LA-UR-16-29131

# Proton and Ion Linear Accelerators

## 1. Basics of Beam Acceleration

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June 7 - July 2, 2021

Los Alamos NATIONAL LABORATORY

# Energy, Velocity, Momentum

Total energy $$E_{part} = \sqrt{(pc)^2 + (mc^2)^2} = mc^2 + W$$

Rest energy $$mc^2$$

Kinetic energy $$W = \sqrt{p^2c^2 + m^2c^4} - mc^2 = mc^2(\gamma - 1)$$

Relativistic particle energy $$\gamma = \frac{E_{part}}{mc^2} = 1 + \frac{W}{mc^2} = \frac{1}{\sqrt{1-\beta^2}}$$

Particle velocity relative to speed of light $$\vec{\beta} = \frac{\vec{\mathrm{v}}}{c}$$

Mechanical (kinetic) particle momentum $$\vec{p} = m\gamma\vec{\mathrm{v}} = mc\vec{\beta}\gamma$$

Particle velocity versus relativistic energy $$\beta = \frac{\sqrt{\gamma^2 - 1}}{\gamma}$$

Mechanical momentum versus velocity and relativistic energy $$\frac{p}{mc} = \beta\gamma = \sqrt{\gamma^2 - 1}$$

# Energy, Velocity, Momentum (cont.)

| | $\beta$ | $\gamma$ | $W$ | $cp$ |
|---|---|---|---|---|
| $\beta$ | $\beta$ | $\frac{\sqrt{\gamma^2-1}}{\gamma}$ | $\frac{\sqrt{(1+W/E_0)^2-1}}{1+W/E_0}$ | $\frac{cp/(mc^2)}{\sqrt{1+[cp/(mc^2)]^2}}$ |
| $\gamma$ | $\frac{1}{\sqrt{1-\beta^2}}$ | $\gamma$ | $1+W/E_0$ | $\sqrt{1+\left(\frac{cp}{mc^2}\right)^2}$ |
| $W$ | $\left(\frac{1}{\sqrt{1-\beta^2}}-1\right)E_0$ | $E_0(\gamma-1)$ | $W$ | $mc^2\left[\sqrt{1+\left(\frac{cp}{mc^2}\right)^2}-1\right]$ |
| $cp$ | $mc^2\frac{\beta}{\sqrt{1-\beta^2}}$ | $E_0(\gamma^2-1)^{1/2}$ | $[W(2E_0+W)]^{1/2}$ | $cp$ |

Some relations concerning first derivatives of relativistic factors:

$$\frac{d\beta}{d\gamma}=\frac{1}{\beta\gamma^3}\ ;\quad \frac{d(1/\beta)}{d\gamma}=-\frac{1}{\beta^3\gamma^3}\ ;\quad \frac{d(\beta\gamma)}{d\beta}=\gamma^3\ ;\quad \frac{d(\beta\gamma)}{d\gamma}=\frac{1}{\beta}\ ;$$

Logarithmic first derivatives:

$$\frac{d\beta}{\beta}=\frac{1}{\beta^2\gamma^2}\frac{d\gamma}{\gamma}=\frac{1}{\gamma(\gamma+1)}\frac{dW}{W}=\frac{1}{\gamma^2}\frac{dp}{p}\ ;\quad \frac{d\gamma}{\gamma}=(\gamma^2-1)\frac{d\beta}{\beta}=\left(1-\frac{1}{\gamma}\right)\frac{dW}{W}=\beta^2\frac{dp}{p}$$

(P. Lapostolle and M. Weiss, CERN-PS-2000-001 DR)

# Vector Operations in Cartesian Coordinates

$$\nabla \psi = \frac{\partial \psi}{\partial x}\,\hat{\mathbf{x}} + \frac{\partial \psi}{\partial y}\,\hat{\mathbf{y}} + \frac{\partial \psi}{\partial z}\,\hat{\mathbf{z}}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{z}}$$

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}$$

# Vector Operations in Cylindrical Coordinates

$$x = r\cos\theta \qquad r = \sqrt{x^2+y^2}$$

$$y = r\sin\theta \qquad \tan\theta = \frac{y}{x}$$

$$z = z \qquad z = z$$

$$\nabla\psi = \frac{\partial\psi}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{\partial\psi}{\partial z}\,\hat{\mathbf{z}}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(r\,A_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial\theta} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\mathbf{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial\theta} - \frac{\partial A_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial}{\partial r}(r\,A_\theta) - \frac{\partial A_r}{\partial\theta}\right)\hat{\mathbf{z}}$$

Note that

$$\nabla\times\mathbf{A} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ A_r & r\,A_\theta & A_z \end{vmatrix}.$$

$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2}$$

# Maxwell's equations

$$rot\,\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$rot\,\vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{j}$$

$$div\,\vec{D} = \rho$$

$$div\,\vec{B} = 0$$

Electric field $\vec{E}$

Electric displacement field $\vec{D} = \varepsilon_o \vec{E}$

Magnetic field $\vec{B} = \mu_o \vec{H}$

Magnetic field strength $\vec{H}$

Permittivity of free space $\varepsilon_o = 8.85 \cdot 10^{-12}\,\mathrm{F/m}$

Permeability of free space $\mu_o = 4\pi \cdot 10^{-7}\,\mathrm{H/m}$

# Units

$$W = eU\ [eV],\ [electronVolt]$$

$$1\ eV = 1.6\cdot 10^{-19}[C]\ \text{x}\ 1\ [V] = 1.6\cdot 10^{-19}\,Joule$$

$$1\ \ Joule = 1\ \ Coulomb\cdot 1\ \ Volt = \frac{kg\cdot m^2}{\mathrm{s}^2}$$

## Electron energy

$$m_{electron} = 9.1\cdot 10^{-31}\,kg$$

$$c = 3\cdot 10^8\,m\,/\,\sec$$

$$e = 1.6\cdot 10^{-19}\,Culomb$$

$$\frac{m_{electron}c^2}{e} = 0.51092\cdot 10^6\,Volt$$

$$m_{electron}c^2 = 0.51092\cdot 10^6\,eV = 0.51092\,MeV$$

## Proton energy

$$m_{proton} = 1.672\cdot 10^{-27}\,kg = 1836\,m_{electron}$$

$$m_{proton}c^2 = 938.27\,MeV$$

$$\frac{m_{proton}c^2}{e} = 938.27\cdot 10^6\,Volt$$

# Units (cont.)

Ion Energy

Atomic mass unit (1/12 the mass of one atom of carbon-12):

1u= 1.660540 x $10^{-27}$ kg

$$E_a = 931.481\, MeV$$

Proton mass: 1.007276 u

Electron mass: 0.00054858 u

$$E_{ion} = 931.481 \cdot A - 0.511 \cdot Z\, [MeV]$$

A-atomic mass number

Z-number of removed electrons (ionization state)

Binding energy of removed electrons is neglected

Negative Ion of Hydrogen

$H^-$ ion mass: 1.00837361135 u

$$E_{H^-} = E_{proton} + 2 \text{x} E_{electron} = 939.28\, MeV$$

# Units (cont.)

Particle momentum $\frac{p}{mc} = \beta\gamma = \sqrt{\gamma^2 - 1}$ $p = \frac{mc^2}{c}\sqrt{\gamma^2 - 1}\ [\frac{GeV}{c}]$

Particle rigidity $B\rho = \frac{p}{q}\ [T \cdot m]$

Example: proton beam with kinetic energy W = 3 GeV:

$$E_{part} = mc^2 + W = 3.938\,GeV \qquad \gamma = \frac{mc^2 + W}{mc^2} = 4.2 \qquad \beta = \frac{\sqrt{\gamma^2 - 1}}{\gamma} = 0.971$$

$$\frac{p}{mc} = \beta\gamma = \sqrt{\gamma^2 - 1} = 4.079 \qquad p = \frac{mc^2}{c}\sqrt{\gamma^2 - 1} = 3.82\ \frac{GeV}{c} \qquad \frac{p}{e} = B\rho = 12.7\ T \cdot m$$

# Equations of Motion in Cartesian Coordinates

$$\frac{d\vec{x}}{dt} = \vec{v}$$

$$\frac{d\vec{p}}{dt} = q(\vec{E} + \vec{v} \times \vec{B})$$

$$\frac{dx}{dt} = \frac{p_x}{m\gamma} \qquad \frac{dp_x}{dt} = q\left(E_x + \frac{p_y}{m\gamma}B_z - \frac{p_z}{m\gamma}B_y\right)$$

$$\frac{dy}{dt} = \frac{p_y}{m\gamma} \qquad \frac{dp_y}{dt} = q\left(E_y - \frac{p_x}{m\gamma}B_z + \frac{p_z}{m\gamma}B_x\right)$$

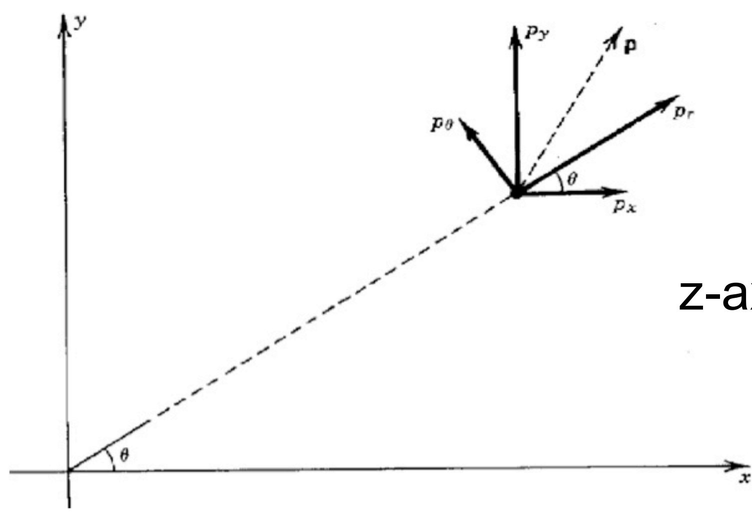
$$\frac{dz}{dt} = \frac{p_z}{m\gamma} \qquad \frac{dp_z}{dt} = q\left(E_z + \frac{p_x}{m\gamma}B_y - \frac{p_y}{m\gamma}B_x\right)$$

# Equations of Motion in Cylindrical Coordinates

$$\frac{dr}{dt} = \frac{p_r}{m\gamma} \qquad \frac{dp_r}{dt} = \frac{p_\theta^2}{m\gamma\, r} + q\,(E_r + \frac{p_\theta}{m\gamma} B_z - \frac{p_z}{m\gamma} B_\theta)$$

$$\frac{d\theta}{dt} = \frac{p_\theta}{m\gamma\, r} \qquad \frac{1}{r}\frac{d(rp_\theta)}{dt} = q\,(E_\theta + \frac{p_z}{m\gamma} B_r - \frac{p_r}{m\gamma} B_z)$$

