection of KV beam on (y-y’)

KV distribution provides two-dimensional elliptical projections at every pair of phase-space coordinates with uniform particle distribution within each ellipse.

Space Charge Potential of Elliptical Beam

Potential of the beam, U_b, is to be found from Poisson's equation:

$$\frac{\partial^2 U_b}{\partial x^2} + \frac{\partial^2 U_b}{\partial y^2} = -\frac{\rho(z)}{\varepsilon_o} \quad , \tag{2.136}$$

where space charge density

$$\rho(z) = \begin{cases} \dfrac{I}{\pi\beta c R_x R_y}, & \dfrac{x^2}{R_x^2} + \dfrac{y^2}{R_y^2} \le 1 \\ 0, & \dfrac{x^2}{R_x^2} + \dfrac{y^2}{R_y^2} \ge 1 \cdot \end{cases} \tag{2.137}$$

Solution of Eq. (2.136) for potential of elliptical charged cylinder with current I and beam envelopes R_x, R_y is:

$$U_b(x, y, z) = -\frac{I}{4\pi\varepsilon_o \beta c R_x R_y}\left[x^2 + y^2 - \frac{R_x - R_y}{R_x + R_y}(x^2 - y^2)\right] \quad , \tag{2.138}$$

and field components $\vec{E} = -grad U_b$ are:

$$E_x = \frac{I}{\pi\varepsilon_o \beta c R_x (R_x + R_y)} x \quad , \qquad E_y = \frac{I}{\pi\varepsilon_o \beta c R_y (R_x + R_y)} y \tag{2.139}$$

Uniformly populated beam with elliptical cross section provides linear space charge forces. Therefore, initial suggestion about linearity of particle equations of motion in presence of space charge forces is correct.

KV Envelope Equations

Hamiltonian of particle motion within the beam with elliptical cross section is:

$$H = \frac{p_x^2 + p_y^2}{2m\gamma} + q\beta c G(z)\frac{(x^2 - y^2)}{2} - \frac{qI}{4\pi\varepsilon_o \beta\gamma^2 c R_x R_y}[x^2 + y^2 - \frac{R_x - R_y}{R_x + R_y}(x^2 - y^2)]. \quad (2.141)$$

Equations of particle motion in presence of space charge forces are:

$$\frac{d^2x}{dz^2} + [k_x(z) - \frac{4I}{I_c \beta^3\gamma^3 R_x(R_x + R_y)}]\, x = 0 \quad , \quad (2.142)$$

$$\frac{d^2y}{dz^2} + [k_y(z) - \frac{4I}{I_c \beta^3\gamma^3 R_y(R_x + R_y)}]\, y = 0 \quad . \quad (2.143)$$

Characteristic current:

$$\boxed{I_c = 4\pi\varepsilon_o \frac{mc^3}{q} = 3.13\cdot 10^7 \frac{A}{Z}[Ampere]}$$

Eqs. (2.142), (2.143) are similar to that without space charge forces, where instead of functions $k_x(z)$, $k_y(z)$ the modified functions $\tilde{k}_x(z)$, $\tilde{k}_y(z)$ are used:

$$\tilde{k}_x(z) = k_x(z) - \frac{4I}{I_c \beta^3\gamma^3 R_x(R_x + R_y)} , \quad (2.144)$$

$$\tilde{k}_y(z) = k_y(z) - \frac{4I}{I_c \beta^3\gamma^3 R_y(R_x + R_y)} \quad (2.145)$$

KV Envelope Equations (cont.)

Substitution of expressions (2.144), (2.145) instead of $k_x(z)$, $k_y(z)$ into envelope equations (2.56), (2.57) gives us the *KV envelope equations* for the beam with space charge forces:

$$\frac{d^2R_x}{dz^2} - \frac{\epsilon_x^2}{R_x^3} + k_x(z)\,R_x - \frac{4\,I}{I_c\,\beta^3\gamma^3(R_x + R_y)} = 0\ , \qquad (2.146)$$

$$\frac{d^2R_y}{dz^2} - \frac{\epsilon_y^2}{R_y^3} + k_y(z)\,R_y - \frac{4\,I}{I_c\,\beta^3\gamma^3(R_x + R_y)} = 0. \qquad (2.147)$$

Equations (2.146), (2.147) are non-linear differential equations of the second order. They can be formally derived from Hamiltonian:

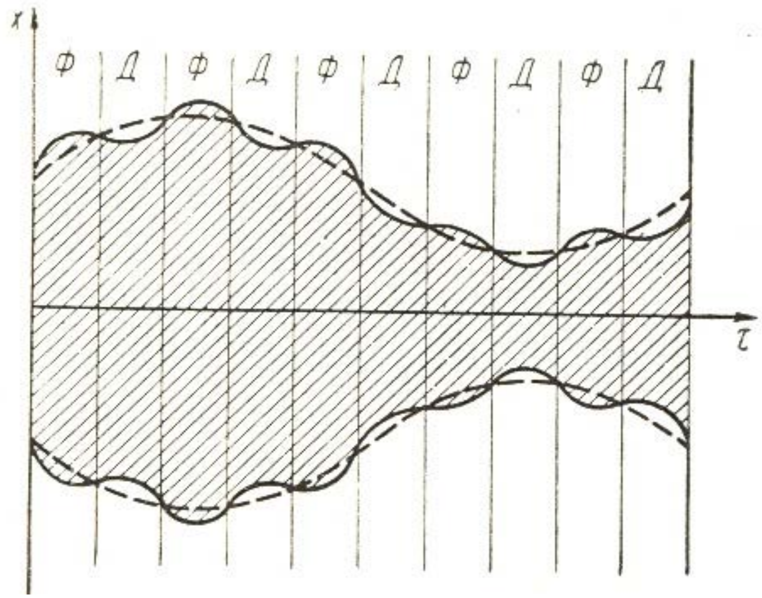
$$H = \frac{(R_x^{'})^2}{2} + \frac{(R_y^{'})^2}{2} + k_x(z)\frac{R_x^2}{2} + k_y(z)\frac{R_y^2}{2} + 2P^2 ln\frac{1}{R_x+R_y} + \frac{\epsilon_x^2}{2R_x^2} + \frac{\epsilon_y^2}{2R_y^2}\ , \qquad (2.148)$$

where parameter P^2 is called the generalized perveance

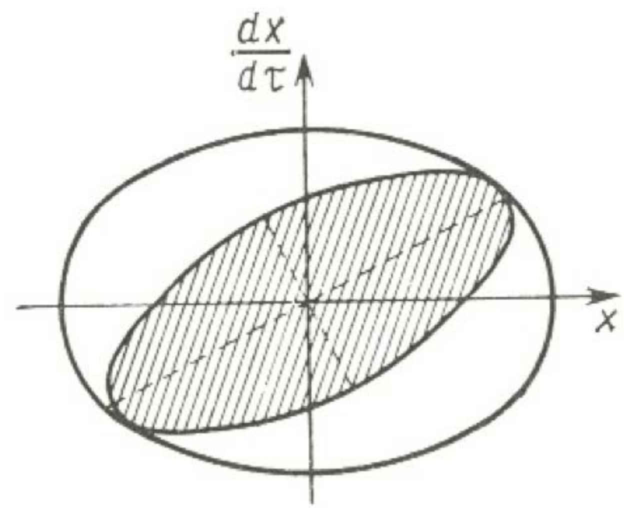
$$P^2 = \frac{2I}{I_c\,\beta^3\gamma^3}\ . \qquad (2.149)$$

KV Envelope Equations (cont.)

In general case, solution of the set of envelope equations, Eqs. (2.146), (2.147) are non-periodic functions, which corresponds to envelopes of unmatched beam. However, if functions $k_x(z)$, $k_y(z)$ are periodic, there is a periodic solution of envelope equations. Envelope equations can be solved numerically at the period of structure via varying the initial conditions $R_x(0), R_x'(0), R_y(0), R_y'(0)$ unless the solution at the end of period coinsides with initial conditions $R_x(L) = R_x(0)$, $R_x'(L) = R_x'(0)$, $R_y(L) = R_y(0)$, $R_y'(L) = R_y'(0)$. Again, as in case of beam with negligible current, this beam is called the matched beam. It occupies the smallest fraction of aperture of the channel.



The envelope of unmatched beam in a quadrupole channel



Effective beam emittance.

Averaged Beam Envelopes

Consider periodic focusing structure with periodic focusing function $k(z) = k_x(z) = -k_y(z)$. For focusing channels, where phase advance $\mu_o \leq 60^o$, one can use smooth approximation to beam envelopes. Let us rewrite envelope equations as

$$\frac{d^2 R_x}{dz^2} = -\frac{qG(z)}{mc\beta\gamma} R_x - \frac{\partial U(R_x, R_y)}{\partial R_x} \qquad \frac{d^2 R_y}{dz^2} = \frac{qG(z)}{mc\beta\gamma} R_y - \frac{\partial U(R_x, R_y)}{\partial R_y}$$

where potential function

$$U(R_x, R_y) = -\frac{4I}{I_c(\beta\gamma)^3} \ln(R_x + R_y) + \frac{\ni_x^2}{2R_x^{\ 2}} + \frac{\ni_y^2}{2R_y^{\ 2}}$$

Analogously to particle trajectories in smoothed approximation, solution for beam envelopes can be represented as

$$R_x(z) = \overline{R_x}(z) + \xi_x(z) \qquad R_y(z) = \overline{R_y}(z) + \xi_y(z)$$

where $\overline{R_x}(z)$, $\overline{R_y}(z)$ are smoothed envelopes, and $\xi_x(z)$, $\xi_y(z)$ are small fast oscillating functions. After averaging, fast oscillating term is substituted as $\dfrac{qG(z)}{mc\beta\gamma} \rightarrow (\dfrac{\mu_o}{S})^2$.

Averaged Beam Envelopes

Equations for smooth envelopes are

$$\frac{d^2\bar{R}_x}{dz^2} - \frac{\epsilon_x^2}{\bar{R}_x^{\,3}} + \frac{\mu_o^2}{S^2}\bar{R}_x - \frac{4I}{I_c(\beta\gamma)^3(\bar{R}_x+\bar{R}_y)} = 0$$

$$\frac{d^2\bar{R}_y}{dz^2} - \frac{\epsilon_y^2}{\bar{R}_y^{\,3}} + \frac{\mu_o^2}{S^2}\bar{R}_y - \frac{4I}{I_c(\beta\gamma)^3(\bar{R}_x+\bar{R}_y)} = 0$$

Small oscillating functions are determined by fast oscillating terms only. Therefore, solution for small oscillating parts are are the same as that for single-particle:

$$\xi_x = \upsilon_{\max}\bar{R}_x \sin(\frac{2\pi\beta ct}{S}) \qquad \xi_y = -\upsilon_{\max}\bar{R}_y \sin(\frac{2\pi\beta ct}{S})$$

Solution of envelope equations in smooth approximations are

$$R_x(z) = \bar{R}_x(z)[1+\upsilon_{\max}\sin(2\pi\frac{z}{S})]$$

$$R_y(z) = \bar{R}_y(z)[1-\upsilon_{\max}\sin(2\pi\frac{z}{S})]$$

Matched Beam with Negligible Current

In the limit of negligible current, $I = 0$, envelope equations are decoupled. Consider matched beam, $\bar{R}_x'' = \bar{R}_y'' = 0$, with equal emittances in both planes $\epsilon_x = \epsilon_y = \epsilon$:

$$-\frac{\epsilon^2}{\bar{R}_x^3} + \frac{\mu_o^2}{S^2}\bar{R}_x = 0, \qquad -\frac{\epsilon^2}{\bar{R}_y^3} + \frac{\mu_o^2}{S^2}\bar{R}_y = 0$$

Equations have the common solution, $\bar{R}_x = \bar{R}_y = R_o$:

$$\boxed{R_o^2 = \frac{\epsilon S}{\mu_o}}$$

It defines the averaged beam radius in quadrupole channel for the beam with negligible space charge forces. Beam envelopes for negligible current:

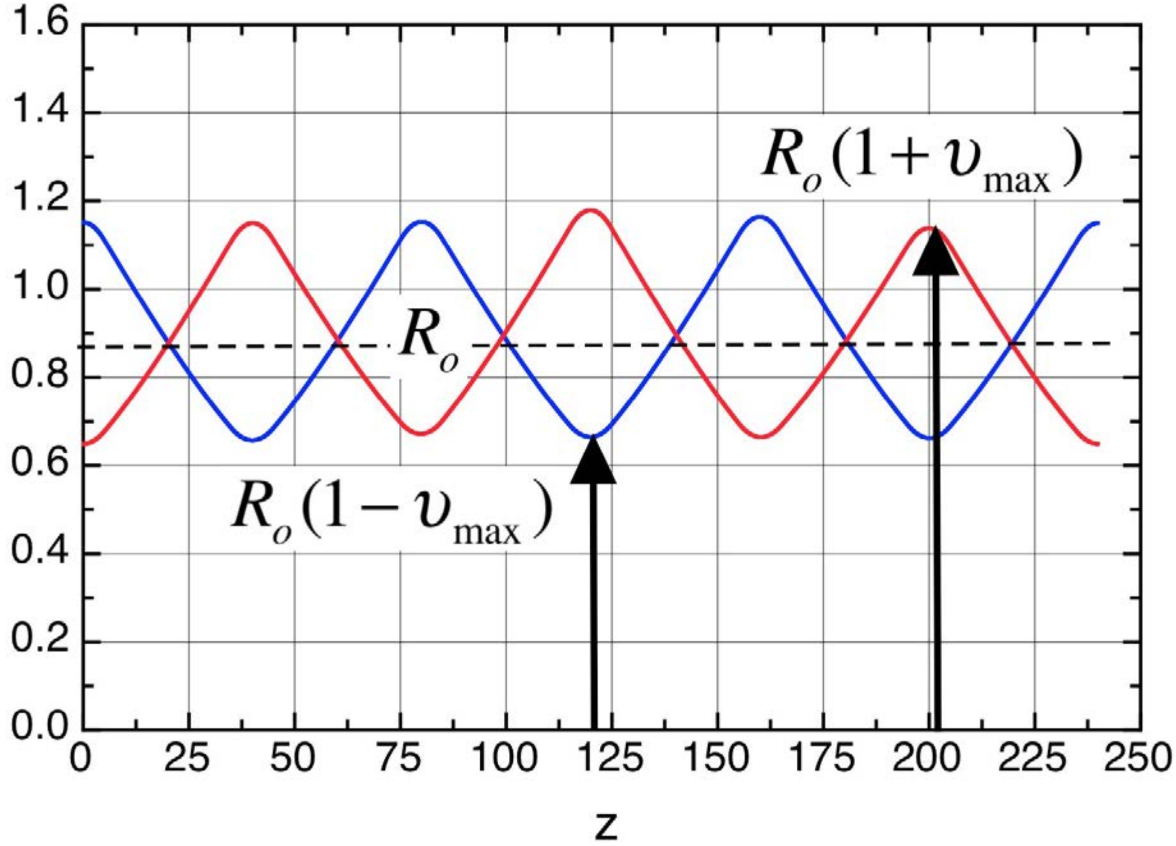
$$\boxed{R_x(z) = R_o[1 + \upsilon_{\max}\sin(2\pi\frac{z}{S})]} \qquad \boxed{R_y(z) = R_o[1 - \upsilon_{\max}\sin(2\pi\frac{z}{S})]}$$

where relative amplitude of envelope oscillation in FODO channel from averaging method:

$$\upsilon_{\max} = \frac{2}{\pi^2\sqrt{1 - \frac{4}{3}\frac{D}{S}}}\frac{\sin(\pi\frac{D}{S})}{(\pi\frac{D}{S})}\mu_o \approx 0.2026\mu_o \quad (\text{for } D << S)$$

Matrix method gave for FODO channel: $R_o = \sqrt{\frac{\epsilon S}{\sin\mu_o}}$, $\upsilon_{\max} \approx \frac{\mu_o}{4}$

Matched Beam with Negligible Current (cont.)