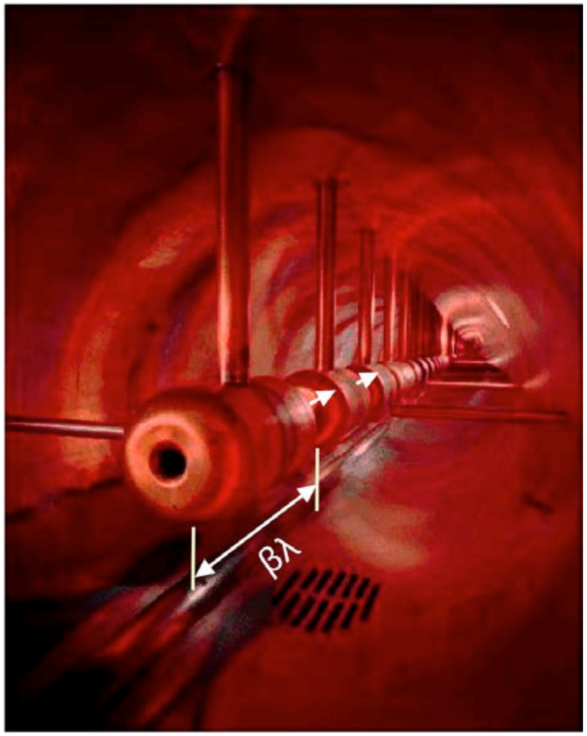
$$\frac{dz}{dt} = \frac{p_z}{m\gamma} \qquad \frac{dp_z}{dt} = q\,(E_z + \frac{p_r}{m\gamma} B_\theta - \frac{p_\theta}{m\gamma} B_r)$$



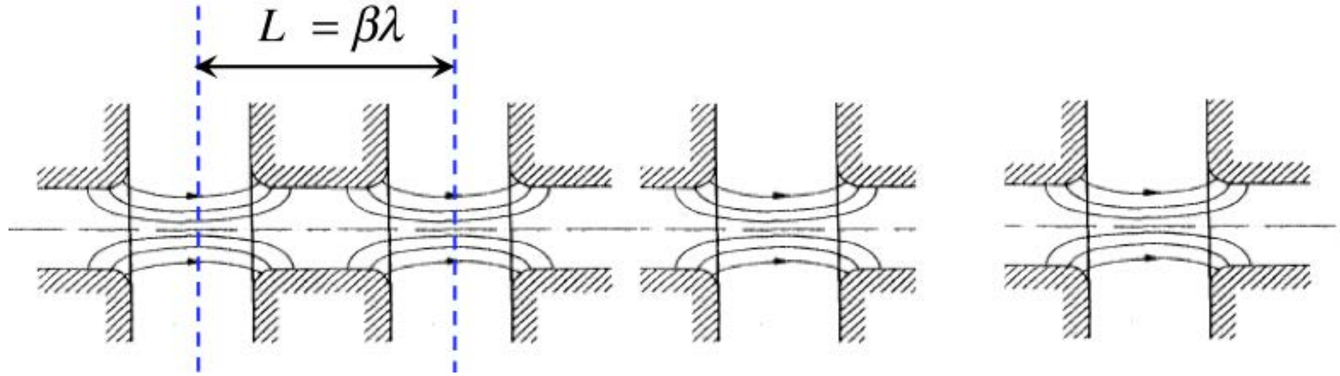

z-axis is directed to the reader.

Relationship between cylindrical and Cartesian coordinates.

# Resonance Principle of Particle Acceleration




Alvarez accelerating structure




Field distribution in RF structure: $E_z(z,r,t) = E_g(z,r)\cos(\omega t)$

Time of flight between RF gaps $t_{flight} = T_{RF\ period} = \frac{1}{f}$ [sec]

Distance between RF gaps $L = n\beta c T_{RF\ period} = n\beta\lambda$ [m]

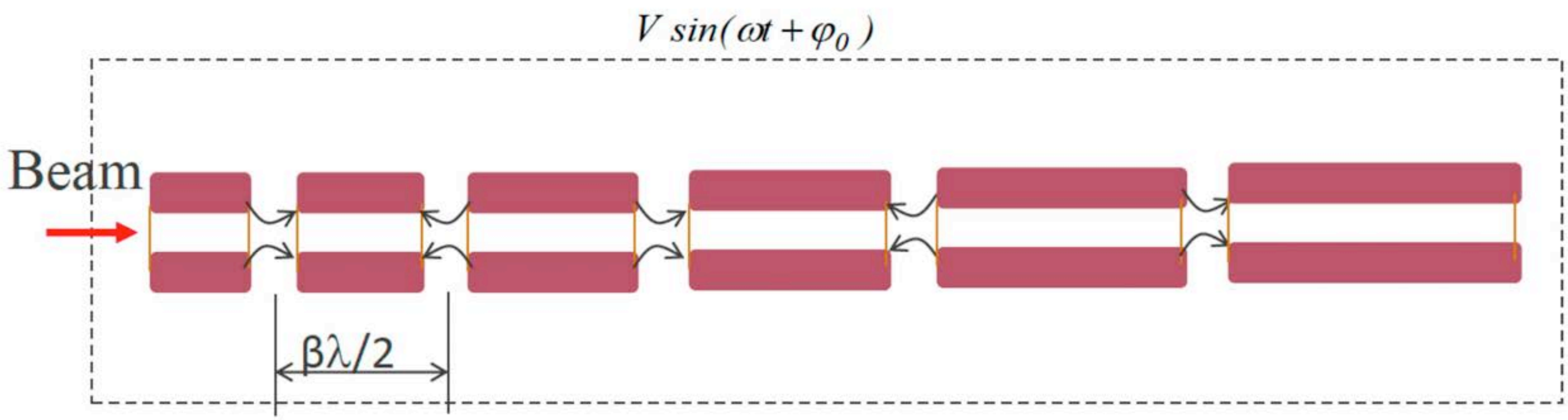
RF Frequency $f$ [Hz], [1/sec]

Circular RF Frequency $\omega = 2\pi f$ [radians/sec]

RF Wavelength $\lambda = \frac{c}{f}$ [m]

Acceleration in linear resonance accelerator is based on synchronism between accelerating field and particles.

# Acceleration in π - Structure




Accelerating structure with π - type standing wave.

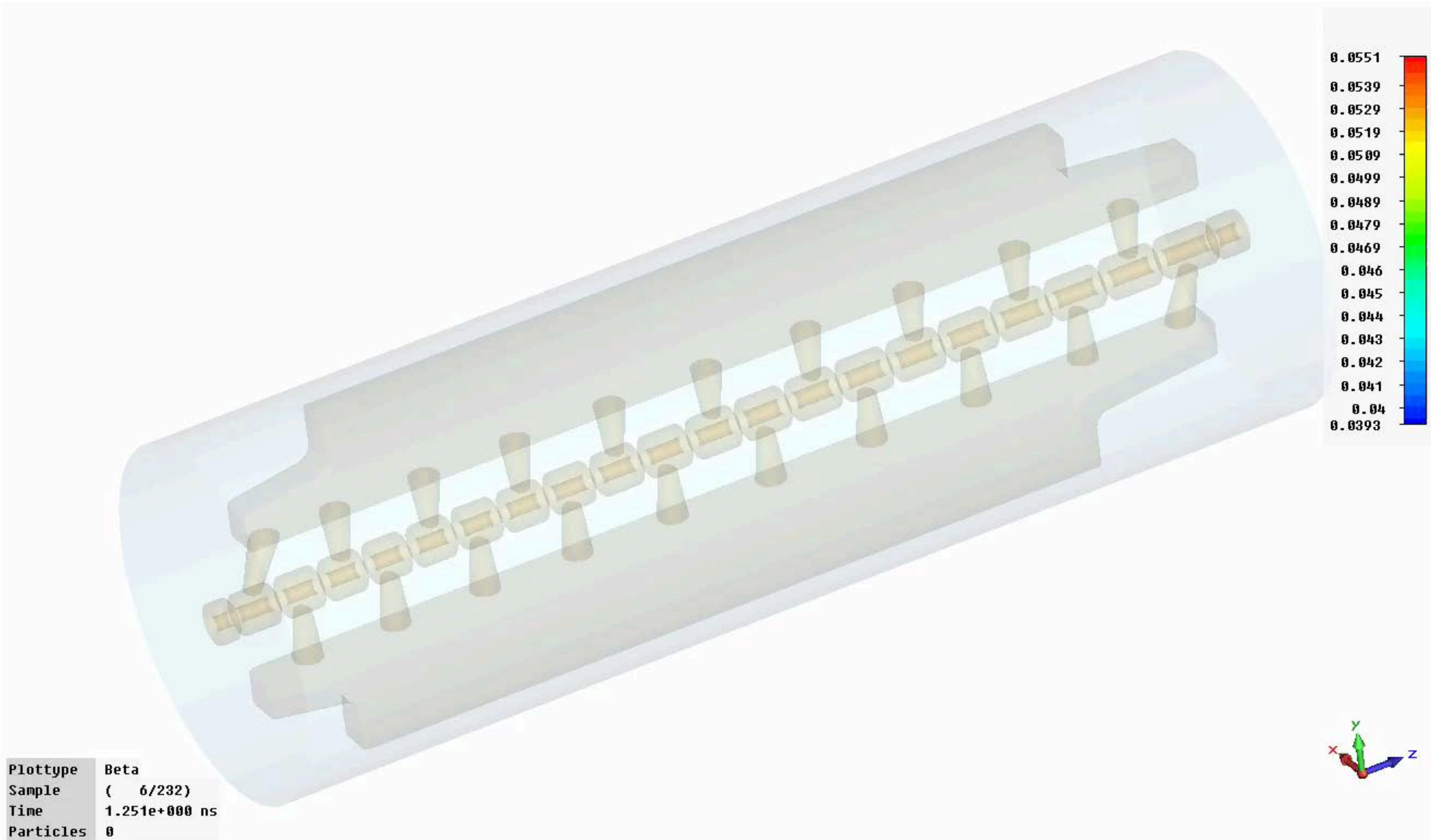
Time of flight between RF gaps of π- structure

$$t_{flight} = \frac{T_{RF\ period}}{2}$$

Distance between RF gaps of π- structure

$$L = \frac{\beta c T_{RF\ period}}{2} = \frac{\beta\lambda}{2}$$

# Acceleration in $\pi$- Structure




Acceleration in $\pi$- structure (Courtesy of Sergey Kurennoy).

# G. Ising Proposal on Linear Acceleration (1924)

Gustav Ising (1883-1960)