Acceptance of the Channel from Envelope Equations

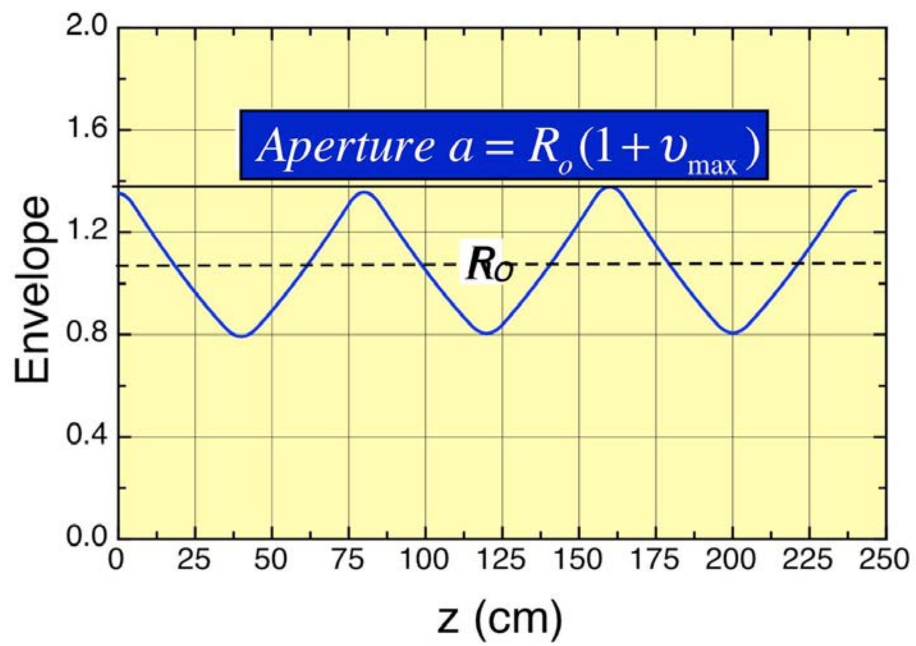
Aperture, a, is reached by the beam with maximum possible emittance in the points where $R_o(1+\upsilon_{\max}) = a$. Acceptance of the channel obtained from envelope equations:

$$A_{env} = \frac{a^2 \mu_o}{S\,(1+\upsilon_{\max})^2}$$

Acceptance of FODO channel with thin lenses, $D/S \ll 1$, estimated from envelope equations, is:

$$A_{env} = \frac{a^2 \mu_o}{S\,(1+0.203\mu_o)^2}$$

Normalized acceptance of the channels: $\varepsilon_{ch} = \beta\gamma A$



<u>Compare with FODO acceptance obtained from matrix method:</u>

$$\boxed{A = \frac{a^2}{S}\frac{\sin\mu_o}{(1+\sin\frac{\mu_o}{2})}}$$

Beam Radius in Space-Charge Dominated Regime

When space charge forces are not negligible, smoothed KV equations

for matched beam, $\overline{R}_x'' = \overline{R}_y'' = 0$:

$$-\frac{\text{э}^2}{\overline{R}_x^3} + \frac{\mu_o^2}{S^2}\overline{R}_x - \frac{4I}{I_c(\beta\gamma)^3(\overline{R}_x + \overline{R}_y)} = 0\,, \qquad -\frac{\text{э}^2}{\overline{R}_y^3} + \frac{\mu_o^2}{S^2}\overline{R}_y - \frac{4I}{I_c(\beta\gamma)^3(\overline{R}_x + \overline{R}_y)} = 0$$

Equations have common solution $\overline{R}_x = \overline{R}_y = R_e$: $-\frac{\text{э}^2}{R_e^3} + \frac{\mu_o^2}{S^2}R_e - \frac{2I}{I_c(\beta\gamma)^3 R_e} = 0$

Which can be rewritten as

$$R_e - \frac{R_o^4}{R_e^3} - \frac{2IR_o^4}{I_c(\beta\gamma)^3 R_e\, \text{э}^2} = 0$$

From the last equation, the averaged beam radius in space – charge regime is expressed via beam radius with negligible space charge forces as

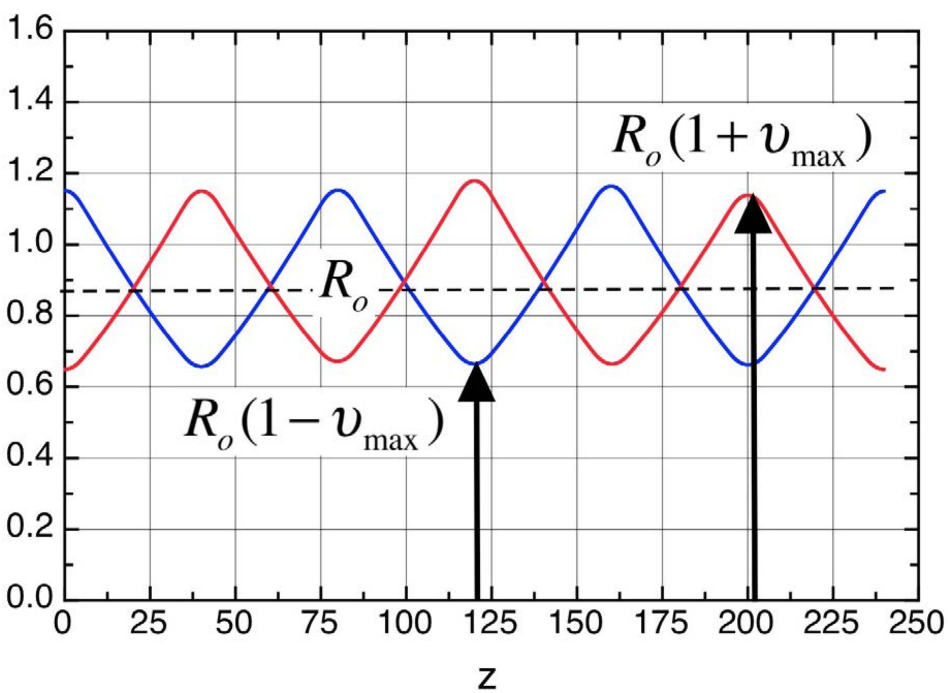
$$\boxed{R_e = R_o\sqrt{b_o + \sqrt{1 + b_o^2}}}$$

where b_o is the space charge parameter

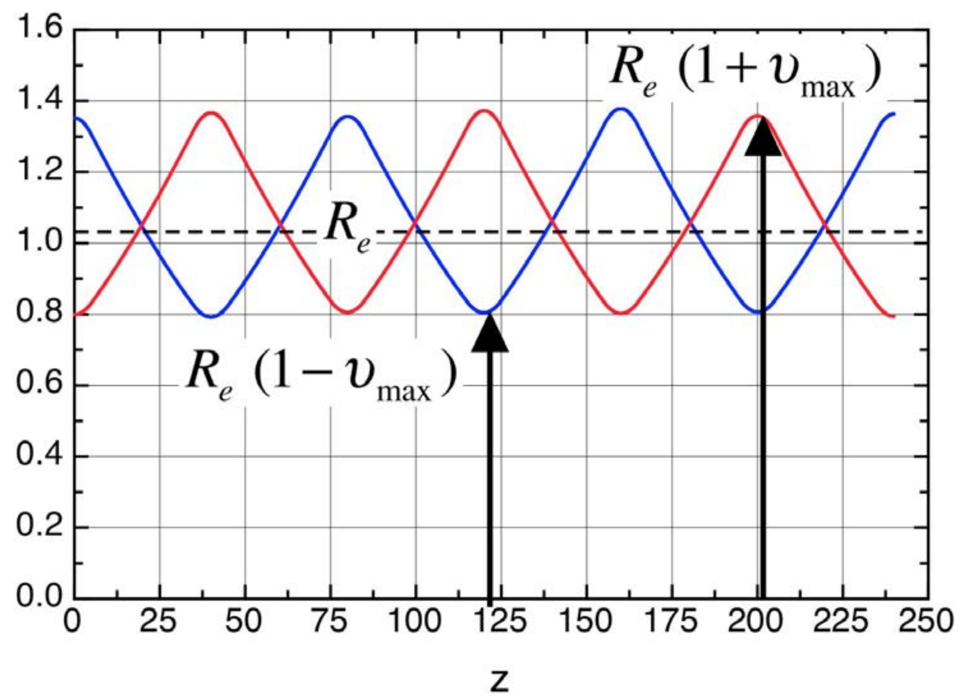
$$\boxed{b_o = \frac{1}{(\beta\gamma)^3}\frac{I}{I_c}\left(\frac{R_o}{\text{э}}\right)^2 = \frac{1}{\beta\gamma}\frac{I}{I_c}\left(\frac{R_o}{\varepsilon}\right)^2}$$

Matched Beam Envelopes Versus Beam Current

$I = 0$

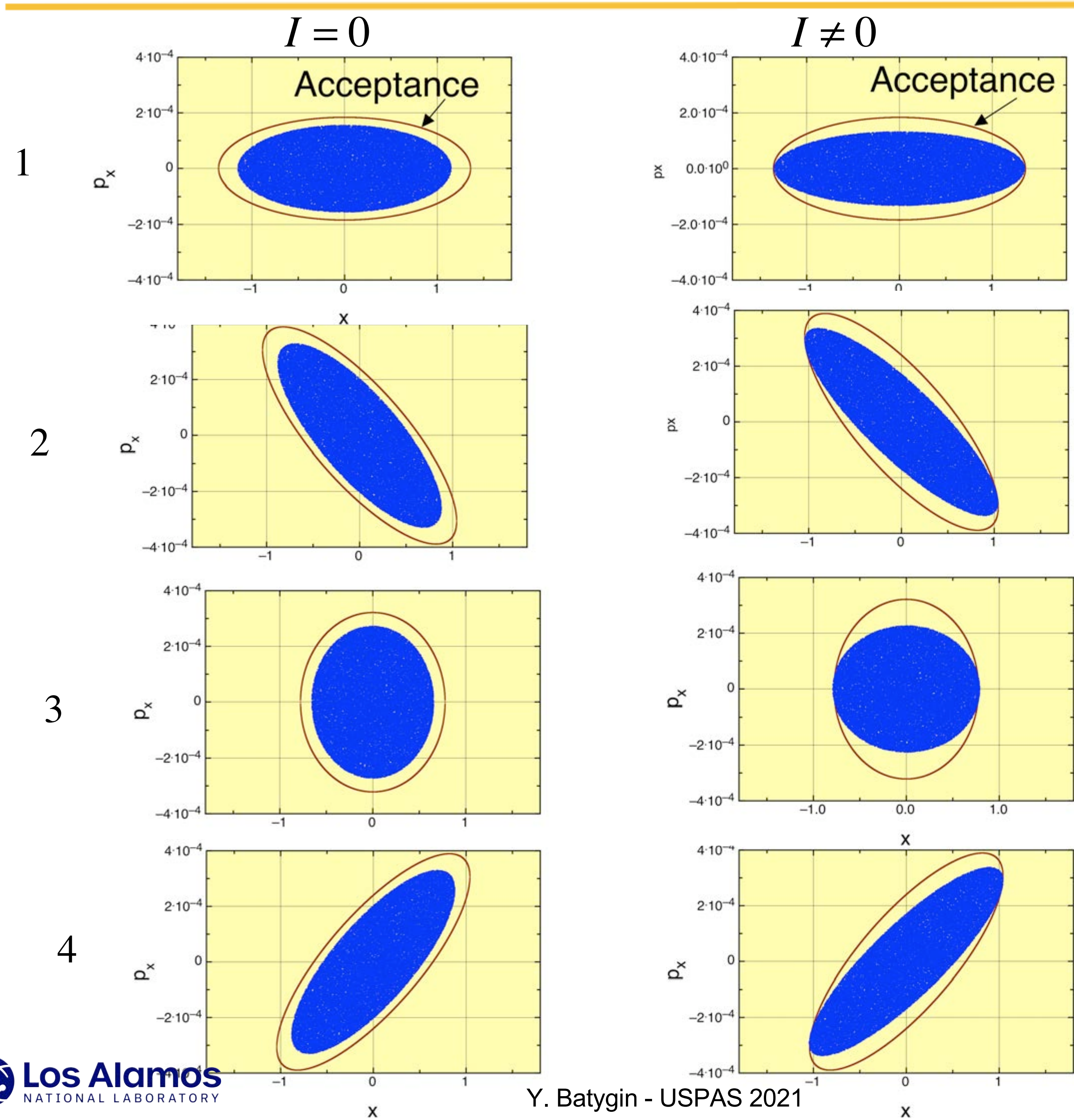


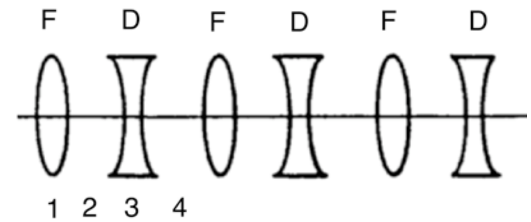
$I \neq 0$



Envelopes of the beam with negligible current and non-negligible current. While average beam radius is different, relative amplitude of envelope oscillations is the same in smooth approximation

Matched Beam Versus Beam Current (cont.)





Ellipses of matched beam with non-zero current no longer coincide with Floquet ellipses.

Depressed Transverse Oscillation Frequency

Eqs. (2.142), (2.143) define particle trajectory in quadrupole channel in presence of space charge field of the uniformly populated beam with elliptical cross-section. Taking $\bar{R}_x \approx \bar{R}_y = R_e$, equation for single particle trajectory in smoothed approximation is

$$\frac{d^2 X}{dz^2} + [\frac{\mu_o^2}{S^2} - \frac{2I}{I_c(\beta\gamma)^3 R_e^2}]X = 0 ,$$

and similar for y - direction. It can be re-written as

$$\frac{d^2 X}{dz^2} + \frac{\mu^2}{S^2} X = 0 ,$$

where μ is the averaged betatron frequency in presence of space charge forces, which is also called the *depressed betaron tune*:

$$\mu^2 = \mu_o^2 - \frac{2I}{I_c(\beta\gamma)^3}(\frac{S}{R_e})^2 .$$

Equation for depressed betatron tune indicates that space charge forces result in reduction of frequency of transverse oscillations. It can be rewritten as

$$\mu = \mu_o(\sqrt{1+b_o^2} - b_o) .$$

Effect of Space Charge on Beam Size and Phase Advance

Transverse oscillation frequency drops with increase of beam current, but remains non-zero. Therefore, *beam stability can be provided at any value of beam current*. However, increase of beam current requires increase of aperture of the channel, and *stability of transverse oscillations can be provided at arbitrary high value of beam current, but in the channel with infinitely large aperture*.

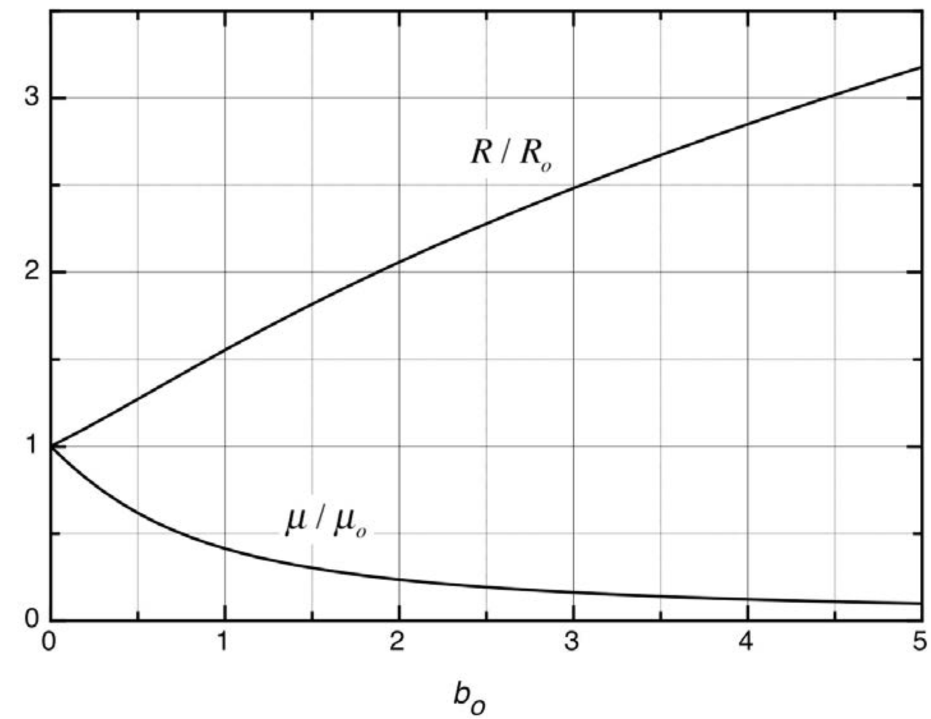
Ratio of depressed to undepressed phase shift

$$\frac{\mu}{\mu_o} = \sqrt{1+b_o^2} - b_o = \frac{1}{\sqrt{1+b_o^2} + b_o}$$

serves as an indication of space charge dominance:

$\mu / \mu_o < 0.7$ space-charge-dominated regime,

$\mu / \mu_o > 0.7$ emittance dominated regime.



Averaged beam radius and transverse oscillation frequency as functions of space charge parameter b_o.