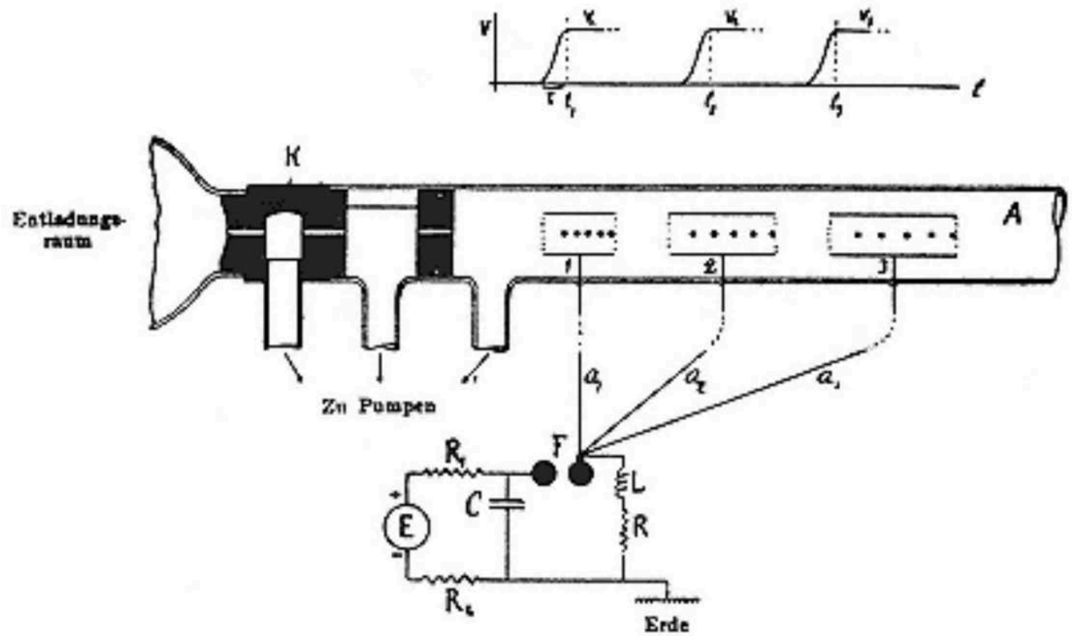
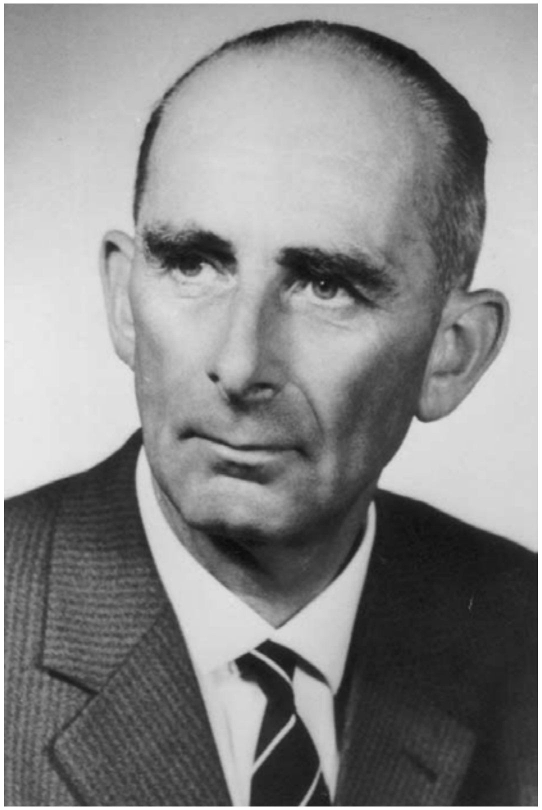
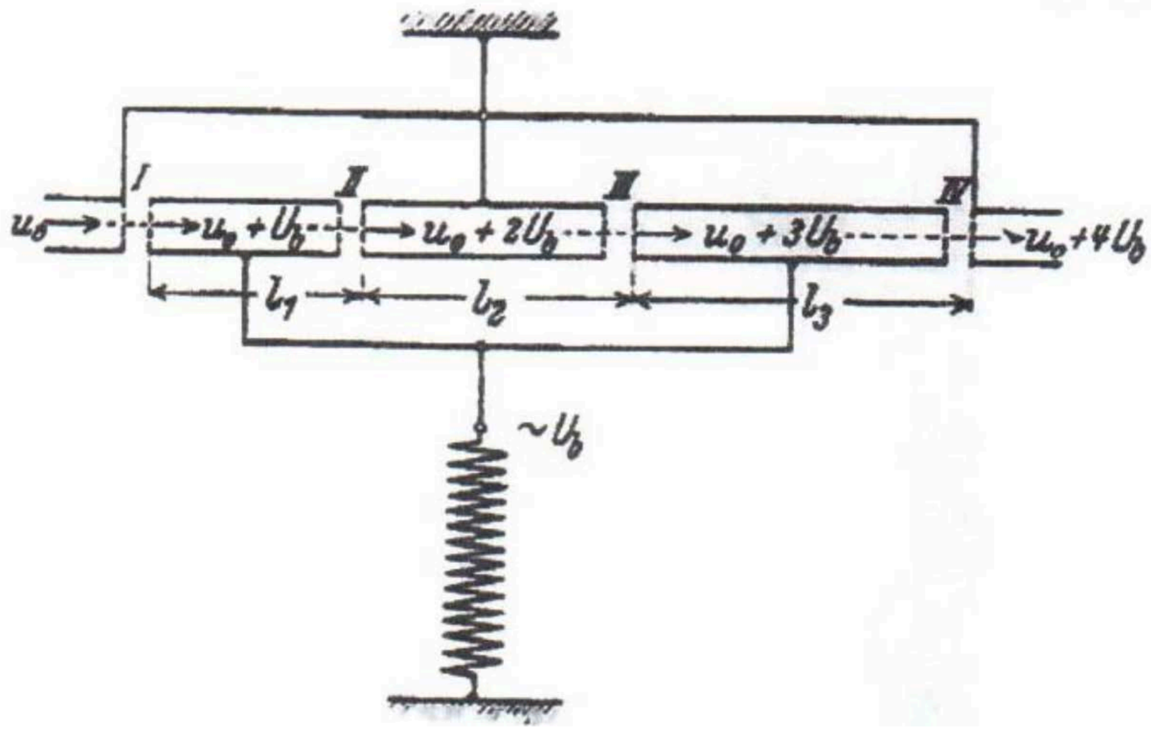



Fig. 2.13
Ising's proposal for a linear particle accelerator. The high-frequency field is supplied by a discharge across the spark gap F; K is the cathode; $a_1$, $a_2$, $a_3$, connections to the drift tubes. Ising, *Kosmos, 11* (1933), 171.

In 1924 G. Ising proposes time-varying fields across drift tubes. This is "resonant acceleration", which can achieve energies above the given highest voltage in the system. G. Ising published an accelerator concept with voltage waves propagating from a spark discharge to an array of drift tubes.

# First Demonstration of RF Linear Acceleration by R. Wideroe (1928)



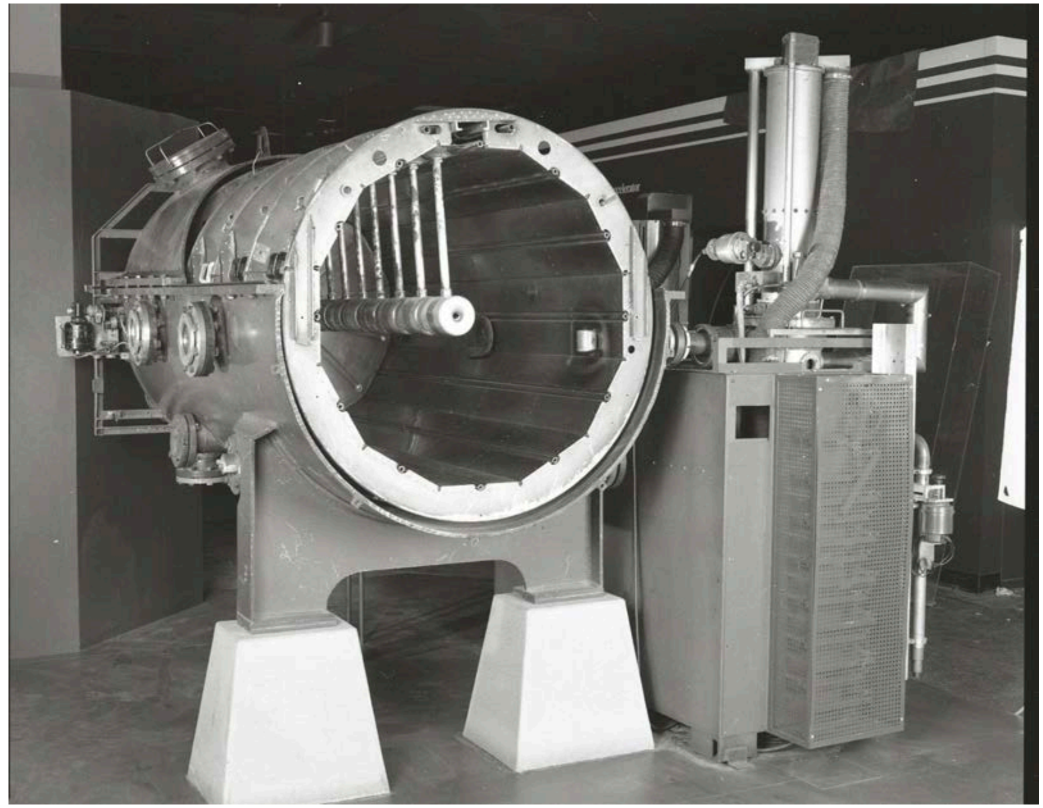



Rolf Wideroe (1902-1996)

In 1928 R. Wideroe demonstrates Ising's principle with 1 MHz, 25 kV oscillator to make 50 keV potassium ions. Wideroe simplified Ising's concept by replacing the spark gap with an ac oscillator.

# First Proton Linac by L. Alvarez (1947)

Luis Alvarez (1911-1988)



In 1947 Luis Alvarez at Berkeley designed a proton drift-tube linac 12-m long, 1-m diameter, 4 MeV to 32 MeV, initially using surplus 200-MHz vacuum tubes. Alvarez introduced a copper resonant cavity for better efficiency, loaded with an array of drift tubes.

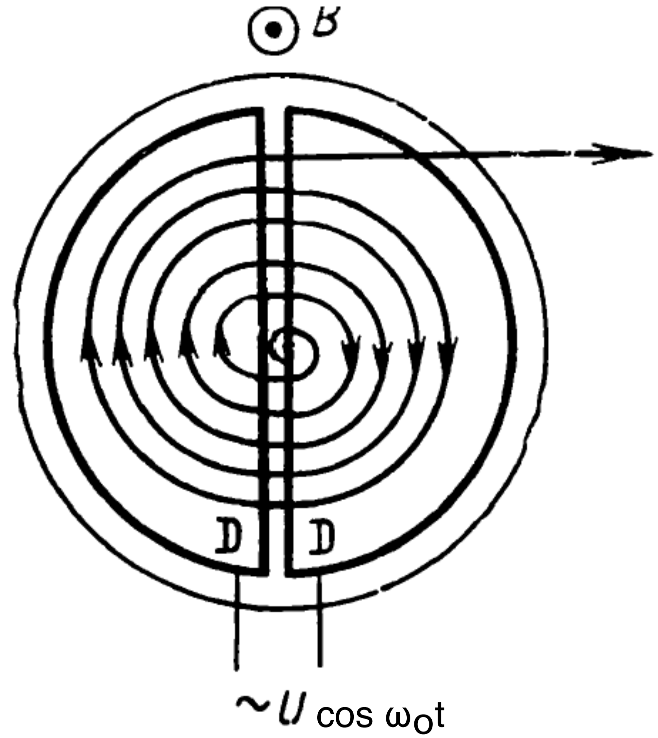
# Circular Resonance Acceleration: Classical Cyclotron

The acceleration of a particle in a circular orbit is determined by Lorentz force

$$\frac{m\gamma v^2}{R} = qvB$$

Rewrite this equation as

$$\boxed{BR = \frac{p}{q}}$$




To provide synchronism the frequency of electric field $\omega_o$ must be equal to frequency of particle rotation in magnetic field. In classical (non-relativistic cyclotron):

$$\omega_o = \omega = \frac{q}{m}$$

Kinetic energy is increasing proportionally to number of turns

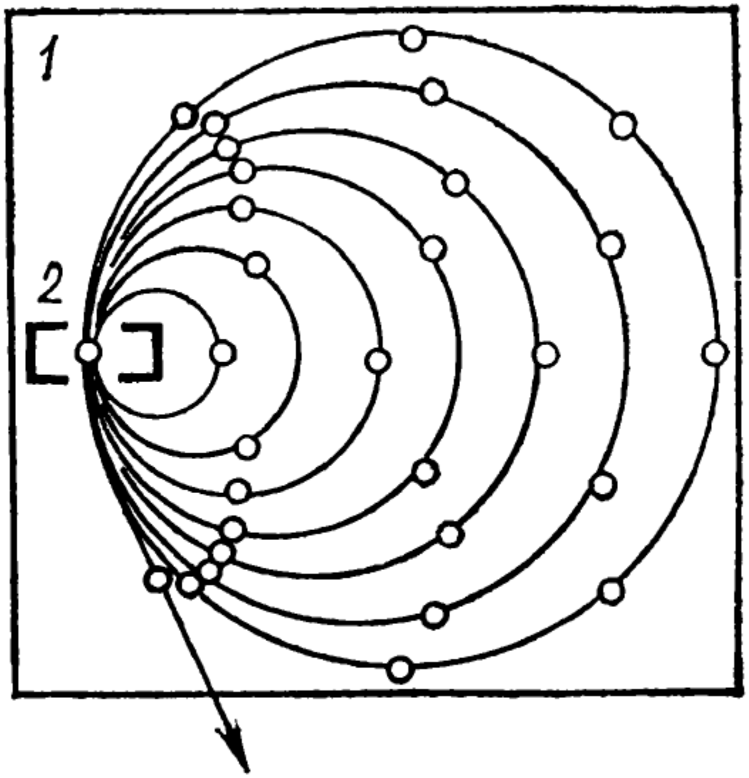
$$W \approx 2qUn$$

Radius of particle orbit $R \approx \frac{2}{B}\sqrt{n\frac{Um}{q}}$

# Circular Resonance Acceleration: Microtron

Cyclotron cannot be used for acceleration of electrons, because electrons become relativistic after energy gain of a few 100 keV.