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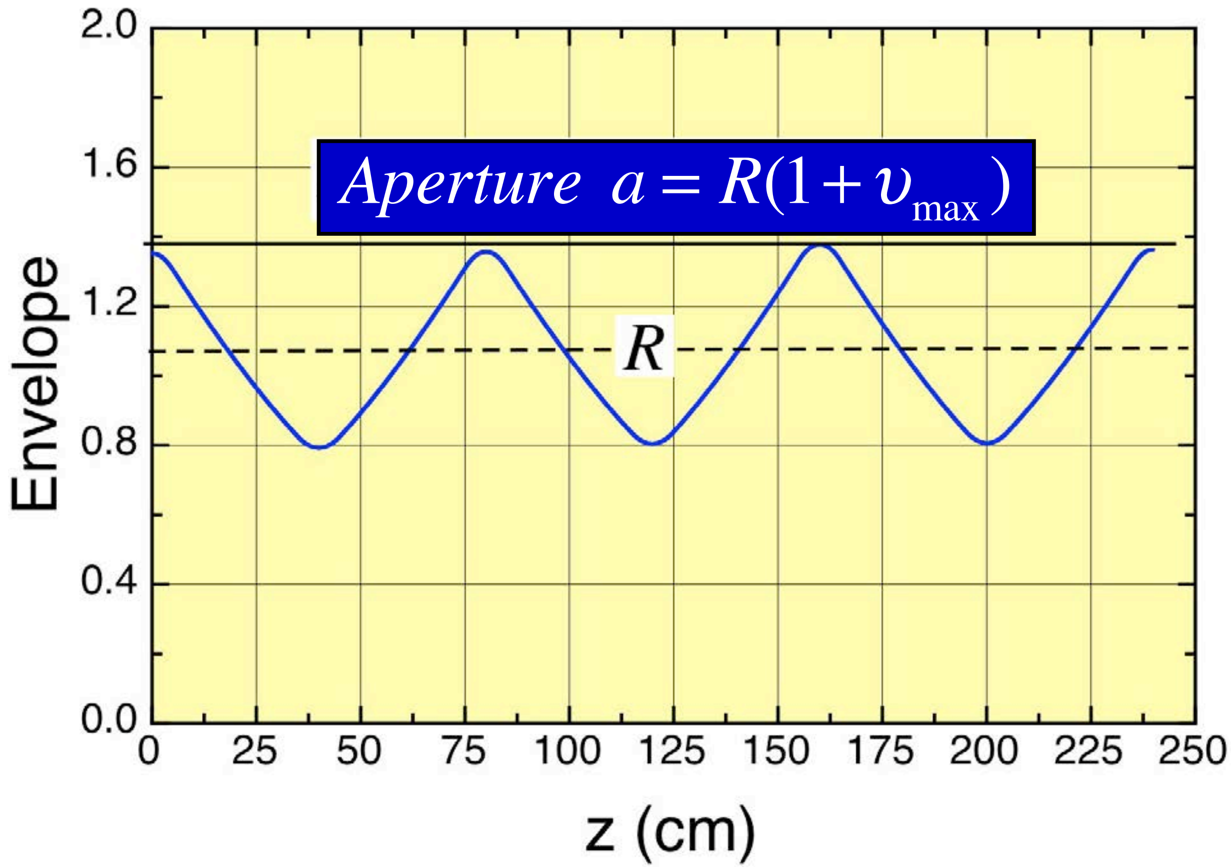
Beam Current Limit



Beam current limit corresponds to the beam, which fills in all available aperture.

Beam Current Limit (cont.)

Maximum beam current in quadrupole focusing channel corresponds to the beam, which fills in all available aperture, $a = R(1+\upsilon_{\max})$:

$$a = \sqrt{\frac{\ni S}{\mu_o}} \sqrt{b_o + \sqrt{1+b_o^2}}\,(1+\upsilon_{\max})$$

For $b_o = 0$, this equation describes the beam with maximum possible emittance in the channel, equal to acceptance of the channel, $\ni = A_{env}$:

$$a = \sqrt{\frac{A_{env} S}{\mu_o}}(1+\upsilon_{\max})$$

Ratio of equations gives us the relationship between acceptance of the channel and the maximum emittance of the beam with non-zero current, which fills in all aperture of the channel:

$$\ni = A_{env}(\sqrt{1+b_o^2} - b_o)$$

Substitution of the expression for space charge parameter b_o gives for maximum transported beam current:

$$\boxed{I_{\max} = \frac{I_c}{2}\frac{\mu_o}{S} A_{env}(\beta\gamma)^3 [1-(\frac{\ni}{A_{env}})^2]}$$

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Non-Uniform Beam Equilibrium in Linear Field

In general case, the Hamiltonian is not a constant of motion, because potentials can depend on time, $\vec{A} = \vec{A}(t)$, $U=U(t)$. Note that even if the potentials of the external field, $\vec{A}_{ext}$, U_{ext}, are time-independent, the beam field potentials, $\vec{A}_b$, U_b, might still depend on time, and the Hamiltonian remains time-dependent. If an additional condition of matching the beam with the channel (where the beam distribution remains stationary) is applied, explicit dependence on time disappears from the beam potentials. In this case, the Hamiltonian becomes time-independent, and therefore, is an integral of motion. The Hamiltonian, can then be used to find the unknown distribution function of the beam via the expression $f = f(H)$ and the subsequent solution of equation for space charge potential (Kapchinsky, 1985).

Hamiltonian corresponding to the motion in averaged linear focusing field is given by

$$H = \frac{p_x^2 + p_y^2}{2m\gamma} + \frac{m\gamma\,\Omega_r^2}{2}(x^2 + y^2) + q\frac{U_b}{\gamma^2}, \tag{4.26}$$

where Ω_r is the frequency of smoothed particle oscillations. If the beam is matched with the continuous channel, space charge potential U_b is constant, and Hamiltonian is a constant of motion.

Non-Uniform Beam Equilibrium in Linear Field (cont.)

Let us transform Hamiltonian, Eq. (4.26), to another one, multiplying Eq. (4.26) by a constant:

$$K = \frac{L^2}{m\gamma\,(\beta c)^2} H \qquad (4.39)$$

It corresponds to changing of independent time variable t for dimensionless time $\tau = t\beta c / L$. New Hamiltonian is given by

$$K = \frac{\dot{x}^2 + \dot{y}^2}{2} + \frac{\mu_o^2}{2}(x^2 + y^2) + \frac{q L^2 U_b}{m c^2 \gamma^3 \beta^2}, \qquad (4.40)$$

where $\dot{x} = dx / d\tau$, $\dot{y} = dy / d\tau$. Let us use particle radius $R^2 = x^2 + y^2$ and total transverse momentum $P^2 = \dot{x}^2 + \dot{y}^2$, where

$$\dot{x} = P \cos\theta, \qquad \dot{y} = P \sin\theta. \qquad (4.41)$$

Hamiltonian, Eq. (4.40), is now

$$K = \frac{P^2}{2} + \frac{\mu_o^2}{2} R^2 + \frac{q L^2 U_b}{m c^2 \gamma^3 \beta^2} \qquad (4.42)$$

Non-Uniform Beam Equilibrium in Linear Field (cont.)

Consider the following distribution:

$$f = \begin{cases} f_o, & K \le K_o \\ 0, & K > K_o \end{cases} . \qquad (4.43)$$

According to transformation, Eq. (4.41), space charge density of the beam is expressed as

$$\rho(R) = 2\pi q f_o \int_o^{P_{max}(R)} PdP = \pi q f_o P_{max}^2(R) . \qquad (4.44)$$

For each value of R, the maximum value of transverse momentum P_{max} *(R)* is achieved for $K = K_o$. From Eq. (4.40)

$$P_{max}^2(R) = 2K_o - \mu_o^2 R^2 - \frac{2qL^2 U_b}{mc^2\gamma^3\beta^2} . \qquad (4.45)$$

Therefore, space charge density, Eq. (4.44), is

$$\rho(R) = \pi q f_o \left(2K_o - \mu_o^2 R^2 - \frac{2qL^2 U_b}{mc^2\gamma^3\beta^2}\right) . \qquad (4.46)$$

Poisson's equation for unknown space charge potential of the beam U_b is

$$\frac{1}{R}\frac{d}{dR}\left(R\frac{dU_b}{dR}\right) = -\frac{\pi q f_o}{\varepsilon_o}\left(2K_o - \mu_o^2 R^2 - \frac{2qL^2 U_b}{mc^2\gamma^3\beta^2}\right) . \qquad (4.47)$$

Non-Uniform Beam Equilibrium in Linear Field (cont.)

Let us introduce notation:

$$R_o = \frac{\varepsilon_o m c^2 \beta^2 \gamma^3}{2\pi q^2 f_o L^2}, \qquad s = \frac{R}{R_o}, \qquad (4.48)$$

Then, Poisson's equation, Eq. (4.47) is

$$\frac{1}{s}\frac{d}{ds}\left(s\frac{dU_b}{ds}\right) - U_b = \frac{m c^2 \beta^2 \gamma^3}{q L^2}\left(\frac{\mu_o^2 s^2 R_o^2}{2} - K_o\right). \qquad (4.49)$$

Solution of differential equation (4.49) is a combination of general solution of the homogeneous equation $U_b^{(u)} = C_o I_o(s)$ and of a particular solution of non-homogeneous equation $U_b^{(n)} = C_1 s^2 + C_2$:

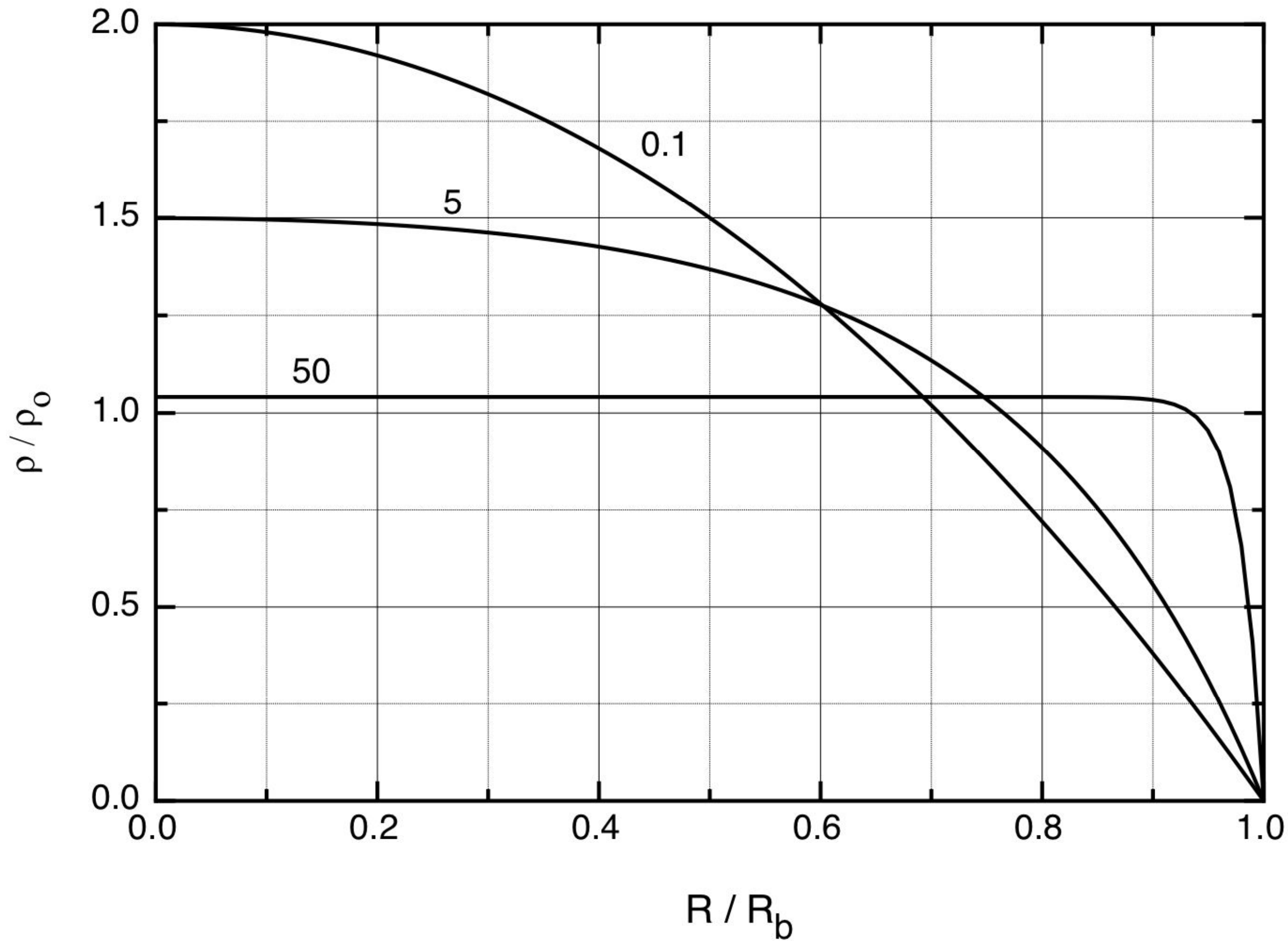
$$U_b = \frac{m c^2 \beta^2 \gamma^3}{q L^2}\left[(2\mu_o^2 R_o^2 - K_o)(I_o(s) - 1) - \frac{\mu_o^2 s^2 R_o^2}{2}\right]. \qquad (4.57)$$

Space charge density profile

$$\rho\left(s_b \frac{R}{R_b}\right) = \frac{\rho_o}{\left[1 - \frac{2 I_1(s_b)}{s_b I_o(s_b)}\right]}\left[1 - \frac{I_o\left(s_b \frac{R}{R_b}\right)}{I_o(s_b)}\right], \qquad (4.74)$$

where the following notation is used: $s_b = \frac{R_b}{R_o}$

Non-Uniform Beam Equilibrium in Linear Field (cont.)



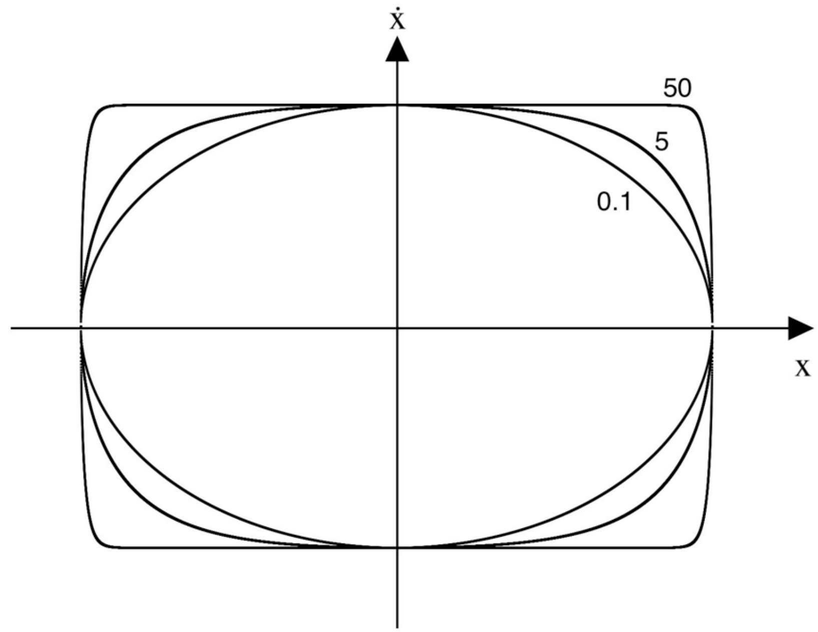
Density profile, Eq. (4.74), for different values of parameter s_{b}.

Non-Uniform Beam Equilibrium in Linear Field (cont.)

Projection of the volume at the phase plane $(x, \dot{x})$:

$$\frac{s_b^2}{4\mu_o^2 R_b^2}\dot{x}^2 + \frac{1}{I_o(s_b)} I_o\left(s_b \frac{x}{R_b}\right) = 1 \quad . \qquad (4.65)$$

Eq. (4.65) describes the boundary of phase space of the beam at the plane $(x, \dot{x})$.

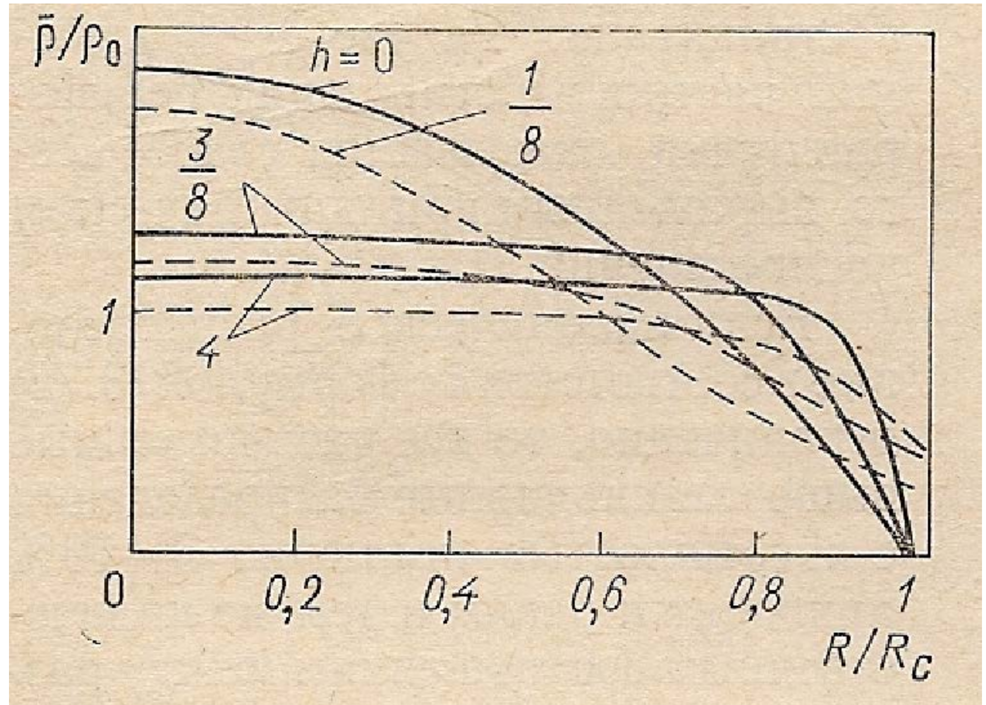


Boundary phase space trajectories of particles, Eq. (4.65), for different values of parameter s_b.

Non-Uniform Beam Equilibrium in Linear Field (cont.)

Similar results can be obtained for another distribution function

$$f = f_o \exp(-\frac{H}{H_o})$$



Space charge density for different distributions:

(solid) $f = f_o,\ H \le H_o$

(dotted) $f = f_o \exp(-H/H_o)$

Performed anlysis shows, that for small values of space charge forces, particle phase space trajectories are close to elliptical, and beam profile density is essentially nonlinear. With increase of space charge forces, boundary particle trajectories become more close to rectangular, and density beam profile becomes more uniform. In space charge dominated regime, stationary beam profile tend to be uniform, and space charge field of the beam compensates for external field.

Rms Beam Envelopes

Set of equations for the first and the second moments of distribution function in x-direction

$$\frac{d}{dt} < x > = < v_x >$$

$$\frac{d}{dt} < v_x > = \frac{1}{m\gamma} < F_x >$$

$$\frac{d}{dt} < x v_x > = < v_x^2 > + \frac{1}{m\gamma} < x F_x >$$

$$\frac{d}{dt} < x^2 > = 2 < x v_x >$$

$$\frac{d}{dt} < v_x^2 > = \frac{2}{m\gamma} < v_x F_x >$$

where the Lorentz force in x-direction is $F_x = q\,(E_x + v_y B_z - v_z B_y)$

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Rms Beam Envelopes (cont.)

Taking into account that $v_x = v_z\, x'$ and introducing notation

$$f_x = \frac{F_x}{m\gamma\,(\beta_z c)^2}$$

the set of moment equations is

$$\frac{d}{dz} < x > = < x' >$$

$$\frac{d}{dz} < x' > = < f_x >$$

$$\frac{d}{dz} < xx' > = < x'^2 > + < x f_x >$$

$$\frac{d}{dz} < x^2 > = 2 < xx' >$$

$$\frac{d}{dz} < x'^2 > = 2 < x' f_x >$$

Conservation of RMS Beam Emittance in Linear Field

Square of 4-rms beam emittance

$$\epsilon_x^2 = 16(< x^2 >< x'^2 > - < xx' >)$$

Derivative of square of 4-rms beam emittance

$$\frac{d\epsilon_x^2}{dz} = 32\,(< x^2 > < x'\, f_x > - < x\, x' > \; < x f_x >)$$

If Lorentz force is linear with coordinate, $f_x = -kx$, the rms beam emittance is a constant of motion

$$\frac{d\epsilon_x^2}{dz} = 32\,k\,(< x^2 > < x'\, x > - < x\, x' > \; < x^2 >) = 0$$

In nonlinear field rms emittance is not conserved.

Rms Beam Envelopes (F.Sacherer, P.Lapostolle, PAC 1971)

Rms envelope equations

$$\frac{d^2\tilde{X}}{dz^2} - \frac{\text{э}_x^2}{\tilde{X}^3} + k_x(z)\,\tilde{X} - \frac{4\,I}{I_c\,\beta^3\gamma^3(\tilde{X}+\tilde{Y})} = 0$$

$$\frac{d^2\tilde{Y}}{dz^2} - \frac{\text{э}_y^2}{\tilde{Y}^3} + k_y(z)\,\tilde{Y} - \frac{4\,I}{I_c\,\beta^3\gamma^3(\tilde{X}+\tilde{Y})} = 0$$

2-rms beam envelopes

$$R_x = \tilde{X} = 2\sqrt{<x^2>} \qquad R_y = \tilde{Y} = 2\sqrt{<y^2>}$$

4-rms beam emittances

$$\text{э}_x = 4\sqrt{<x^2><x'^2> - <xx'>^2}$$

$$\text{э}_y = 4\sqrt{<y^2><y'^2> - <yy'>^2}$$

RMS envelope equations are valid for arbitrary distribution, but rms emiattnce is no longer constant. RMS envelope equations are not closed.

Particle Distributions with Elliptical Symmetry in 4D Phase Space

Consider quadratic from of 4-dimensional phase space variables:

$$I = (\sigma_x x^{'} - \sigma_x^{'} x)^2 + (\frac{x}{\sigma_x})^2 + (\sigma_y y^{'} - \sigma_y^{'} y)^2 + (\frac{y}{\sigma_y})^2$$

Consider different distributions $f = f(I)$ in phase space which depend on quadratic form:

Water Bag:

$$f = \begin{cases} \dfrac{2}{\pi^2 F_o^2}, & I \le F_o \\ 0, & I > F_o \end{cases}$$

Parabolic:

$$f = \frac{6}{\pi^2 F_o^2}(1 - \frac{I}{F_o})$$

Gaussian:

$$f = \frac{1}{\pi^2 F_o^2} exp(-\frac{I}{F_o})$$

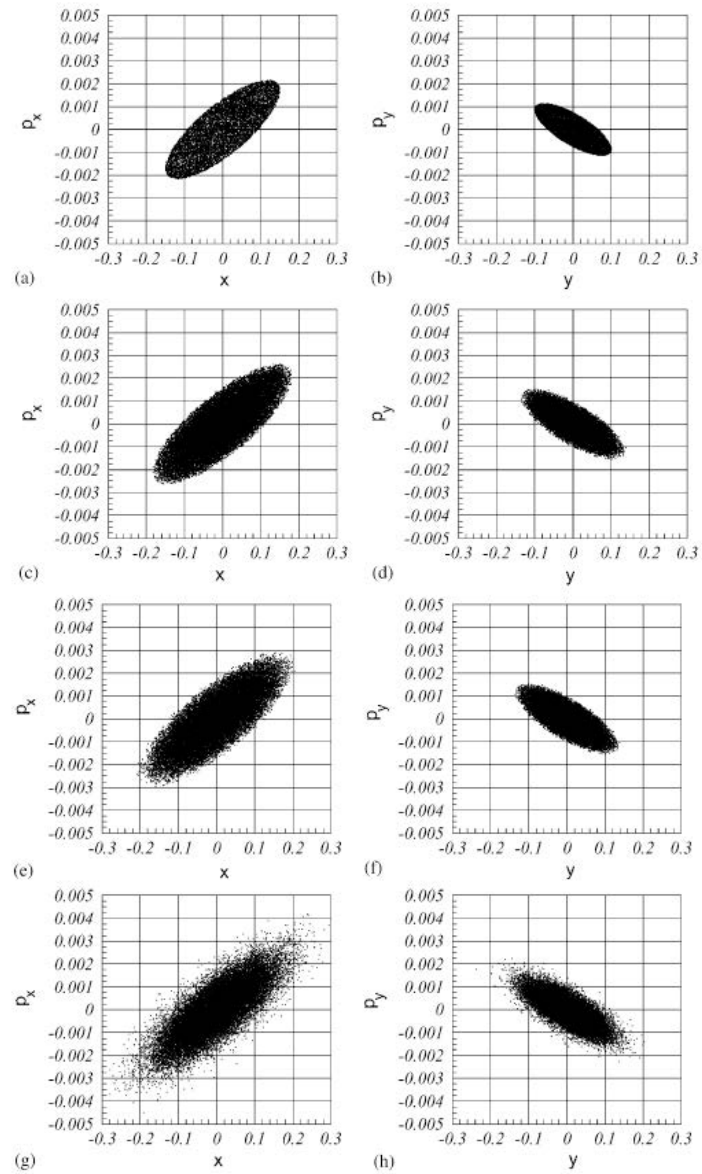
Normalization: $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\, dx\, dx'\, dy\, dy' = 1$$

Characteristics of 4D Beam Distributions

Distribution	Definition $\rho(x,x',y,y')=\rho(I)$ $I=r_x^2+r_y^2$ $r_x^2=\gamma_x x^2+2\alpha_x xx'+\beta_x x'^2$ $r_y^2=\gamma_y y^2+2\alpha_y yy'+\beta_y y'^2$	Distribution in phase space $\rho(x,x')=\rho(r_x^2)$	Space charge density	Space charge field
KV	$\frac{1}{\pi^2 F_o}\delta(I-F_o)$	$\frac{1}{\pi \backepsilon_x}$	$\frac{I}{\pi R^2 \beta c}$	$\frac{I}{2\pi\varepsilon_o R^2 \beta c} r$
Water Bag	$\frac{2}{\pi^2 F_o^2},\ I \le F_o$	$\frac{4}{3\pi \backepsilon_x}(1-\frac{2}{3}\frac{r_x^2}{\backepsilon_x})$	$\frac{4I}{3\pi R^2 \beta c}(1-\frac{2}{3}\frac{r^2}{R^2})$	$\frac{2I}{3\pi\varepsilon_o \beta c}\frac{r}{R^2}(1-\frac{r^2}{3R^2})$
Parabolic	$\frac{6}{\pi^2 F_o^2}(1-\frac{I}{F_o})$	$\frac{3}{2\pi \backepsilon_x}(1-\frac{r_x^2}{2 \backepsilon_x})^2$	$\frac{3I}{2\pi R^2 \beta c}(1-\frac{r^2}{2R^2})^2$	$\frac{3I}{4\pi\varepsilon_o \beta c}\frac{r}{R^2}(1-\frac{r^2}{2R^2}+\frac{r^4}{12R^4})$
Gaussian	$\frac{1}{\pi^2 F_o^2} exp(-\frac{I}{F_o})$	$\frac{2}{\pi \backepsilon_x}\exp(-2\frac{r_x^2}{\backepsilon_x})$	$\frac{2I}{\pi R^2 \beta c}\exp(-2\frac{r^2}{R^2})$	$\frac{I}{2\pi\varepsilon_o \beta c r}[1-\exp(-2\frac{r^2}{R^2})]$

Projections of 4D Distributions on Phase Planes

Y.K. Batygin / Nuclear Instruments and Methods in Physics Research A 539 (2005) 455–489



KV

$$\varepsilon_{\max} = 4\varepsilon_{rms}$$

Water Bag

$$\varepsilon_{\max} = 6\varepsilon_{rms}$$

Parabolic

$$\varepsilon_{\max} = 8\varepsilon_{rms}$$

Gaussian

$$\varepsilon_{\max} = \infty$$

Particle distributions with equal values of rms emittance.

Projections of 4D Distributions on Phase Plane

Projections of 4D Distributions on phase plane (x-x') is the integral of distribution over remaining variables

$$\rho_x (x,x') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, x', y, y')\, dy\, dy'$$

Let us change the variables (y, y') for new variables T, ψ

$$\sigma_y y' - \sigma_y' y = T \cos \psi$$

$$\frac{y}{\sigma_y} = T \sin \psi$$

Phase space element $dy\, dy'$ is transformed as

$$dy\, dy' = \begin{vmatrix} \frac{\partial y}{\partial T} & \frac{\partial y}{\partial \psi} \\ \frac{\partial y'}{\partial T} & \frac{\partial y'}{\partial \psi} \end{vmatrix} dT\, d\psi = T\, dT\, d\psi\, .$$

The quadratic form is $I = r_x^2 + T^2$ where the following notation is used:

$$r_x^2 = (\sigma_x x' + \sigma_x' x)^2 + \left(\frac{x}{\sigma_x}\right)^2.$$

With new variables, the projection on phase space is

$$\rho_x (x, x') = \pi \int_{o}^{\infty} f(r_x^2 + T^2)\, dT^2.$$

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Projections of 4D Distributions on Phase Plane (cont.)

Water Bag distribution

$$f = \begin{cases} \dfrac{2}{\pi^2 F_o^2}, & I = r_x^2 + T^2 \le F_o \\ 0, & I > F_o \end{cases}$$

is restricted by surface

$$r_x^2 + T_1^2 = F_o, \quad T_1^2 = F_o - r_x^2$$

Projection of *Water Bag* distribution on (x-x')

$$\rho_x(x, x') = \frac{2}{\pi F_o^2} \int_o^{T_1^2} dT^2 = \frac{2}{\pi F_o}\left(1 - \frac{r_x^2}{F_o}\right)$$

For *Parabolic* distribution, projection on (x-x') is:

$$\rho_x(x, x') = \frac{6}{\pi F_o^2} \int_0^{T_1^2} \left(1 - \frac{r_x^2 + T^2}{F_o}\right) dT^2 = \frac{3}{\pi F_o}\left(1 - \frac{r_x^2}{F_o}\right)^2$$

For *Gaussian* distribution projection on x, x' plane is

$$\rho_x(x, x') = \frac{1}{\pi F_o^2} \int_o^{\infty} exp\left(- \frac{r_x^2 + T^2}{F_o}\right) dT^2 = \frac{1}{\pi F_o} exp\left(- \frac{r_x^2}{F_o}\right)$$

Rms Emittance of 4D Beam Distributions

Four rms beam emittance

$$\text{э}_x = 4\pi \int_o^\infty r_x^3 \rho_x (r_x^2)\, dr_x$$

Water bag distribution:

$$\text{э}_x = \frac{8}{F_o} \int_o^{\sqrt{F_o}} r_x^3 (1 - \frac{r_x^2}{F_o})\, dr_x = \frac{2}{3} F_o$$

Parabolic distribution

$$\text{э}_x = \frac{12}{F_o} \int_o^{\sqrt{F_o}} r_x^3 (1 - \frac{r_x^2}{F_o})^2\, dr_x = \frac{F_o}{2}$$

Gaussian distribution:

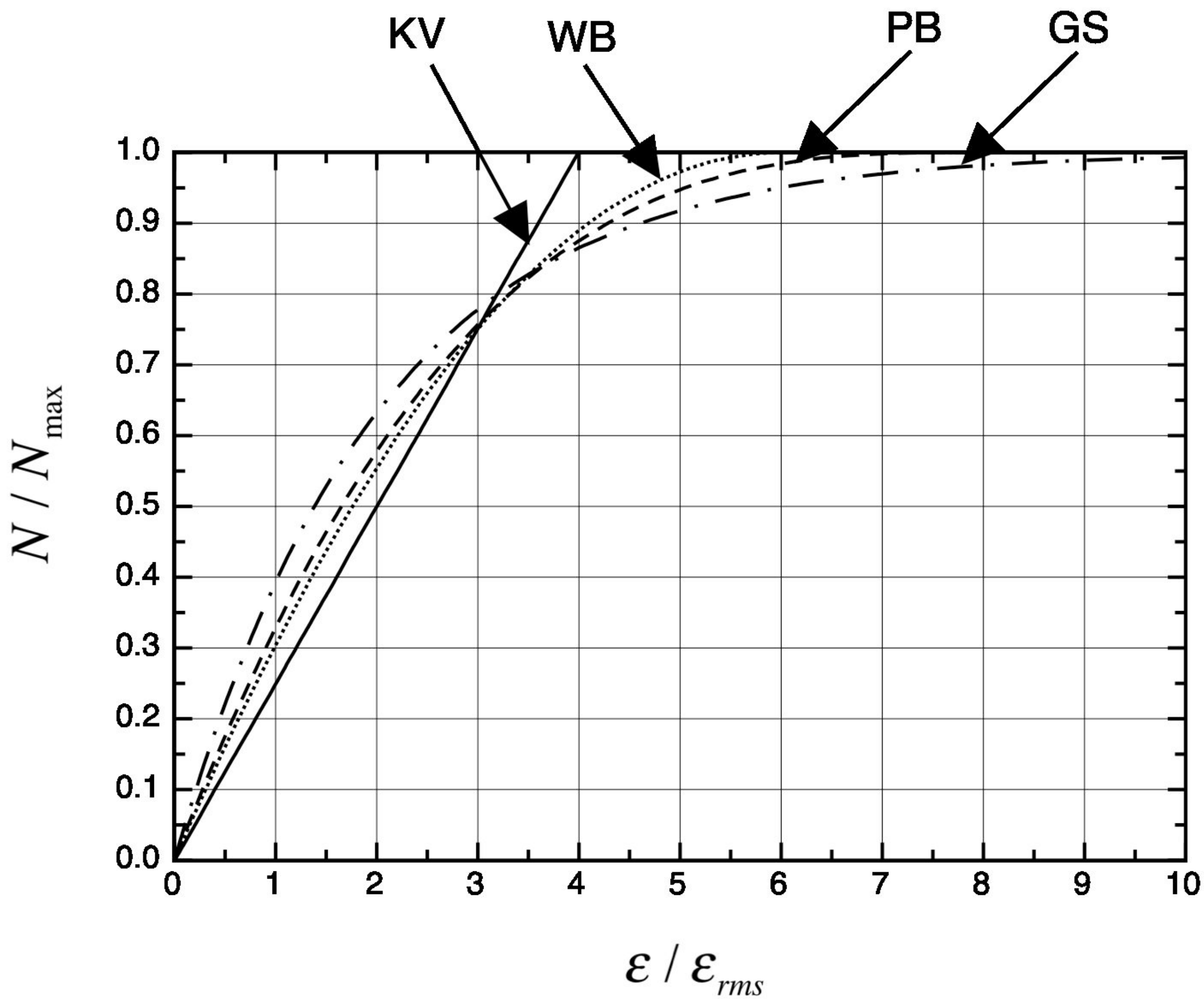
$$\text{э}_x = \frac{4}{F_o} \int_o^\infty r_x^3\, exp\,(- \frac{r_x^2}{F_o})\, dr_x = 2\, F_o$$