In Microtron, particles arrive to RF gap after multiple integer number of RF periods




Condition for particle acceleration in microtron: frequency of particle rotation in magnetic field must be equal to RF frequency divided by integer number:

$$\omega = \frac{qB}{m\gamma} = \frac{\omega_{RF}}{k}$$

Layout of microtron: 1 – magnet, 2- accelerating cavity

# Circular Resonance Acceleration: Synchrotron

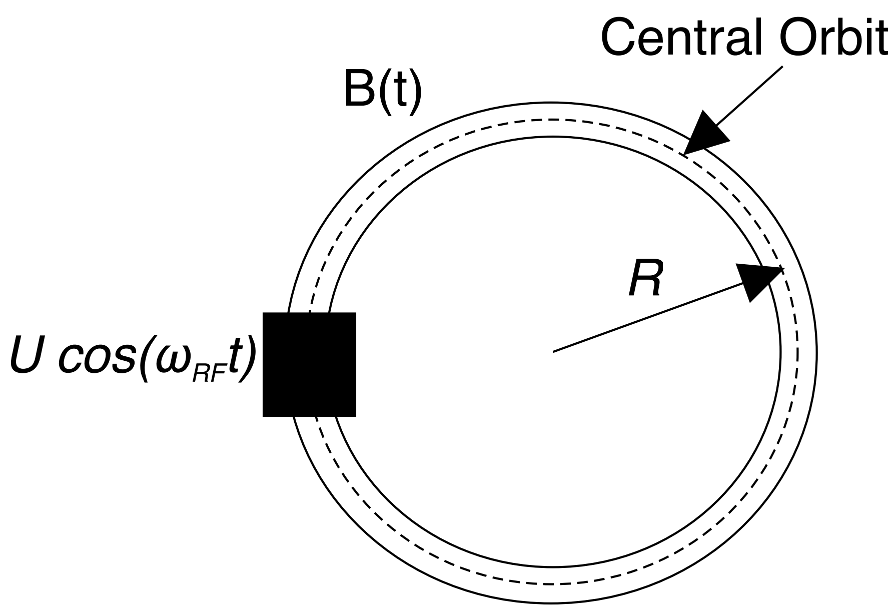
Acceleration with constant orbit radius:

$$R = \frac{p(t)}{B(t)q} = const$$

For acceleration at R = const, RF frequency must be strongly related to magnetic field at the orbit.

Total energy of equilibrium particle

$$E_s = \sqrt{(mc^2)^2 + (pc)^2} = \sqrt{(mc^2)^2 + [qB(t)Rc]^2}$$




Revolution frequency in magnetic field:

$$\omega = \frac{v}{R} = \frac{qvB}{p} = \frac{qB}{m\gamma} = \frac{qBc^2}{E_s}$$

Resonance condition between RF field and revolution frequency in magnetic field (k-integer):

$$\omega_{RF}(t) = k\omega(t) = k\frac{qB(t)c^2}{E_s}$$

# Induction Acceleration

Maxwell's equation for time-dependent electric field

$$rot\,\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Stock's Theorem:

$$\oint_{ABCA} \vec{E}\,d\vec{r} = -\int_S \frac{\partial \vec{B}}{\partial t} d\vec{S} = -\frac{\partial \Phi}{\partial t}$$

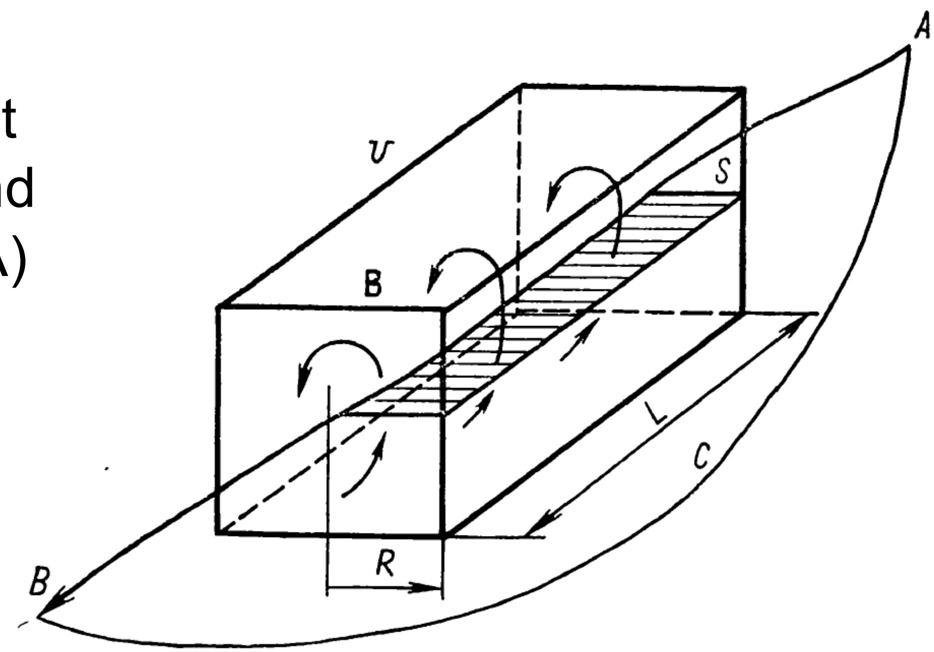
Magnetic flux through shaded area S

$$\Phi = \int \vec{B}\,d\vec{S}$$

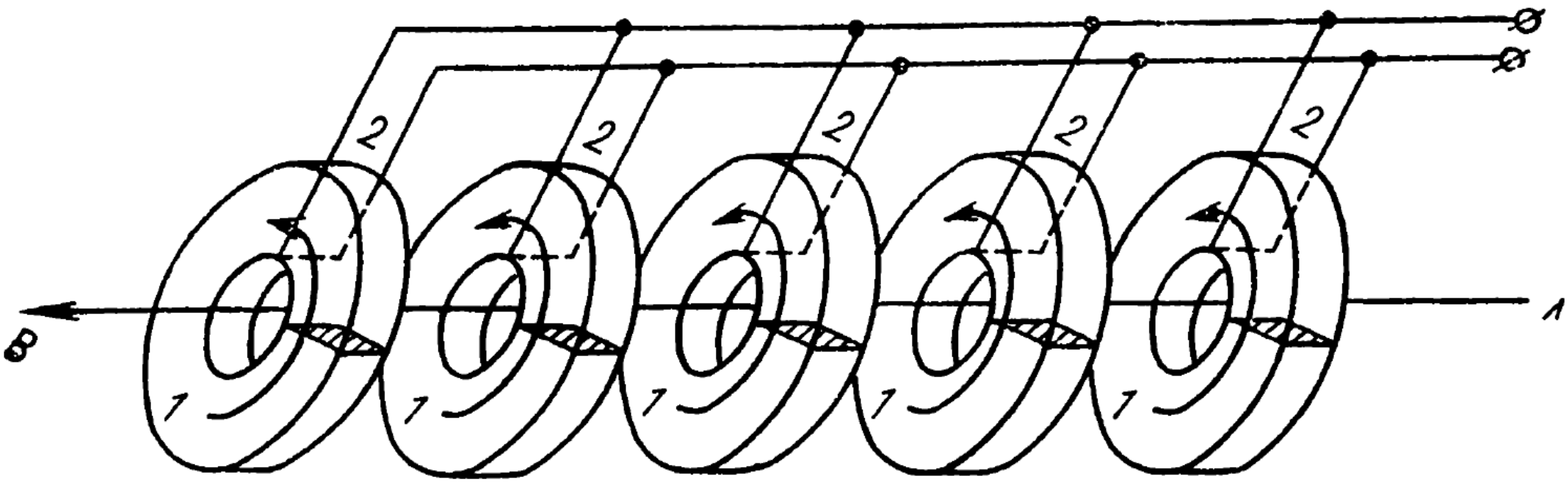
Let us integrate equation for increment of particle energy between points A and B (there is no electric field along B-C-A)

Increment of particle energy:

$$\Delta W = -q \frac{\partial \Phi}{\partial t}$$



# Linear Induction Acceleration




1 – Ferrite inductors, 2 – Coils

Beam propagates between A and B. Induction accelerator is in fact a transformer, where secondary coil is a beam itself.