Fraction of Particles Residing within a Specific Emittance

Fraction of particles within specific emittance $$\frac{N(\epsilon)}{N} = \pi \int_o^{\epsilon} \rho_x (r_x^2)\, dr_x^2$$

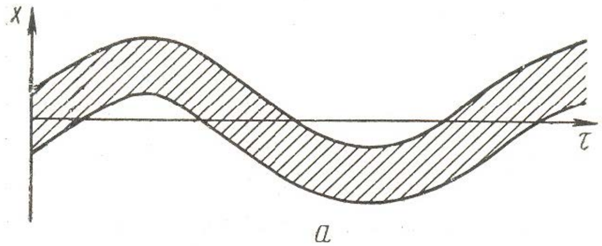
	Beam distribution	Fraction of particles within emittance ϵ (ϵ_x = 4rms emittance)
Water bag	$\rho_x(r_x^2) = \frac{4}{3\pi\epsilon_x}(1 - \frac{2}{3}\frac{r_x^2}{\epsilon_x})$	$\frac{N(\epsilon)}{N_o} = \frac{4}{3}(\frac{\epsilon}{\epsilon_x})(1 - \frac{1}{3}\frac{\epsilon}{\epsilon_x})$
Parabolic	$\rho_x(r_x^2) = \frac{3}{2\pi\epsilon_x}(1 - \frac{r_x^2}{2\epsilon_x})^2$	$\frac{N(\epsilon)}{N_o} = \frac{3}{2}(\frac{\epsilon}{\epsilon_x})[1 - \frac{1}{2}\frac{\epsilon}{\epsilon_x} + \frac{1}{12}(\frac{\epsilon}{\epsilon_x})^2]$
Gaussian	$\rho_x(r_x^2) = \frac{2}{\pi\epsilon_x} exp(-2\frac{r_x^2}{\epsilon_x})$	$\frac{N(\epsilon)}{N_o} = 1 - exp(-2\frac{\epsilon}{\epsilon_x})$

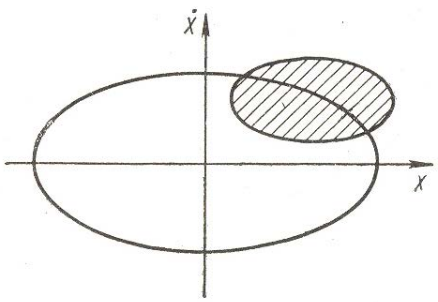
Fraction of Particles Residing within a Specific Emittance

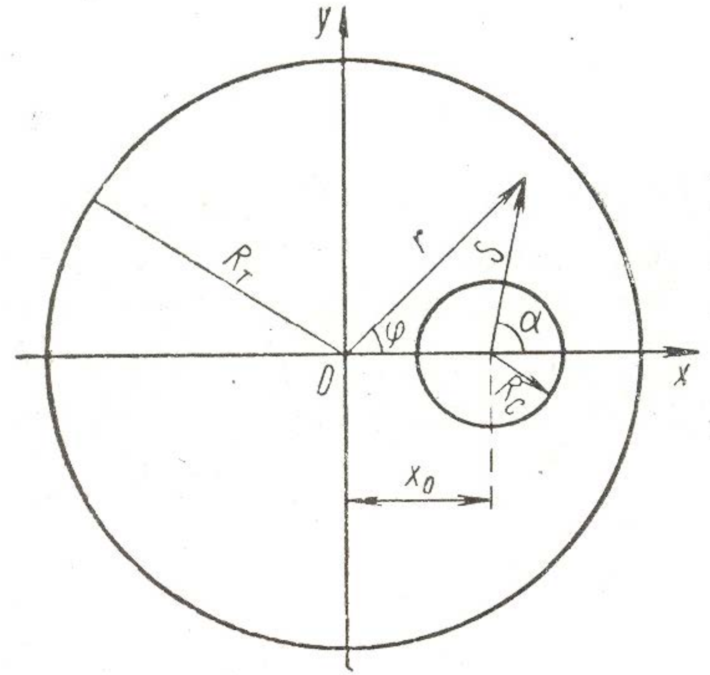


Fraction of particles versus rms emittacnes for different particle distributions.

Coherent Beam Oscillations







Misalignments of the channel results in oscillation of center of beam gravity

Potential of the beam shifted from axis

$$\boxed{U_b(r,\theta) = -\frac{\rho}{4\varepsilon_o}(r^2 - 2x_o\cos\theta) - \frac{\rho R_c^2}{2\varepsilon_o}\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{r}{R_T}\right)^m\left(\frac{x_o}{R_T}\right)^m\cos m\theta}$$

Frequency of oscillations of center of gravity

$$\boxed{\mu_{coh}^2 = \mu_o^2 - \left(\frac{S}{R_T}\right)^2\frac{2I}{I_c(\beta\gamma)^3}}$$

Envelope Instability

Averaging procedure (smooth approximation) was based on assumption that solution of envelope equations are stable

Envelope Equations

$$\frac{d^2 R_x}{dz^2} - \frac{\varepsilon_x^2}{R_x^3} + k(z) R_x - \frac{2P^2}{(R_x + R_y)} = 0$$

$$\frac{d^2 R_y}{dz^2} - \frac{\varepsilon_y^2}{R_y^3} - k(z) R_y - \frac{2P^2}{(R_x + R_y)} = 0$$

Let us represent solution as a combination of periodic solutions $\tilde{R}_x(z),\ \tilde{R}_y(z)$ and deviations from that $\xi_x(z),\ \xi_y(z)$

$$R_x(z) = \tilde{R}_x(z) + \xi_x(z)$$

$$R_y(z) = \tilde{R}_y(z) + \xi_y(z)$$

Equations for deviations from periodic solution:

$$\xi_x'' + \xi_x a_1(z) + \xi_y a_o(z) = 0$$

$$\xi_y'' + \xi_y a_2(z) + \xi_x a_o(z) = 0$$

$$a_o(z) = \frac{2P^2}{(\tilde{R}_x + \tilde{R}_y)^2} \qquad a_1(z) = k(z) + 3\frac{\varepsilon_x^2}{\tilde{R}_x^4} + \frac{2P^2}{(\tilde{R}_x + \tilde{R}_y)^2} \qquad a_2(z) = -k(z) + 3\frac{\varepsilon_y^2}{\tilde{R}_y^4} + \frac{2P^2}{(\tilde{R}_x + \tilde{R}_y)^2}$$

Envelope Oscillations Modes

In smooth approximation $\tilde{R}_x = \tilde{R}_y = \bar{R}$ and equations for deviations from periodic solution, where coefficients

$$\xi_x'' + \xi_x a_1 + \xi_y a_o = 0$$

$$\xi_y'' + \xi_y a_1 + \xi_x a_o = 0$$

$$a_o = \frac{P^2}{2\bar{R}^2} \qquad a_1 = a_2 = \frac{\mu_o^2}{L^2} + 3\frac{э_x^2}{\bar{R}^4} + \frac{P^2}{2\bar{R}^2}$$

Taking into account expression for phase advances (depressed and undepressed), as well as expression for unnormalized beam emittance, we get equations for oscillations of two envelope modes

$$\mu^2 = \mu_o^2 - P^2 (\frac{L}{R})^2$$

$$э = \frac{\mu R^2}{L}$$

Symmetric envelope mode

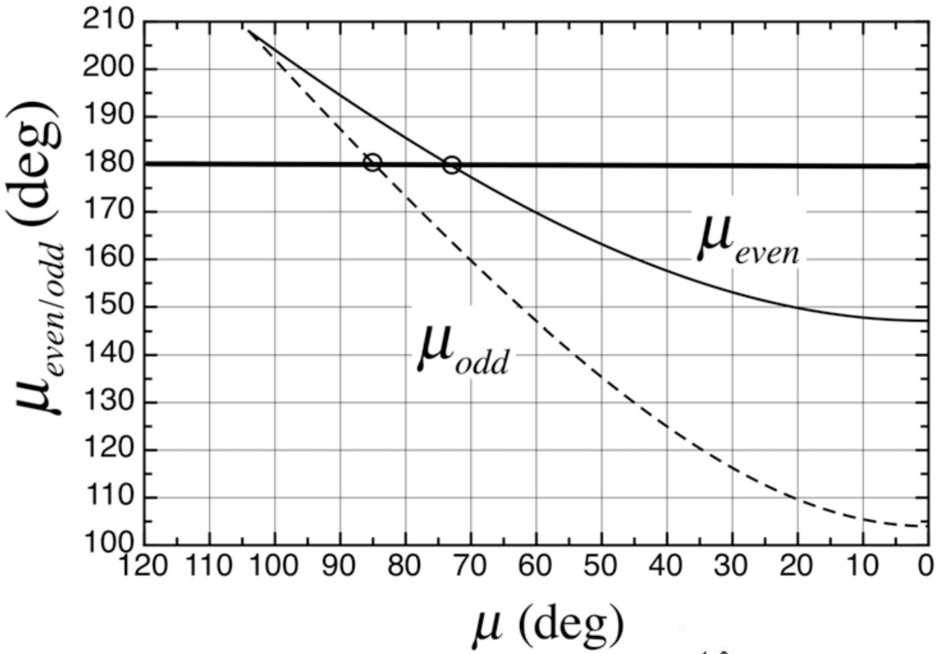
$$\boxed{(\xi_x + \xi_y)'' + \frac{\sigma_{even}^2}{L^2}(\xi_x + \xi_y) = 0} \qquad \sigma_{even} = \sqrt{2(\mu_o^2 + \mu^2)}$$

Anti-symmetric envelope mode

$$\boxed{(\xi_x - \xi_y)'' + \frac{\sigma_{odd}^2}{L^2}(\xi_x - \xi_y) = 0} \qquad \sigma_{odd} = \sqrt{\mu_o^2 + 3\mu^2}$$

Envelope Instability: $\sigma_{even} = 180^o$ or $\sigma_{odd} = 180^o$

Envelope Instability (cont.)

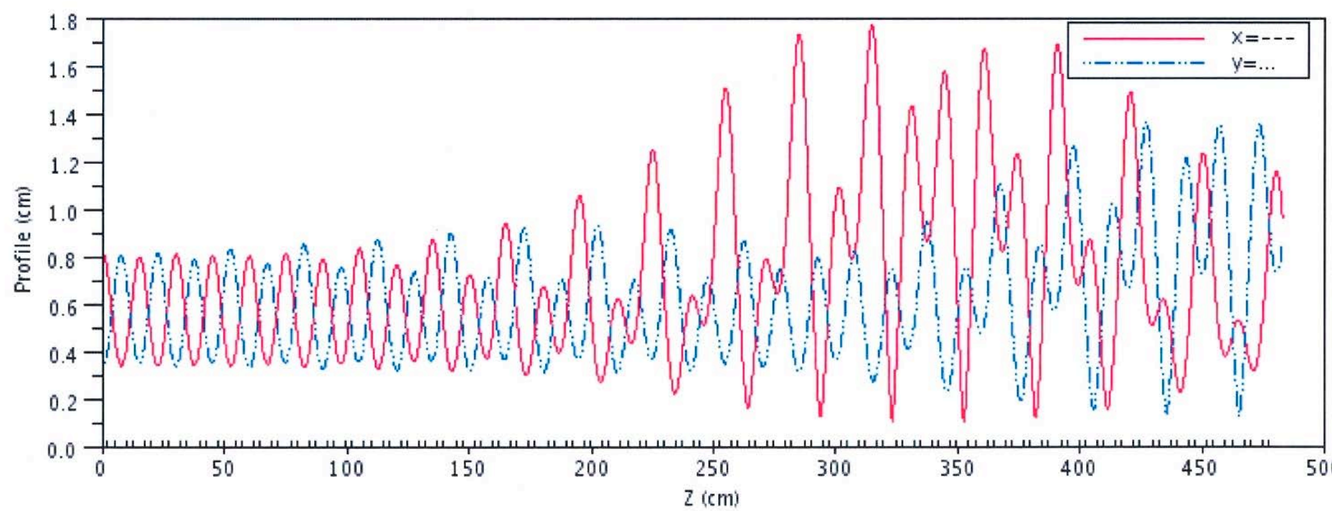


$$\sigma_{even} = 2\mu_o \sqrt{\frac{1}{2} + \frac{1}{2}\left(\frac{\mu}{\mu_o}\right)^2} < 2\mu_o$$

$$\sigma_{odd} = 2\mu_o \sqrt{\frac{1}{4} + \frac{3}{4}\left(\frac{\mu}{\mu_o}\right)^2} < 2\mu_o$$

No instability for $\mu_o < 90^o$

Mismatching envelope modes for μ_o=104^o as functions of space-charge depressed phase advance, μ.



Envelope instability in FODO channel with μ_o=104^o, μ=72^o.

Multipole KV Beam Instability Modes (I.Hofmann, L.Laslett, L.Smith, 1983)

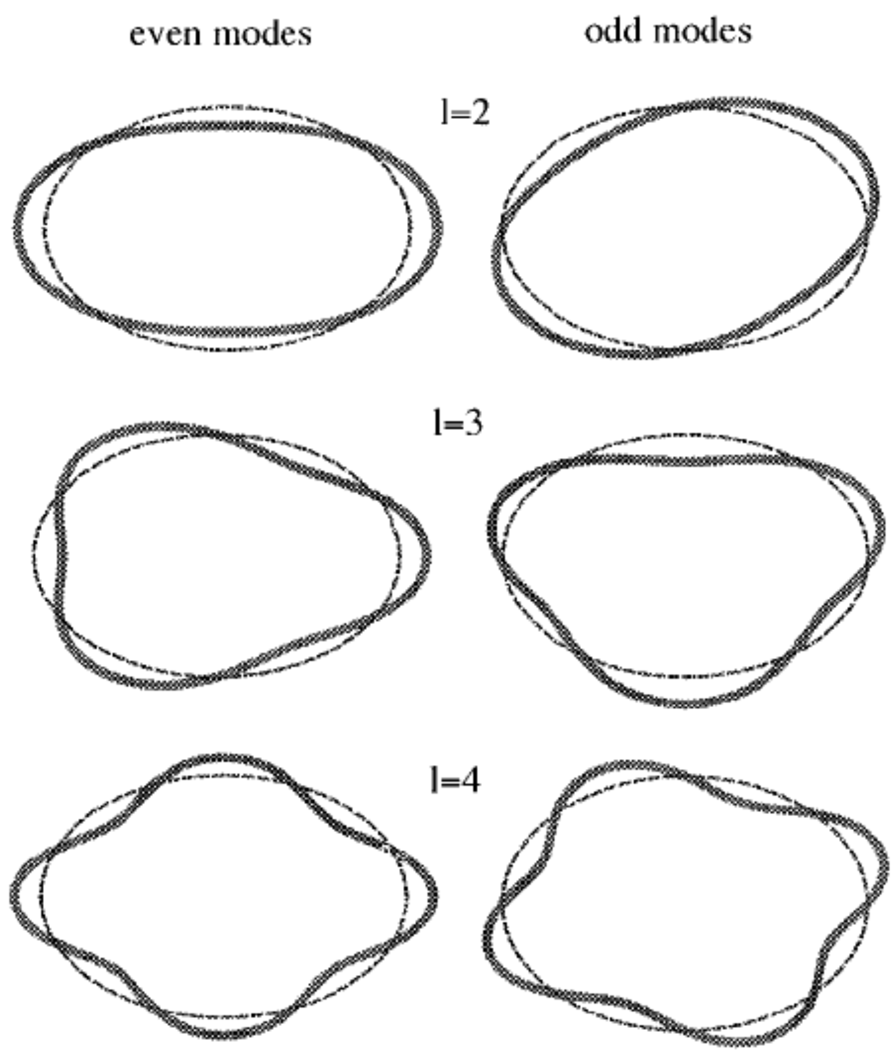


FIG. 1. Beam cross sections for second, third and fourth order even and odd modes (schematic, with x horizontal and y vertical coordinates).

Multipole KV Beam Instability Modes

Eigenmode grequencies and amplitude growth rates can be derived analytically from analysis of small perturbtion of 4D distribution function:

$$f(x,p_x,y,p_y,t) = f_0(x,p_x,y,p_y) + f_1(x,p_x,y,p_y,t) =$$
$$= f_0(H_{0x},H_{0y}) + f_1(x,p_x,y,p_y,t)$$
$$\frac{df_1}{dt} = 0$$

Poisson's equation for perturbed electrostatic potential created by perturbed space charge density:

$$\nabla^2\Phi = -\frac{q}{\epsilon_0}n_1 = -\frac{q}{\epsilon_0}\int f_1 dp_x dp_y$$

The solution for perturbed distribution function and beam potential is being searched as

$$f_1 = f_1(\varphi)e^{-i\omega t}, \quad \Phi = \Phi(\varphi)e^{-i\omega t}$$

TRANSPORT OF HIGH-INTENSITY BEAMS 93

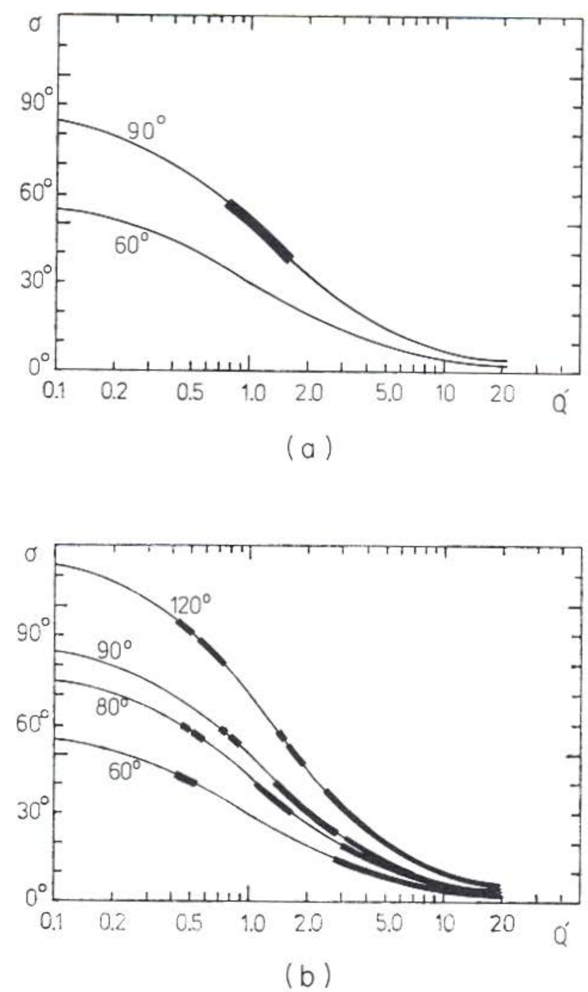
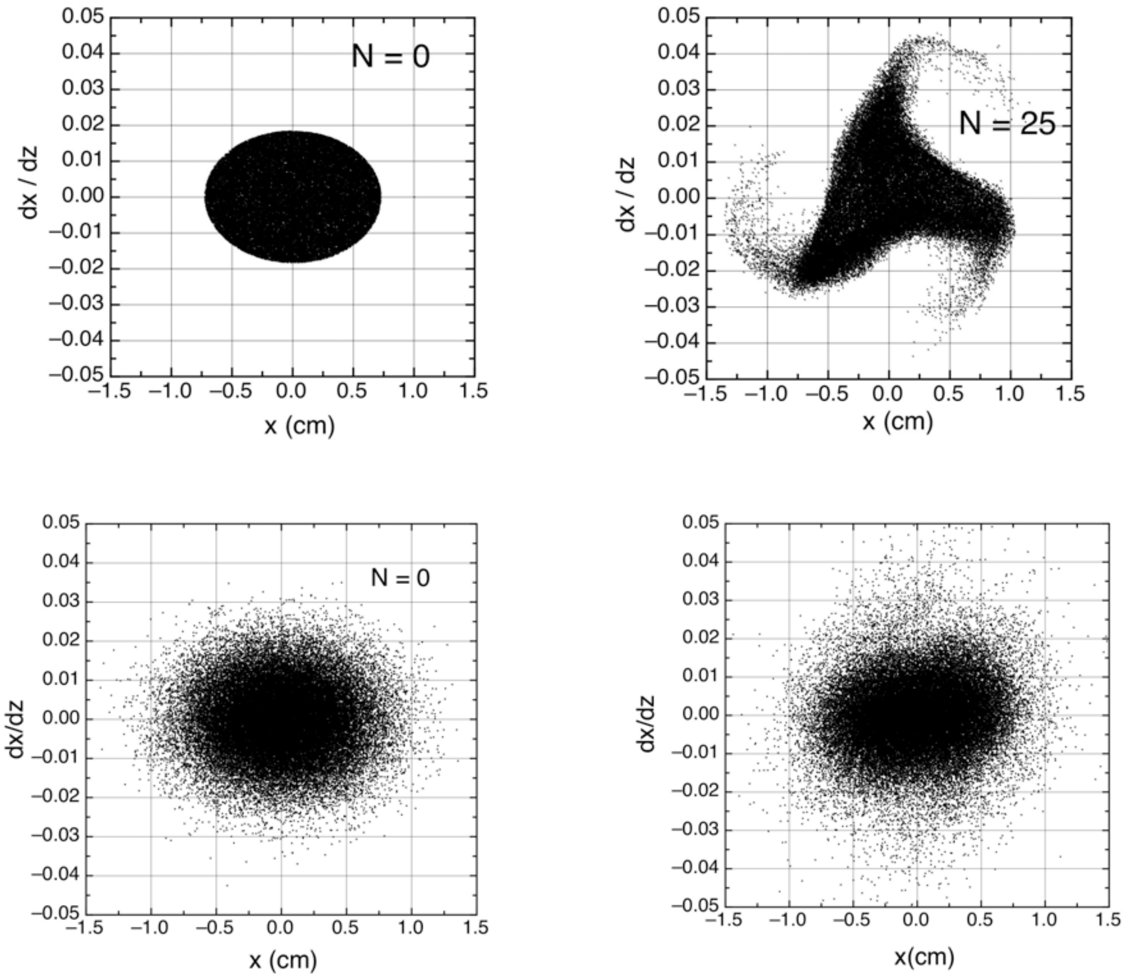


FIG. 19. Instability bands in the phase advance σ for a FODO channel ($\eta = \frac{1}{2}$) and different σ_0: (a) "third-order" modes and (b) "fourth-order" modes.

3rd Order KV Beam Instability in FODO Channel

KV

Gaussian

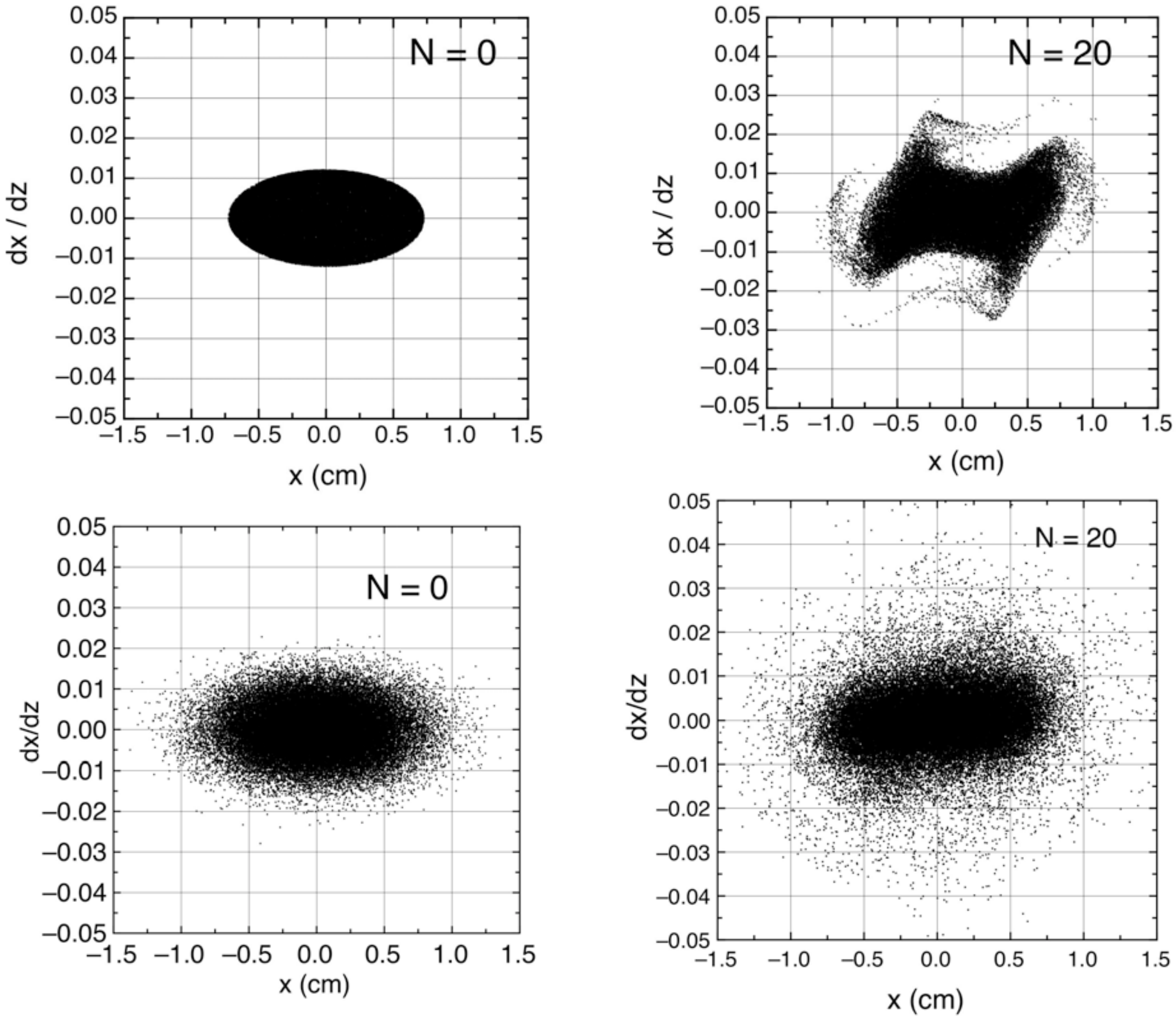


Third-order instability of KV beam in FODO structure with μ_o=90°, μ=45° . Numbers indicate FODO period.

4^{th} Order KV Beam Instability in FODO Channel

KV

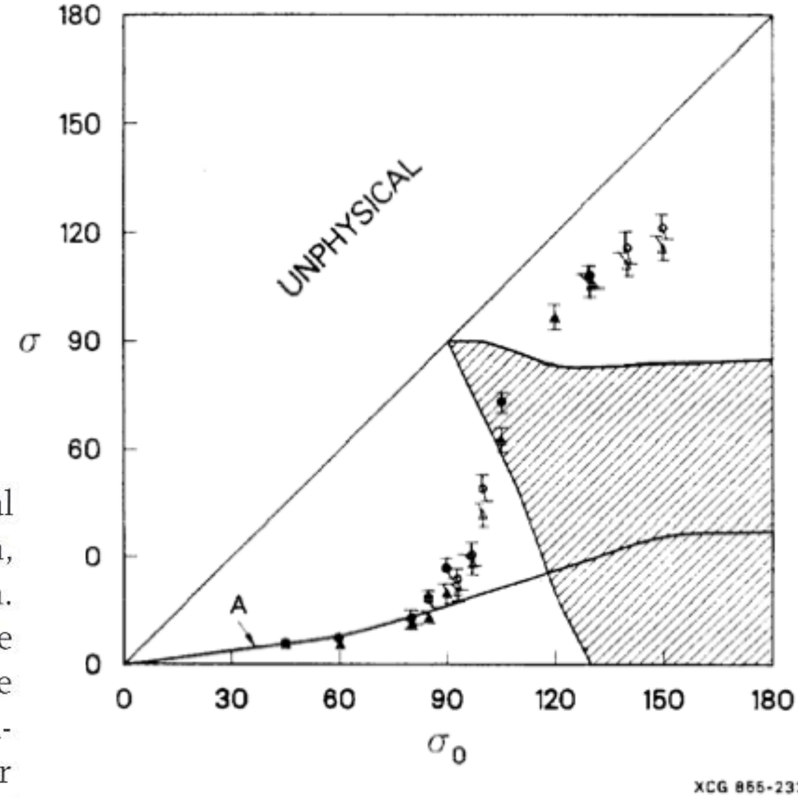
Gaussian



Fourth-order instability of KV beam in FODO structure with μ_o=90°, μ=30° . Numbers indicate FODO period.

Experiments on Stability of Transport Beam at LBNL (1985) and University of Maryland (1995)

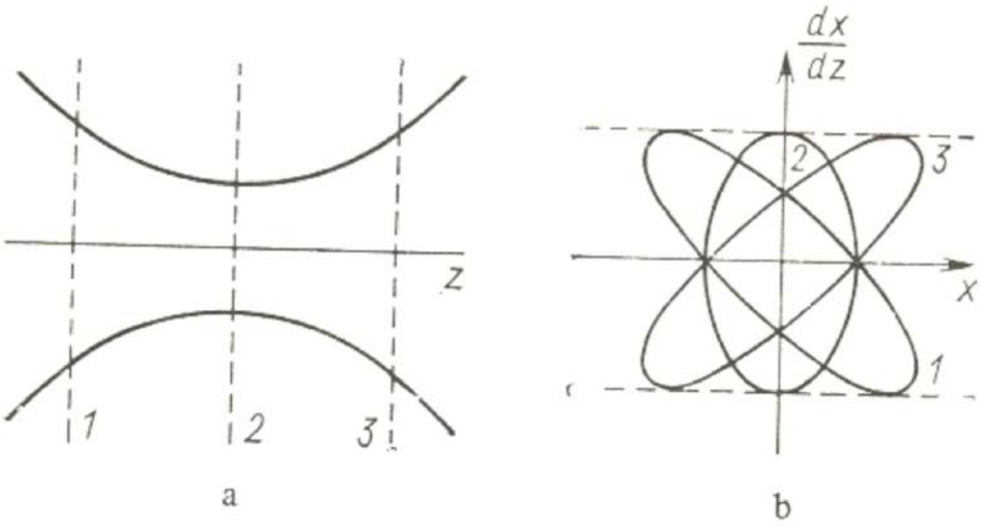
single beam transport channel was constructed at Lawrence Berkeley National Laboratory using 82 electrostatic quadrupole lenses in a FODO configuration, using a cesium beam, as part of the heavy-ion inertial-fusion program. Systematic experiments were conducted by Tiefenback and Keefe, [40] where the beam was matched in both transverse planes, and both σ_0 and σ/σ_0 were varied. The envelope instability predicted by K–V periodic-focusing beam-transport theory for a phase advance per period of $\sigma_0 > 90^\circ$ led to major beam degradation with beam loss. No instability modes predicted by K–V theory, below the $\sigma_0 = 90^\circ$ envelope instability were observed. Similarly, in a systematic experimental study carried out in a solenoid focusing lattice at University of Maryland, [41] the envelope instability was also observed with major beam loss. This was investigated systematically by varying σ/σ_0 and changing σ_0 from below to above 90°. Below 90°, no other instability predicted by the theory, including the third-order (sextupole) mode for $\sigma_0 > 60^\circ$, was found. The conclusion is that for real beams in periodic-focusing channels, the envelope instability predicted by theory for a phase advance per period of $\sigma_0 > 90^\circ$ is the only instability of this theory that leads to emittance growth.