# Linear Induction Accelerator

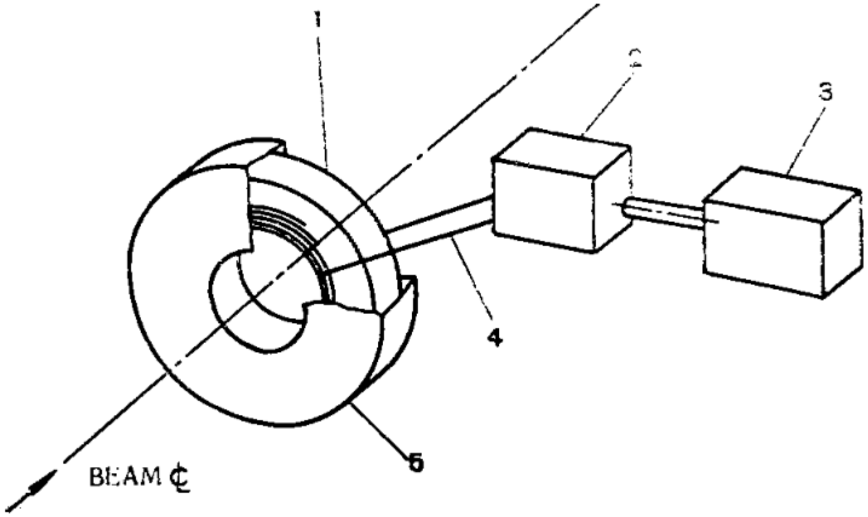
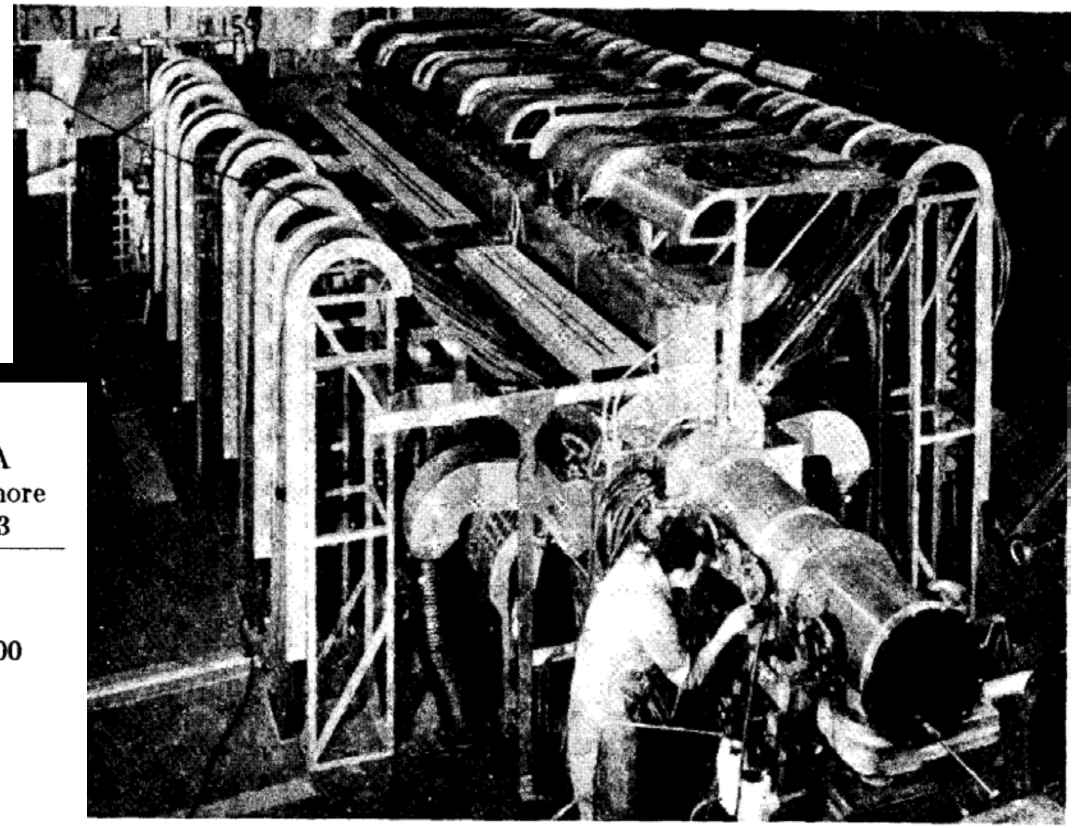



Fig. 1. Induction accelerator principle:
*1* — laminated iron core; *2* — switch; *3* — pulse forming network; *4* — primary loop; *5* — secondary (case).

$$\oint \vec{E}\cdot\vec{dl} = -\frac{1}{c}\int_s \frac{\vec{dB}}{dt}\cdot\vec{ds},$$

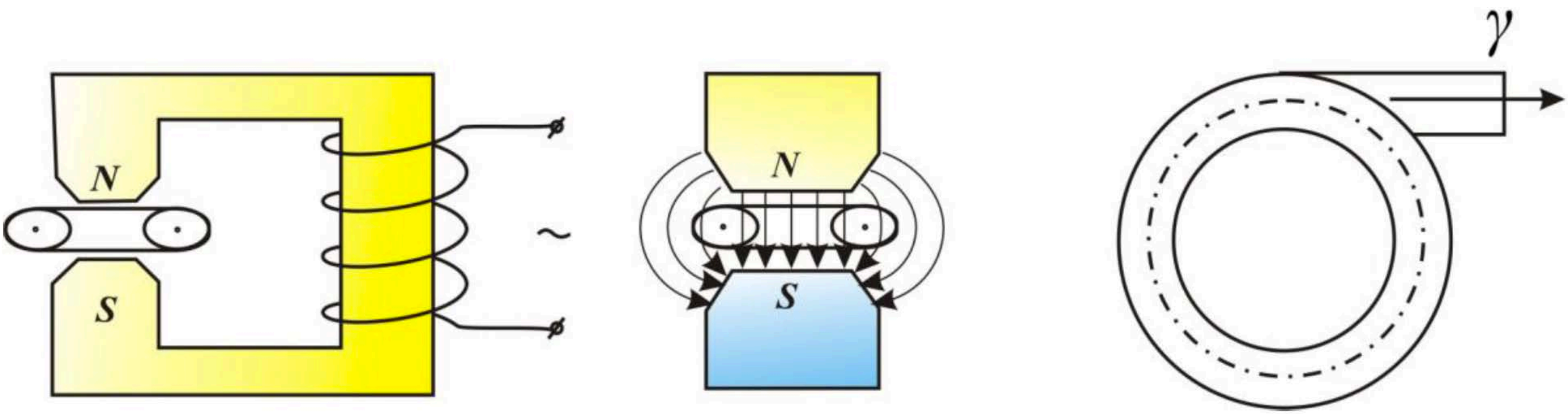


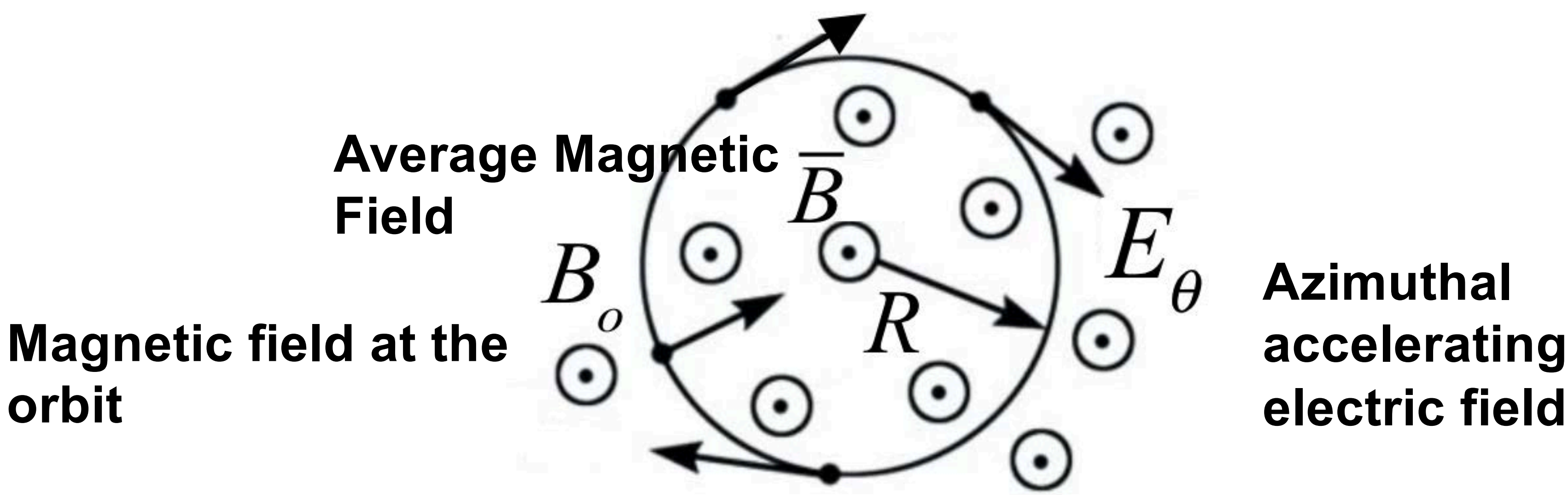
Overhead view of the Astron accelerator as it appeared when first put into operation.

**Table 3. Parameters for Typical Induction Accelerators**

| Accelerator | Astron Injector Livermore 1963 | ERA Injector Berkeley 1971 | NEP 2 Injector Dubna 1971 | ATA Livermore 1983 |
|---|---|---|---|---|
| Kinetic energy, MeV | 3.7 | 4.0 | 30 | 50 |
| Beam current on target, A | 350 | 900 | 250 | 10,000 |
| Pulse duration, ns | 300 | 2-45 | 500 | 50 |
| Pulse energy, kJ | 0.4 | 0.1 | 3.8 | 25 |
| Rep rate, pps | 0-60 | 0-5 | 50 | 5 |
| Number of switch modules | 300 | 17 | 750 | 200 |

# Circular Induction Accelerator: Betatron



# High – Voltage Acceleration

Equation of motion: $$\frac{d\vec{p}}{dt} = q\vec{E} + q[\vec{v}\vec{B}]$$

Let us multiply equation of motion by $\vec{v}$ :

$$\vec{v}\,d\vec{p} = dW \qquad \vec{v}\,dt = d\vec{r}$$

Increment of energy: $$dW = q\vec{E}d\vec{r}$$

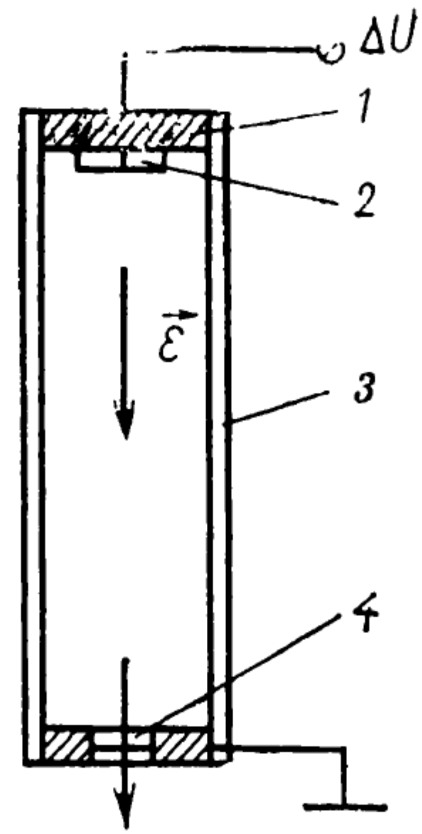
*If electric field is electrostatic, (expressed as gradient of potential)*

$$\vec{E} = -gradU$$

Conservation law: $$W + qU = const$$

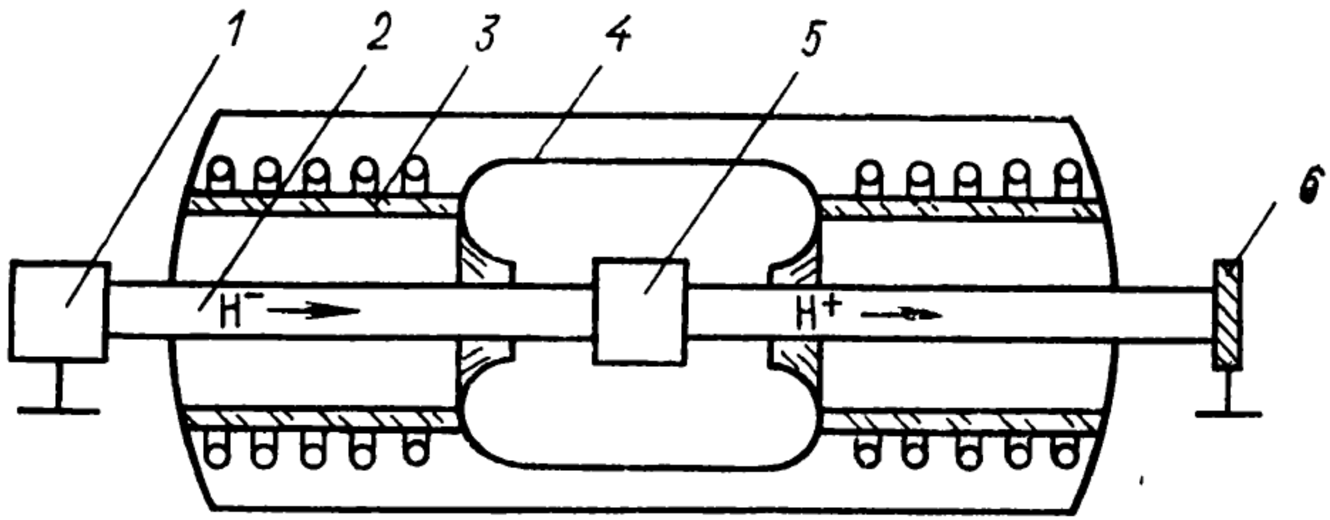
Increment of particle energy is determined by electrostatic potential difference

$$\Delta W = q\Delta U$$




1- High-voltage electrode
2- Particle source
3- Vacuum chamber,
4 – Exit window

# High Voltage Accelerator with Charge Exchange




1- source of negatively charged particles, 2- accelerating tube, 3- mounting of high-voltage electrodes, 4- - high-voltage electrode, 5 – stripper , 6- target