4. Plotted are calculated σ values for stable and apparently stable beams for various σ_0. Filled-in symbols represent beams with the same current and emittance at the beginning and end of the lattice. Hollow symbols mark σ values derived from beams reproducing ϵ and current over at least the last 10 periods, as illustrated in Fig. 3 for σ_0 =100°. Circles mark σ values derived using full beam distribution RMS emittance. Triangles mark calculations using central 95% current of the phase space distribution. The shaded region marks the calculated instability of the envelope equations. Curve A marks the region of equivalent σ attainable at injection with our limited source emittance.

Structure resonances of 3rd, 4th, etc . order are not observed in real beams.

Beam Drift in Free Space



Drift of the beam with finite value of phase space (a) beam envelope, (b) phase space deformation.

Important case is propagation of the beam in the area without any external fields. Consider transport of a round beam $R_x = R_y = R$ in drift space, described by envelope equation

$$\frac{d^2R}{dz^2} - \frac{\epsilon^2}{R^3} - \frac{P^2}{R} = 0 . \qquad \text{(D-1)}$$

Equation (D-1) has the first integral:

$$\left(\frac{dR}{dz}\right)^2 = \left(\frac{dR}{dz}\right)_o^2 + \left(\frac{\epsilon}{R_o}\right)^2\left(1 - \frac{R_o^2}{R^2}\right) + P^2 \ln\left(\frac{R}{R_o}\right)^2 \qquad \text{(D-2)}$$

which determines divergence of the beam as a function of initial beam parameters, beam current, and beam emittance. Eq. (D-2) can be further integrated to determine distance, where beam with initial radius of R_o and initial divergence $R_o^{'}$ is evolved up the radius R

$$z = \frac{R_o^2}{2\epsilon}\int_1^{(\frac{R}{R_o})^2} \frac{ds}{\sqrt{\left[1 + \left(\frac{R_o R_o^{'}}{\epsilon}\right)^2\right]s + \left(\frac{P R_o}{\epsilon}\right)^2 s \ln s - 1}} \qquad \text{(D-3)}$$

Eq. (D-3) can be integrated in case of negligible current, *P = 0:*

$$\frac{R}{R_o} = \sqrt{\left(1 + \frac{R_o^{'}}{R_o} z\right)^2 + \left(\frac{\epsilon}{R_o^2}\right)^2 z^2} \qquad \text{(D-4)}$$

Drift of Space-Charge Dominated Beam

Another case is drift of the beam with negligible beam emittance, but non-zero beam current. Eq. (D-2) has the form

$$\left(\frac{dR}{dz}\right)^2 = \left(\frac{dR}{dz}\right)_o^2 + P^2 \ln\left(\frac{R}{R_o}\right)^2 \qquad \text{(D-5)}$$

To determine expansion of the beam from waist point, let us put initial beam divergence $R_o' = 0$, then Eq. (D-5) becomes

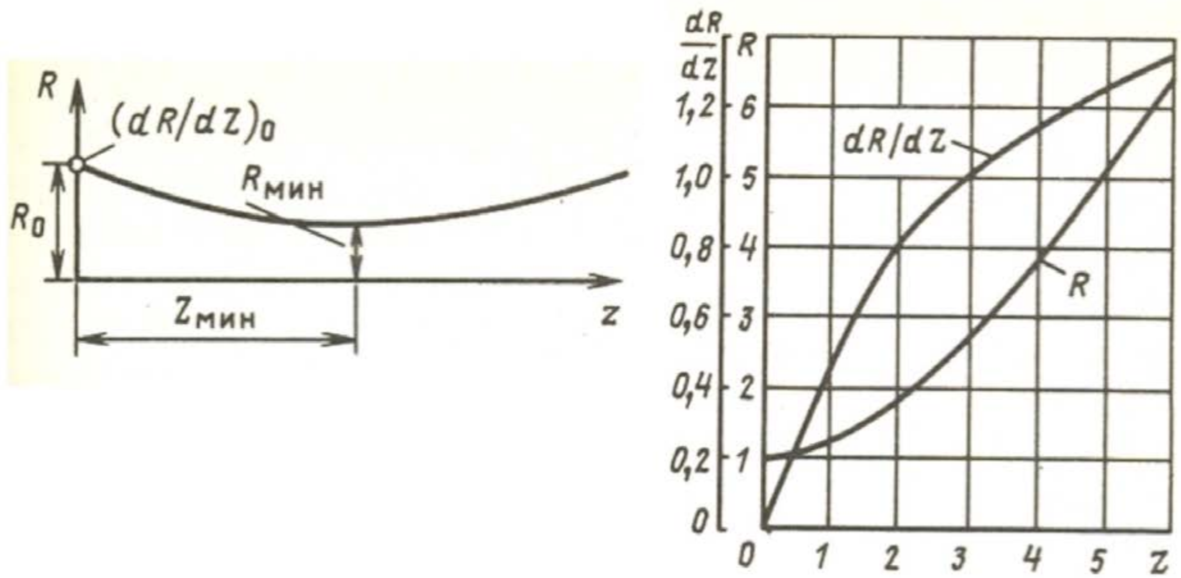
$$\left(\frac{dR}{dz}\right)^2 = P^2 \ln\left(\frac{R}{R_w}\right)^2 \qquad \text{(D-6)}$$

Eq. (D-6) has an approximate solution

$$\boxed{\frac{R}{R_w} \approx 1 + 0.25Z^2 - 0.017Z^3} \qquad \text{(D-7)}$$

$$Z = 2\frac{z}{R_w}\sqrt{\frac{I}{I_c(\beta\gamma)^3}} \qquad \text{(D-8)}$$

where z is counted from the waist point. Eq. (D-7) gives good results for for $0 < Z < 3.2$ and $1 < R/R_w < 3$.



Envelope of an axial-symmetric beam in drift space (Molokovsky, Sushkov, 2005).

Maximum Beam Current Transported Through the Tube

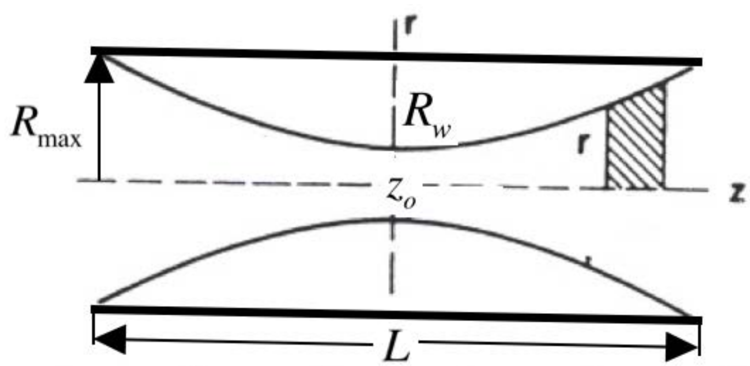
In practical applications, it is important to know the maximum beam current which can be transported through the tube of length L and radius R_{max}. From symmetry point, it is clear that beam should have a waist size $R = R_w$ and zero divergence in the middle of the tube $z = z_o$. Thus, equation (D-6) can be integrated in this case to determine beam expansion from minimal size $R = R_w$ to max size of $R = R_{max}$:

$$\frac{1}{\bar{R}_{\max}} \int_{1}^{\bar{R}_{\max}} \frac{d\bar{R}}{\sqrt{\ln \bar{R}}} = \sqrt{2} P \frac{(z - z_o)}{R_{\max}}. \qquad \bar{R} = R / R_w \qquad \text{(D-9)}$$

The left hand side of Eq. (D-9) has a maximum value of 1.082 for $\bar{R}_{\max} = R_{max} / R_w = 2.35$. The maximum radius is achieved at $z - z_o = L/2$, which in turn yields $P_{\max} L / (\sqrt{2} R_{\max}) = 1.082$. From this expression, the maximum transported current through the tube is

$$I_{\lim} = 1.17 I_c (\beta\gamma)^3 (\frac{R_{\max}}{L})^2 . \qquad \text{(D-10)}$$

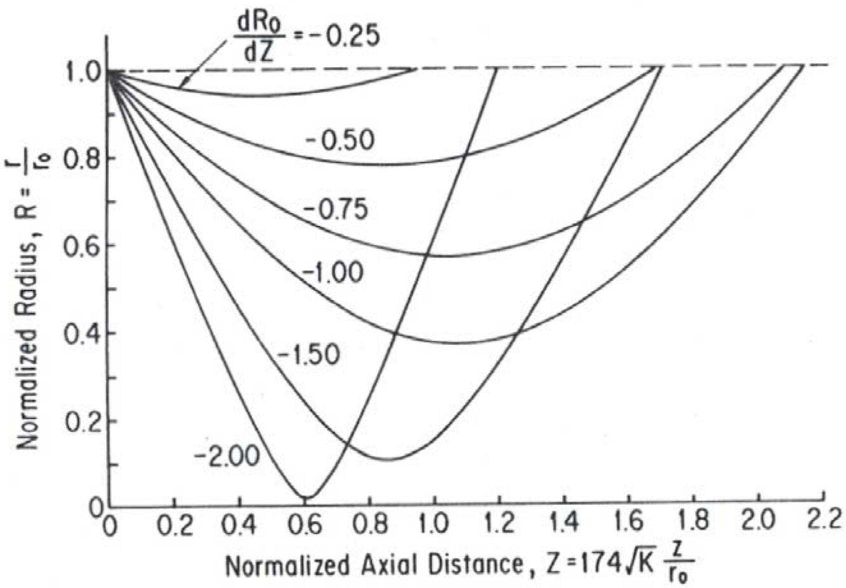
Required beam slope at the entrance of the tube can be determined from Eq. (D-6):



$$\frac{dR}{dz} = \sqrt{\frac{4 I_{\lim}}{I_c (\beta\gamma)^3} \ln(\frac{R_{\max}}{R_w})} \approx 2 \frac{R_{\max}}{L} \qquad \text{(D-11)}$$

On maximum current transported through the tube

Optimization of Beam Drift Space



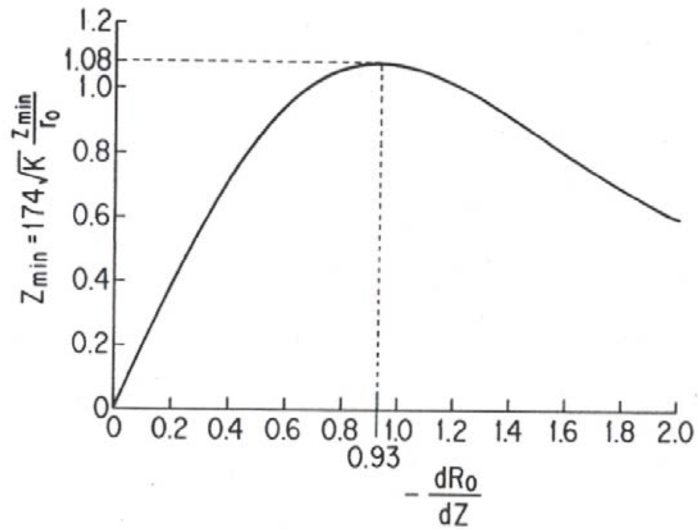


Figure 11: The position of the minimum beam radius as a function of dR_0/dZ. From A.S. Gilmour, Jr.[5]

Beam radius at waist point, $R = R_w$, can be determined from Eq. (D-5) as a function of beam radius R_o and initial beam convergence $R_o^{'}$ assuming in waist point $dR/dz = 0$:

$$R_w = R_o \exp[-(\frac{R_o^{'}}{\sqrt{2P}})^2] \tag{D-12}$$

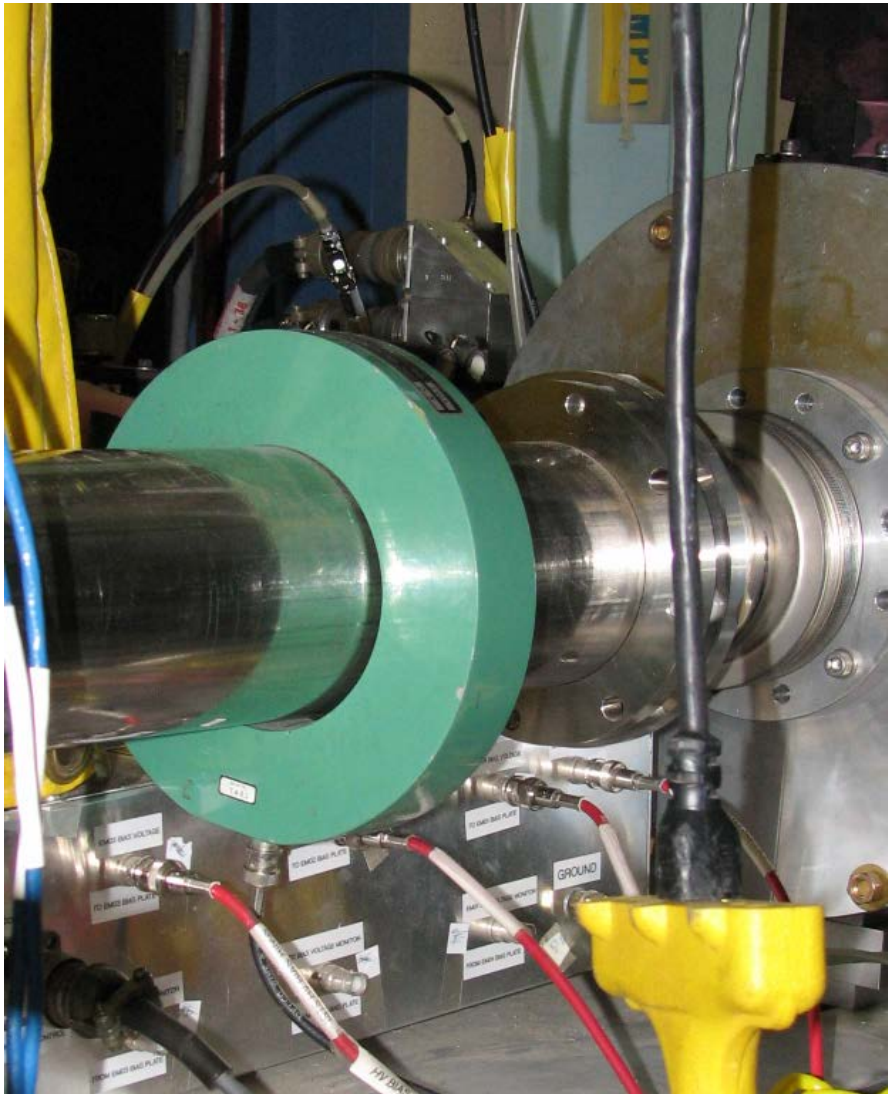
To determine distance, where the beam reaches it's waist, let us rewrite Eq. (D-5) including notations, Eq. (D-8):

$$Z = \int_1^{\bar{R}} \frac{d\bar{R}}{\sqrt{\ln \bar{R} + (d\bar{R}_o / dZ)^2}} \tag{D-13}$$

Using substitution $u = \sqrt{\ln \bar{R} + (d\bar{R}_o / dZ)^2}$, Eq. (D-15) for waist point, where $\bar{R} = 1$, is reduced to

$$Z_{\min} = 2\,e^{-(d\bar{R}_o/dZ)^2} \int_o^{|d\bar{R}_o/dZ|} \exp(u^2)\, du \tag{D-14}$$

Beam Current Measurement



LANL beam current monitor

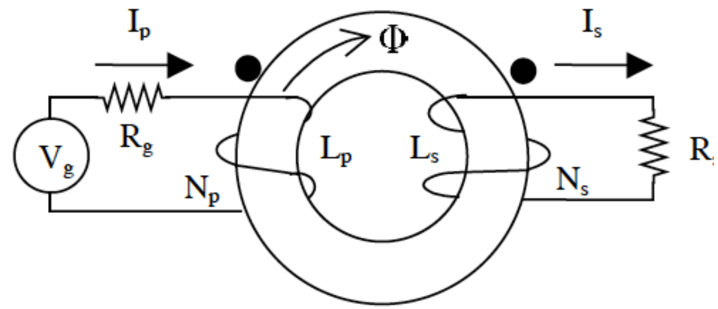


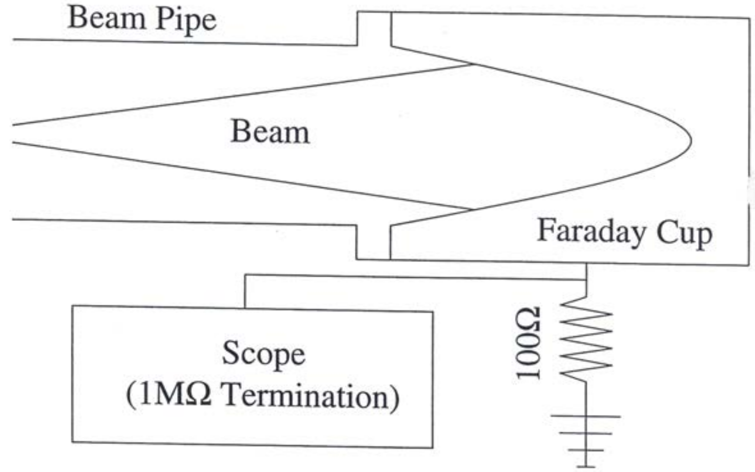
FIGURE 6. Classical transformer circuit.