Maximal potential difference $\Delta U \approx 15 kV$

Maximal energy gain due to charge exchange: $\Delta W \approx 30 kV$

# Electromagnetic Wave Equations

In the absence of charges, $\vec{j}=0,\ \rho=0$, Maxwell's equations are

$$rot\,\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad div\,\vec{E} = 0$$

$$rot\,\vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \qquad div\,\vec{B} = 0$$

speed of light in free space:

$$c = \frac{1}{\sqrt{\varepsilon_o \mu_o}} = 2.99792458 \cdot 10^8\, m/\sec$$

Taking the *rot* of the *rot* equations gives:

$$rot\ \ rot\,\vec{E} = -\frac{\partial}{\partial t}(rot\,\vec{B}) = -\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$$

$$rot\ \ rot\,\vec{B} = \frac{1}{c^2}\frac{\partial}{\partial t}(rot\,\vec{E}) = -\frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2}$$

By using the vector identity $\quad rot\ \ rot\,\vec{A} = grad\ div\,\vec{A} - \Delta\vec{A}$

Taking into account that $div\,\vec{E}=0,\ div\,\vec{B}=0$ we receive wave equations:

$$\boxed{\Delta\vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0} \qquad \boxed{\Delta\vec{B} - \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} = 0}$$

# Components of Electromagnetic Field

Most of RF cavities are excited at a fundamental mode containing three components $E_z$ , $E_r$ , $B_\theta$ . They are connected through Maxwell's equations, therefore it is sufficient to find solution for one component only. Taking into account condition for axial-symmetric field ($\partial / \partial\theta = 0$), wave equation for $E_z$ component is

$$\frac{\partial^2 E_z}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right) - \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2} = 0$$

Radial component $E_z$ can be determined from $div\vec{E} = 0$ as

$$div\,\vec{E} = \frac{1}{r}\frac{\partial}{\partial r}(rE_r) + \frac{\partial E_z}{\partial z} = 0$$

which gives

$$E_r(r) = -\frac{1}{r}\int_o^r \frac{\partial E_z}{\partial z} r' dr'$$

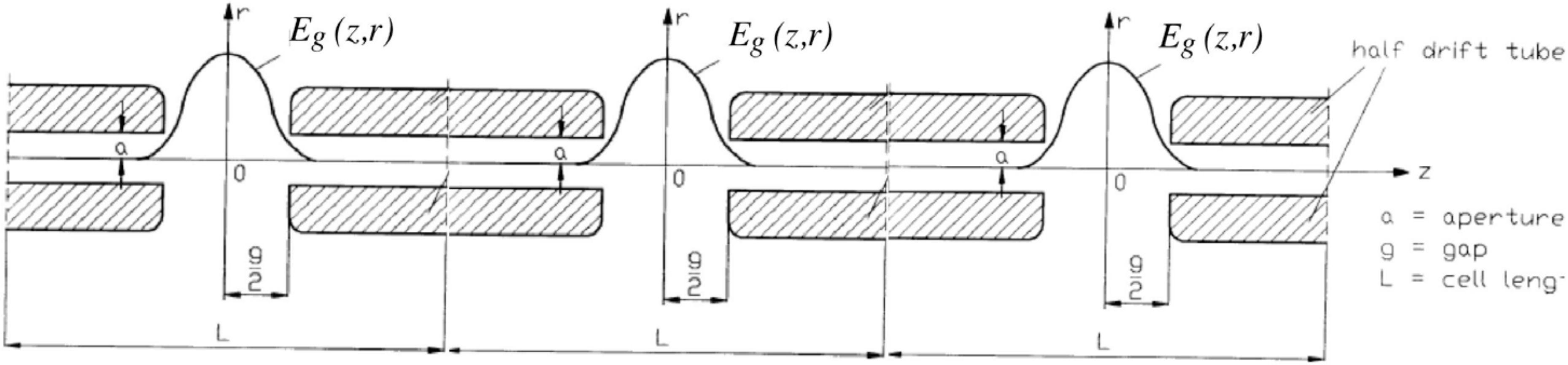
Azimuthal component of magnetic field is determined from

$$rot\,\vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

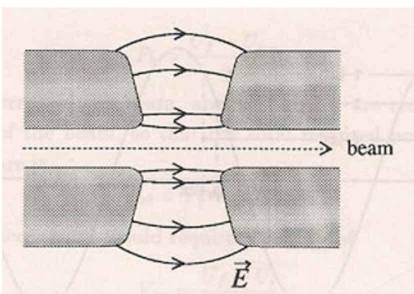
which gives

$$B_\theta = \frac{1}{c^2 r}\int_o^r \frac{\partial E_z}{\partial t} r' dr'$$

# Expansion of RF Field in Alvarez Structure




Periodic distribution of RF field.




Electric field lines between the ends of drift tubes.

Field in RF Gap: $E_z(z,r,t) = E_g(z,r)\cos(\omega t)$

Wave Equation for Field Distribution in RF Gap:

$$\frac{\partial^2 E_g}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_g}{\partial r}\right) + \left(\frac{\omega}{c}\right)^2 E_g = 0$$

Fourier Expansion of Field Distribution in RF Gap:

$$E_g(r,z) = A_o(r) + \sum_{m=1}^{\infty} A_m(r)\cos\left(\frac{2\pi m z}{L}\right)$$

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# Expansion of RF Field (cont.)

Equations for Fourier coefficients of RF gap expansion:

$$\frac{1}{r}\frac{\partial A_o(r)}{\partial r}+\frac{\partial^2 A_o(r)}{\partial r^2}+\left(\frac{\omega}{c}\right)^2 A_o(r)=0, \qquad m=0$$

$$\frac{1}{r}\frac{\partial A_m(r)}{\partial r}+\frac{\partial^2 A_m(r)}{\partial r^2}-k_m^2 A_m(r)=0, \qquad m>0$$

Transverse wave number:

$$k_m=\left(\frac{2\pi m}{L}\right)\sqrt{1-\left(\frac{L}{m\lambda}\right)^2}$$

Solutions are Bessel functions:

$$A_o(r)=A_o J_o\left(\frac{r\omega}{c}\right), \qquad m=0$$

$$A_m(r)=A_m I_o(k_m r), \qquad m>0$$

Finally, expressions for spatial z-component $E_g$ *(z,r)*

$$E_g(r,z)=A_o J_o\left(2\pi\frac{r}{\lambda}\right)+\sum_{m=1}^{\infty}A_m I_o(k_m r)\cos\left(\frac{2\pi m z}{L}\right)$$

# Bessel Functions

Bessel functions of the order *n* are solutions $y = J_n(z)$ of differential Bessel equation:

$$\frac{d^2y}{dz^2} + \frac{1}{z}\frac{dy}{dz} + (1 - \frac{n^2}{z^2})y = 0$$

Power representation of Bessel function:

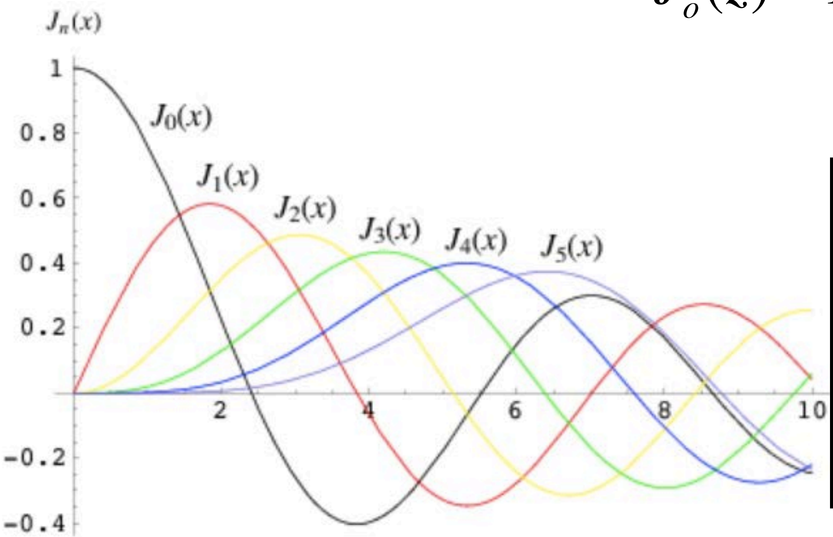
$$J_n(z) = \frac{1}{n!}(\frac{z}{2})^n - \frac{1}{1!(n+1)!}(\frac{z}{2})^{n+2} + \frac{1}{2!(n+2)!}(\frac{z}{2})^{n+4} - ... = (\frac{z}{2})^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(n+k+1)}(\frac{z}{2})^{2k}$$

Integral representation of Bessel functions:

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - z\sin\theta)\ d\theta$$

Special cases for $n$ = 0, 1:

$$J_o(z) = 1 - \frac{z^2}{4} + \frac{z^4}{64} - .....\qquad J_1(z) = -J_o'(z) = \frac{z}{2} - \frac{z^3}{16} + ....$$




Zeros $\nu_{nm}$ of Bessel function $J_n(z) = 0$ .

| | $m = 1$ | $m = 2$ | $m = 3$ | $m = 4$ |
|---|---|---|---|---|
| $n = 0$ | 2.405 | 5.52 | 8.654 | 11.792 |
| $n = 1$ | 3.832 | 7.016 | 10.173 | 13.323 |
| $n = 2$ | 5.136 | 8.417 | 11.62 | 14 |
| $n = 3$ | 6.38 | 9.761 | 13.015 | |

# Modified Bessel Functions

Modified Bessel functions of the *n*-th order $I_n(z) = i^{-n} J_n(iz)$ are solutions of modified Bessel differential equation:

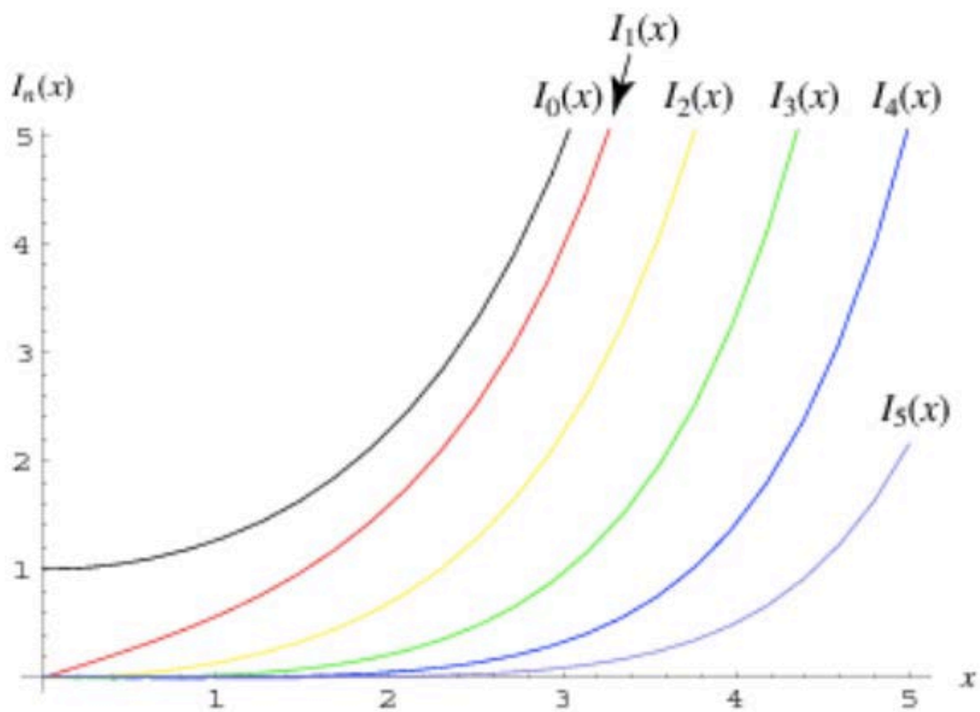
$$\frac{d^2y}{dz^2} + \frac{1}{z}\frac{dy}{dz} - (1 + \frac{n^2}{z^2})y = 0$$

Power representation of modified Bessel functions: $I_n(z) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(n+k+1)} (\frac{z}{2})^{n+2k}$

Special cases for *n* = 0, 1:

$$I_o(z) = 1 + \frac{z^2}{4} + \frac{z^4}{64} + \frac{z^6}{2304} + ...$$

$$I_1(z) = I_o^{'}(z) = \frac{z}{2} + \frac{z^3}{16} + \frac{z^5}{384} + ...$$




Modified Bessel functions of 1st kind, $I_n(x)$.

# Integrals and Derivatives of Bessel Functions

Let $Z_n(x)$ to be an arbitrary Bessel function:

$$\frac{dZ_n(x)}{dx} = -\frac{n}{x}Z_n(x) + Z_{n-1}(x) = \frac{n}{x}Z_n(x) - Z_{n+1}(x)$$

$$\int x^{n+1} Z_n(x)dx = x^{n+1}Z_{n+1}(x)$$

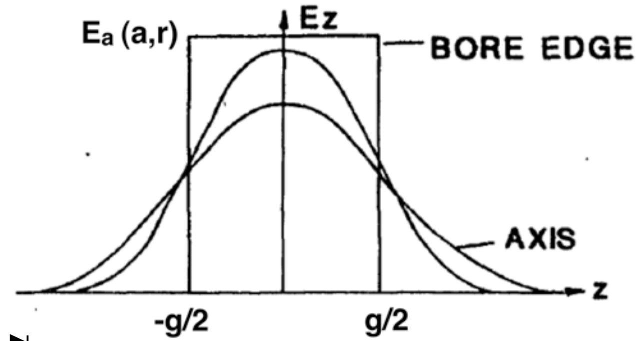
Particularly

$$Z_o^{'}(x) = -Z_1(x)$$

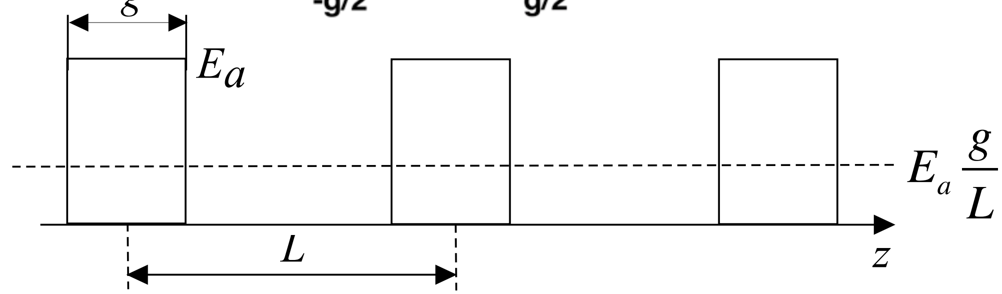
$$Z_1^{'}(x) = Z_o(x) - \frac{Z_1(x)}{x}$$

# Expansion of RF Field (cont.)

To get an approximate expression for coefficients $A_m$ , let us assume the step-function distribution of component inside RF gap of width at bore radius $r = a$




$$E_g(z,a) = \begin{cases} E_a & 0 \le |z| \le \dfrac{g}{2} \\ 0 & |z| > \dfrac{g}{2} \end{cases}$$




Expansion of periodic step-function

$$E_g(a,z) = E_a\left[\frac{g}{L} + \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\sin\left(\pi m\frac{g}{L}\right)\cos\left(2\pi m\frac{z}{L}\right)\right]$$

Field expansion in RF gap

$$E_g(r,z) = A_o J_o\left(2\pi\frac{r}{\lambda}\right) + \sum_{m=1}^{\infty} A_m I_o(k_m r)\cos\left(\frac{2\pi m z}{L}\right)$$

Coefficients in field expansion:

$$A_o = \frac{E_a}{J_o\left(2\pi\dfrac{a}{\lambda}\right)}\frac{g}{L}$$

$$A_m = \frac{2E_a}{I_o(k_m a)}\frac{g}{L}\frac{\sin\left(\pi m\dfrac{g}{L}\right)}{\pi m\dfrac{g}{L}}$$

# Energy Gain of Synchronous Particle in RF Gap

Equation for change of longitudinal particle momentum

$$\frac{dp_z}{dt} = qE_z(z, r, t)$$

From relativistic equations $p_z = mc\sqrt{\gamma^2 - 1}$

$$dp_z = mc^2 d\gamma / (\beta c) \qquad dW = mc^2 d\gamma$$

the equation for change of particle energy

$$\frac{dW}{dz} = qE_z(z, r, t)$$

Increment of energy of synchronous particle per RF gap

$$\Delta W_s = q \int_{-L/2}^{L/2} E_g(z) \cos \omega t_s(z)\, dz$$

Particle velocity is *βc = dz/dt*. Integration gives:

$$t(z) = t_o + \int_0^z \frac{dz}{\beta(z)c}$$

When synchronous particle arrive in the center of the gap, $z$ = 0, the RF phase is equal to $\varphi_s$ . The time of arrival of synchronous particle in point with coordinate $z$ is

$$t_s(z) = \frac{\varphi_s}{\omega} + \frac{z}{\beta c} \quad \text{or} \quad \omega t_s(z) = \varphi_s + k_z z$$

where $k_z = \dfrac{2\pi}{\beta\lambda}$

# Energy Gain of Synchronous Particle in RF Gap (cont.)

Using equity $\cos\omega t_s = \cos\varphi_s \;\cos k_s z - \sin\varphi_s \;\sin k_s z$
the increment of synchronous particle energy per RF gap:

$$\Delta W_s = q\cos\varphi_s [\int_{-L/2}^{L/2} E_g(z)\;\cos(k_z z)dz - tg\varphi_s \int_{-L/2}^{L/2} E_g(z)\;\sin(k_z z)dz]$$

Let us multiply and divide this expression by $E_o L$ , where we introduce average field $E_o$ of the accelerating gap across accelerating period (note that $E_o$=$A_o$):

$$\boxed{E_o = \frac{1}{L}\int_{-L/2}^{L/2} E_g(z)dz} = \frac{E_a}{J_o(2\pi\frac{a}{\lambda})}\frac{g}{L} \approx E_a\frac{g}{L}$$

Effective voltage applied to RF gap:

$$\boxed{U = E_o L}$$

# Transit Time Factor

The increment of synchronous particle energy gain per RF gap can be written as

$$\boxed{\Delta W_s = qE_o TL\cos\varphi_s}$$

where *transit time factor* is

$$T = \frac{1}{E_o L}\left[\int_{-L/2}^{L/2} E_g(z)\ \cos(k_z z)dz - tg\varphi_s \boxed{\int_{-L/2}^{L/2} E_g(z)\ \sin(k_z z)dz}\right]$$

$\approx 0$

First approximation to transit time factor

$$\boxed{T = \frac{\int_{-L/2}^{L/2} E_g(z)\ \cos\left(\frac{2\pi n z}{L}\right)dz}{\int_{-L/2}^{L/2} E_g(z)dz}}$$

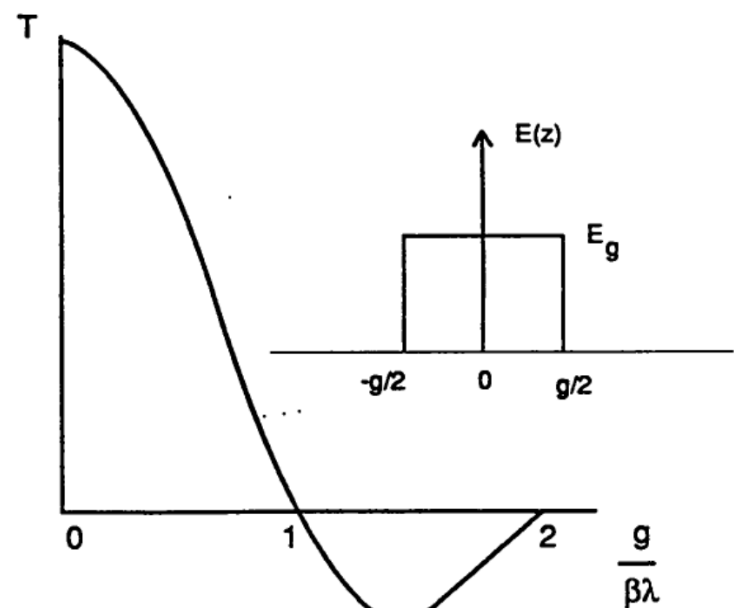
# Transit Time Factor (cont.)

Transit time factor indicates effectiveness of transformation of RF field into particle energy. It mostly depends on field distribution within the gap, which is determined by RF gap geometry.