Torus radii	$r_i = 70$ mm, $r_o = 90$ mm
Torus thickness	$l = 16$ mm
Torus material	Vitrovac 6025: $(CoFe)_{70\%}(MoSiB)_{30\%}$
Torus permeability	$\mu_r \simeq 10^5$ for $f < 100$ kHz, $\mu_r \propto 1/f$ above
Number of windings	10
Sensitivity	4 V/A at $R = 50\ \Omega$, 10^4 V/A with amplifier
Resolution for $S/N = 1$	40 μA_{rms} for full bandwidth
$\tau_{droop} = L/R$	0.2 ms
$\tau_{rise} = \sqrt{L_S C_S}$	1 ns
Bandwidth	2 kHz to 300 MHz

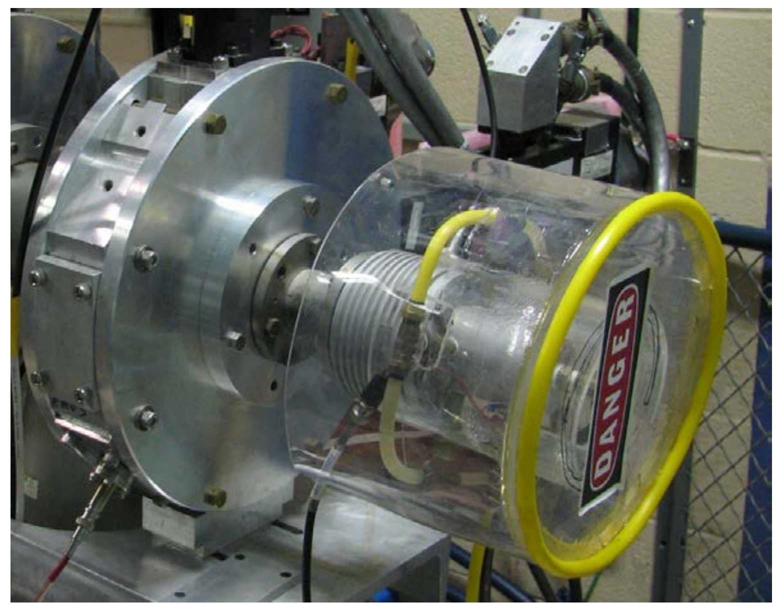
Table 2.1: Some basic specification of the GSI passive transformer.

Faraday Cups

Used as a beam stop for low energy beam and as a fast current monitor.

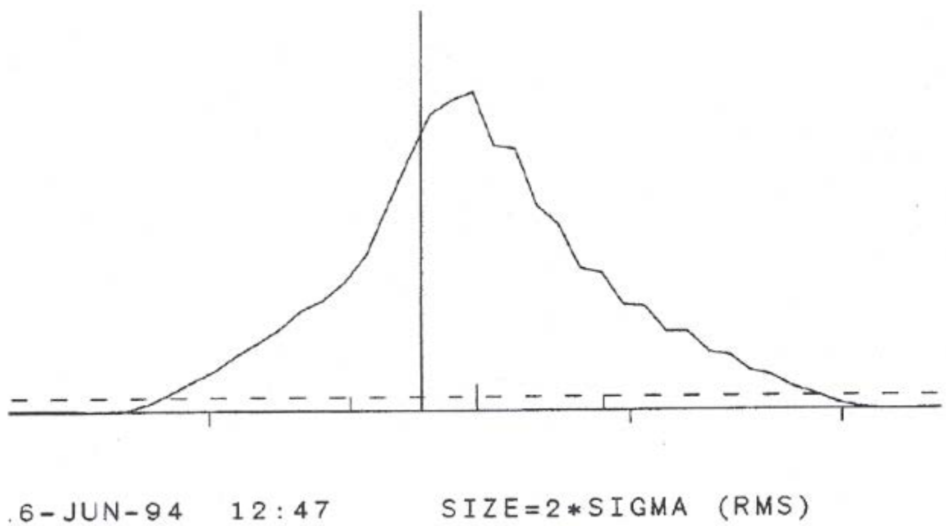


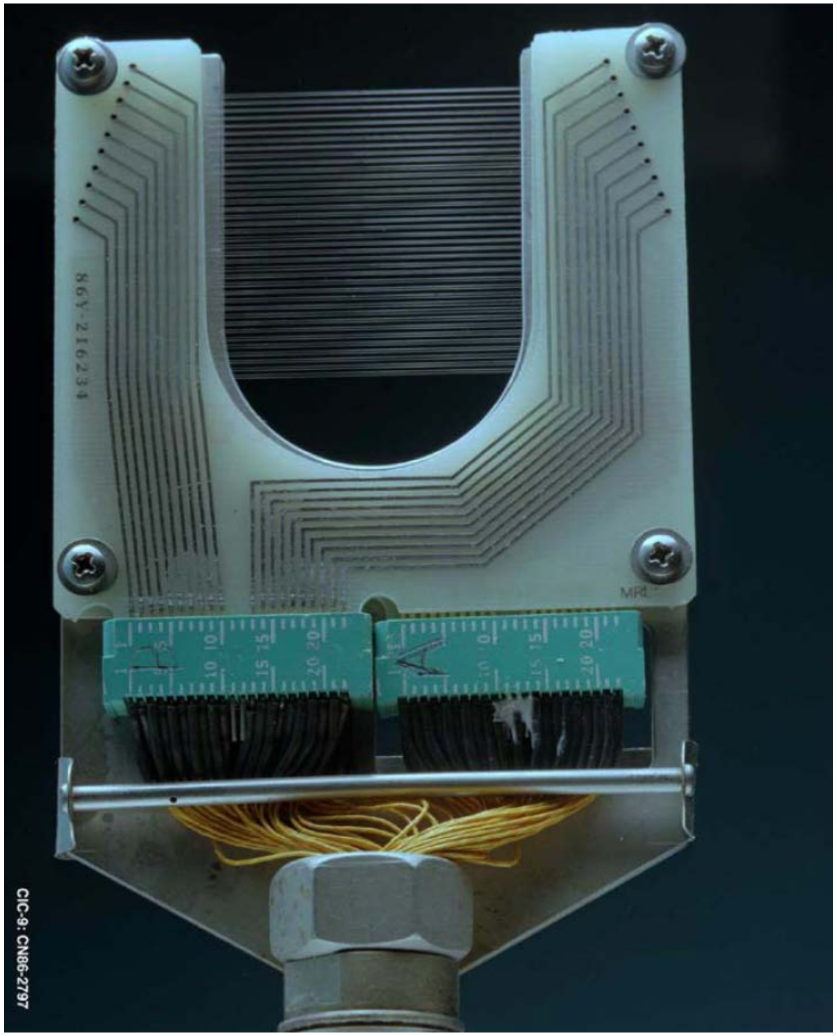
$I_{beam} = V(\text{volts})/100\,\Omega$



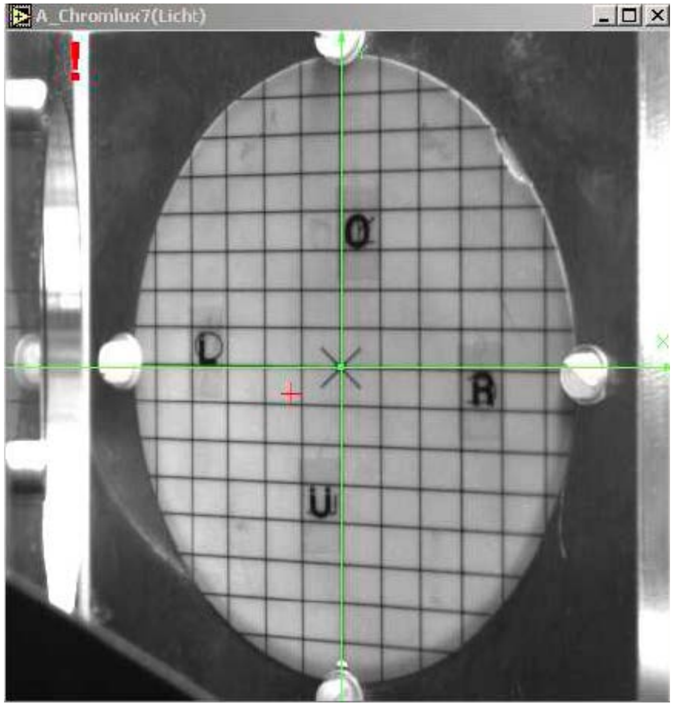
Harps (Profile Monitors)

- 1.3 mil carbon wires
- 76 wires
- 20 mil spacing
- Soldered on to g-10 board
- 1.5” aperture

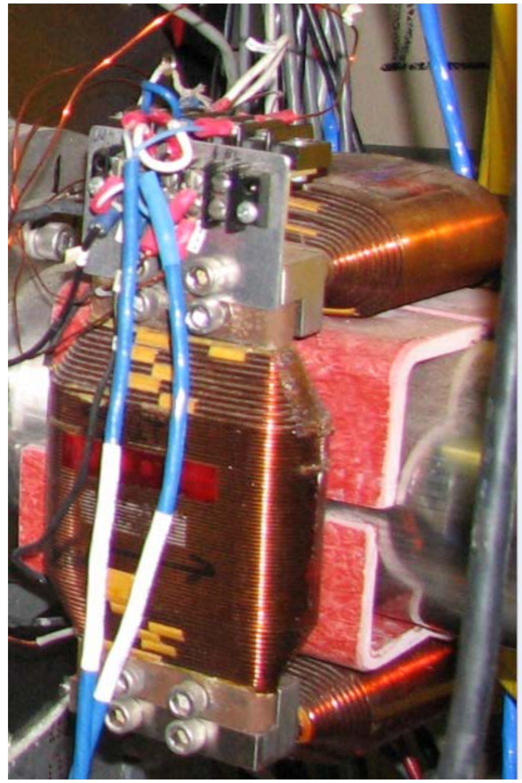




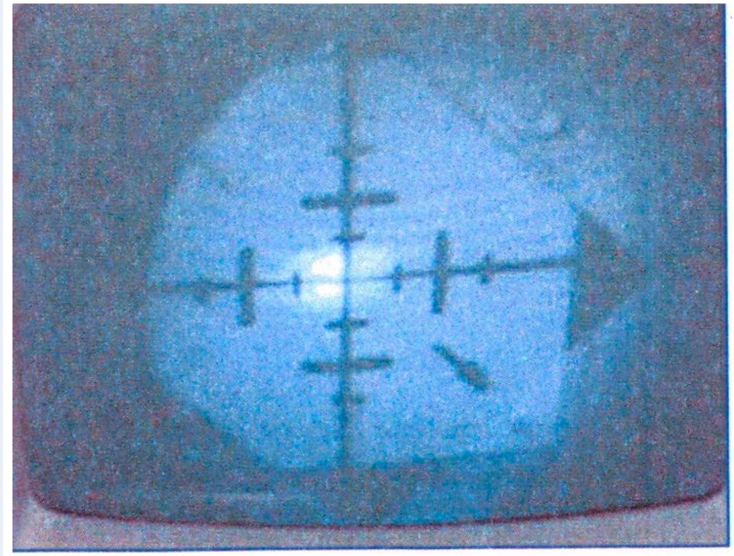
Scintillation Screens and Steering Magnets

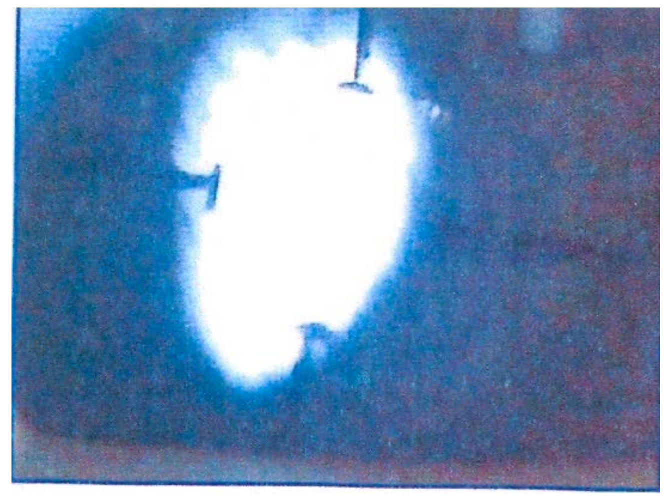


View of a Chromolux screen with a camera. The screen is illuminated by an external light. The lines have a separation of 5mm (P.Forck, 2011).



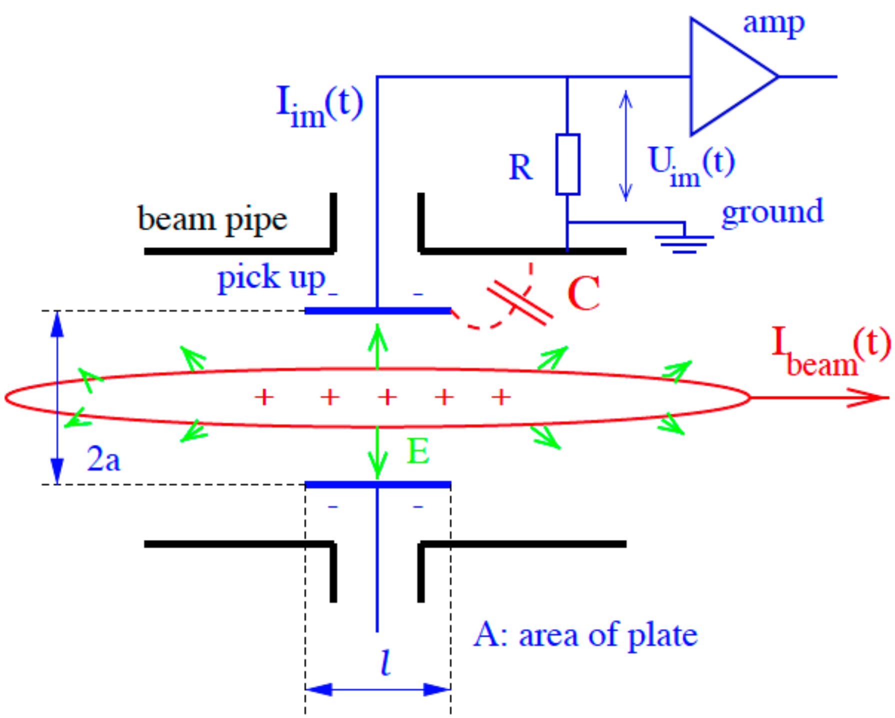
Steering magnet



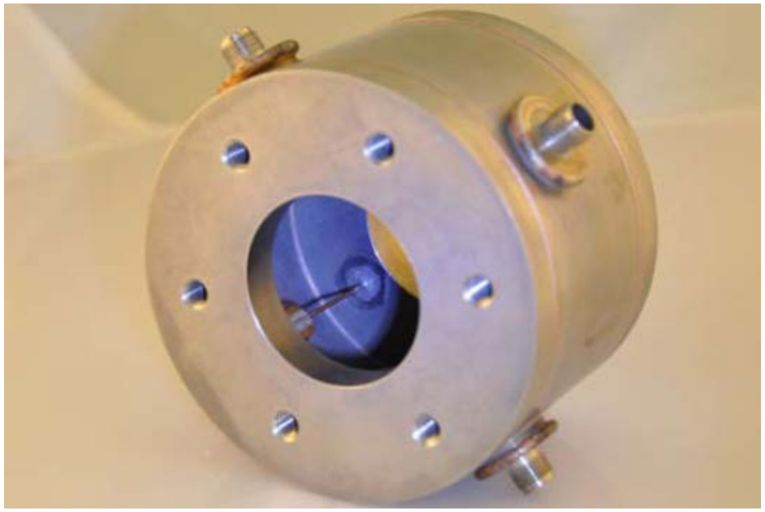


LANSCE phosphor screens illuminated by 800 MeV proton beam.

Beam Position Monitors



Scheme of pick-up electrode (P.Forck, 2011).



LANSCE BPM

Parameter	Value
Frequency of Measurement	201.25 MHz
System Response Time	50 ns
Averaging Window for System Resolution Specifications	100 µs
Position Resolution (% of radius, RMS)	0.46% (0.1mm)
Position Accuracy (% of radius)	±4.6
Position Range (% of inner electrode radius)	±60
Phase Resolution (RMS)	0.25°
Phase Linearity	±2°
Beam Current Resolution (RMS)	0.05 mA
Beam Current Accuracy	N/A
Beam Current Range	0.9 to 21 mA
Timing Uncertainty	±50 ns