Transit time factor $T = \frac{A_n}{2E_o}$ , where $A_n$ is the amplitude of *n*-th harmonics of Fourier field expansion

In most accelerators, synchronism is provided for *n* = 1, therefore:

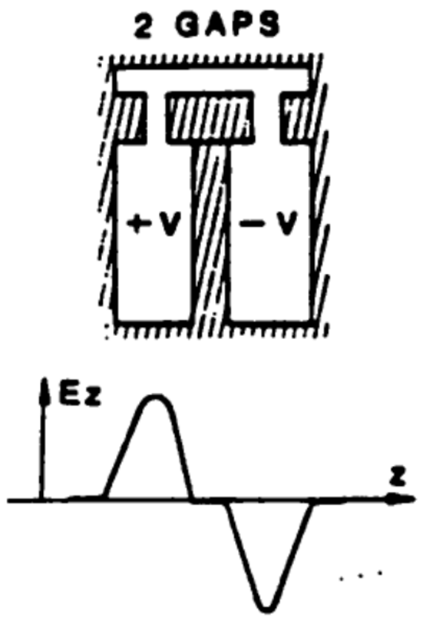
$$T = \frac{J_o(2\pi \frac{a}{\lambda})}{I_o(\frac{2\pi a}{\beta\gamma\lambda})} \frac{\sin(\frac{\pi g}{\beta\lambda})}{\frac{\pi g}{\beta\lambda}}$$



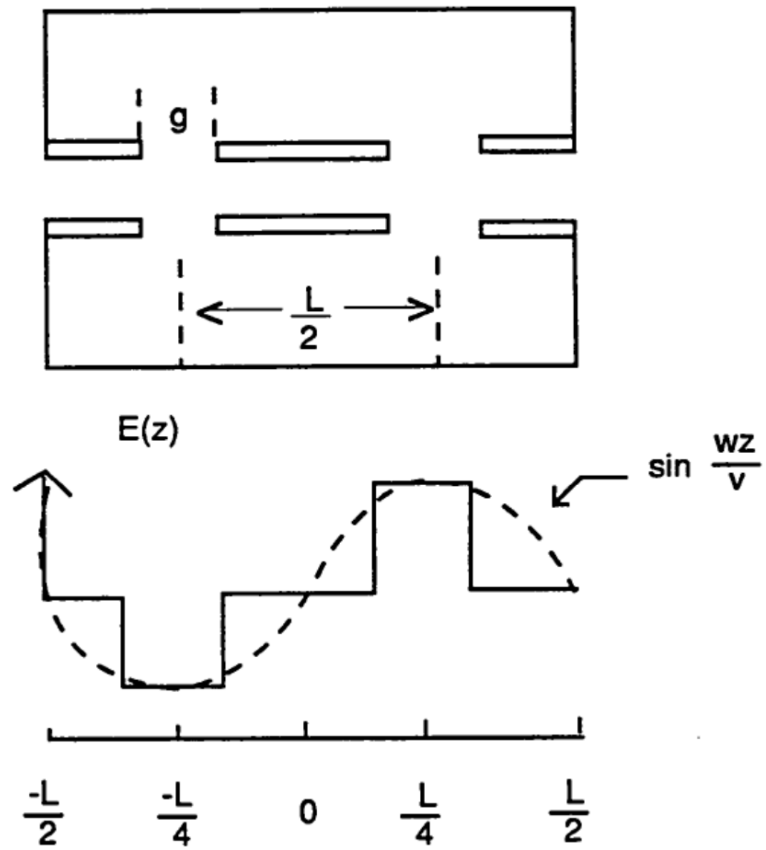

In accelerators usually aperture of the channel is substantially smaller than wavelength, $a << \lambda$ , then $J_o(2\pi a / \lambda) \approx 1$ , and transit time factor is

$$T = \frac{1}{I_o(\frac{2\pi a}{\beta\gamma\lambda})} \frac{\sin(\frac{\pi g}{\beta\lambda})}{\frac{\pi g}{\beta\lambda}}$$

# Transit Time Factor for Two-Gap Cavity




Two gap cavity



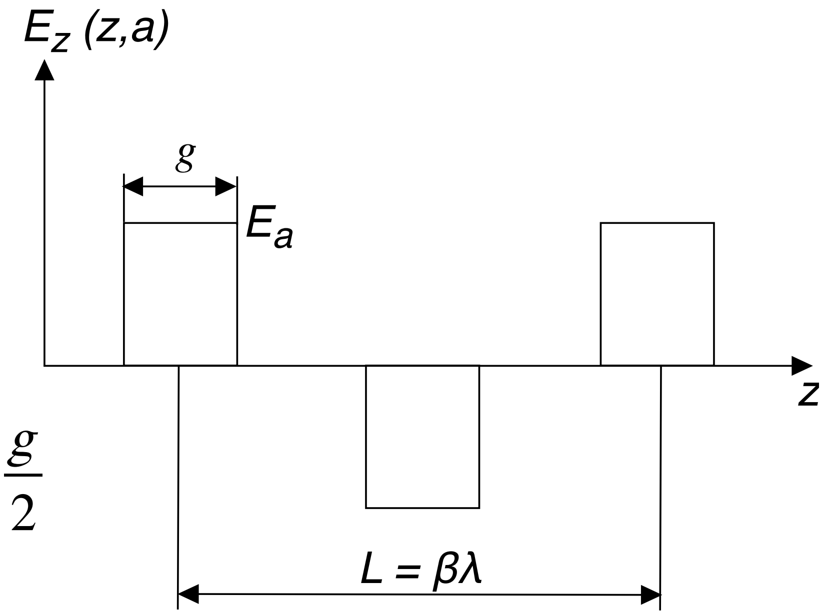

Field expansion in two-gap cavity

Transit time factor for two-gap cavity

$$T = \frac{1}{I_o\left(\frac{2\pi a}{\beta\lambda}\right)} \frac{\sin\left(\frac{\pi g}{\beta\lambda}\right)}{\frac{\pi g}{\beta\lambda}} \sin\frac{\pi L}{2\beta\lambda}$$

# Expansion of RF Field in π - Structure

Like in analysis of Alvarez structure, let us assume the step-function distribution of Ez component inside RF gap at bore radius *r = a*




$$E_g(z,a) = \begin{cases} E_a, & -\frac{g}{2} \le z \le \frac{g}{2}, \quad \beta\lambda - \frac{g}{2} \le z \le \beta\lambda \\ 0, & \frac{g}{2} < z < \frac{\beta\lambda - g}{2}, \quad \frac{\beta\lambda + g}{2} < z < \beta\lambda - \frac{g}{2} \\ -E_a, & \frac{\beta\lambda - g}{2} \le z \le \frac{\beta\lambda + g}{2} \end{cases}$$

Expansion of periodic step-function

$$E_g(z,a) = \frac{4E_a}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{2m-1}\sin[\pi(2m-1)\frac{g}{L}]\ \cos[2\pi(2m-1)\frac{z}{L}]$$

Field expansion in RF gaps

$$E_g(r,z) = \sum_{m=1}^{\infty} A_m I_o(k_m r)\cos(\frac{2\pi m z}{L})$$

Coefficients in field expansion:

$$A_m = \frac{4E_a}{\pi}\frac{(-1)^{m-1}}{(2m-1)}\frac{1}{I_o(k_m a)}\sin[\pi(2m-1)\frac{g}{L}]$$

# Energy Gain of Synchronous Particle in RF Gap and Transit Time Factor of π - Structure

Increment of energy of synchronous particle per RF gap

$$\Delta W_s = q \cos\varphi_s \int_{-L/4}^{L/4} E_g(z)\ \cos(k_s z) dz$$

After integration, increment of energy is

$$\Delta W_s = q(E_a g)\ \cos\varphi_s\ [\frac{1}{I_o(\frac{2\pi a}{\beta\gamma\lambda})} \frac{\sin(\frac{\pi g}{\beta\lambda})}{\frac{\pi g}{\beta\lambda}}]$$

Increment of energy can be written as

$$\boxed{\Delta W_s = qUT\ \cos\varphi_s}$$

Effective voltage applied to RF gap:

$$\boxed{U = E_a g}$$

Transit time factor

$$\boxed{T = \frac{1}{I_o(\frac{2\pi a}{\beta\gamma\lambda})} \frac{\sin(\frac{\pi g}{\beta\lambda})}{\frac{\pi g}{\beta\lambda}}}$$

Average field within the gap of π - type structure

$$\boxed{E_o = \frac{2U}{\beta\lambda}}$$

# Design of Accelerator Structure

Specify dependence of transit time factor on velocity: $T = T(\beta)$.

From equation for energy gain one can express $dz_s$

$$\frac{dW_s}{dz_s} = qE_o T\cos\varphi_s \quad \rightarrow \quad dz_s = \frac{dW_s}{qE_o T\cos\varphi_s}$$

Second equation:

$$dt_s = \frac{dz_s}{\beta_s c}$$

Using equation $dW_s = mc^2 \beta\gamma^3 d\beta$ we can rewrite them as

$$dz_s = (\frac{mc^2}{qE_o\cos\varphi_s})\ \frac{\beta d\beta}{T(\beta)(1-\beta^2)^{3/2}}$$

$$dt_s = (\frac{mc}{qE_o\cos\varphi_s})\ \frac{d\beta}{T(\beta)(1-\beta^2)^{3/2}}$$

# Design of Accelerator Structure (cont.)

Integration gives:

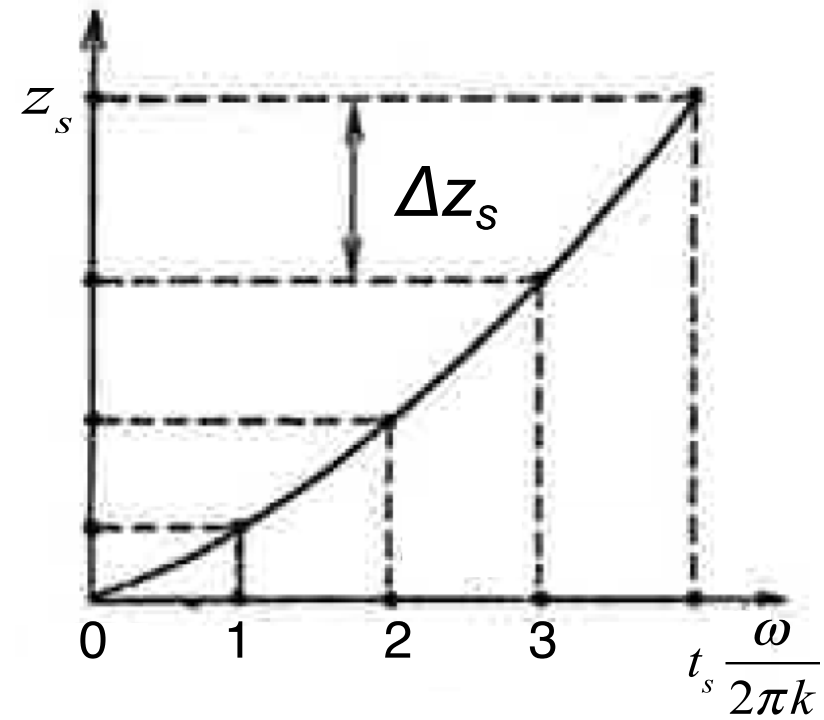
$$z_s = \left(\frac{mc^2}{qE_o \cos\varphi_s}\right)\int_{\beta_o}^{\beta} \frac{\beta d\beta}{(1-\beta^2)^{3/2} T(\beta)}$$

$$t_s = \left(\frac{mc}{qE_o \cos\varphi_s}\right)\int_{\beta_o}^{\beta} \frac{d\beta}{(1-\beta^2)^{3/2} T(\beta)}$$

Using β as independent variable, one can get parametric dependence $z_s(t_s)$. Increment in time $\Delta t_s = k(2\pi/\omega)$ corresponds to distance between centers of adjacent gaps $\Delta z_s$. Gap and drift tube length are determined by adjustment of the value of transit time factor $T=T(\beta, \lambda, a, g)$.

For Alvarez structure k = 1

For π – structure k = 1/2




Calculation the lengths of accelerating periods.

# Simplified Method of Design of Accelerator Structure

Increment of energy of synchronous particle per RF gap

$$\Delta W_s = qE_o TL\cos\varphi_s$$

Increment of energy through increment of relativistic factor

$$dW = mc^2 d\gamma$$

$$d\gamma = \beta\gamma^3 d\beta$$

Increment of velocity of synchronous particle per RF gap:

$$\boxed{\beta_n \approx \beta_{n-1} + k\frac{qE_o T(\beta_s)\lambda}{mc^2\gamma_s^3}\cos\varphi_s}$$

Average velocity at RF gap:

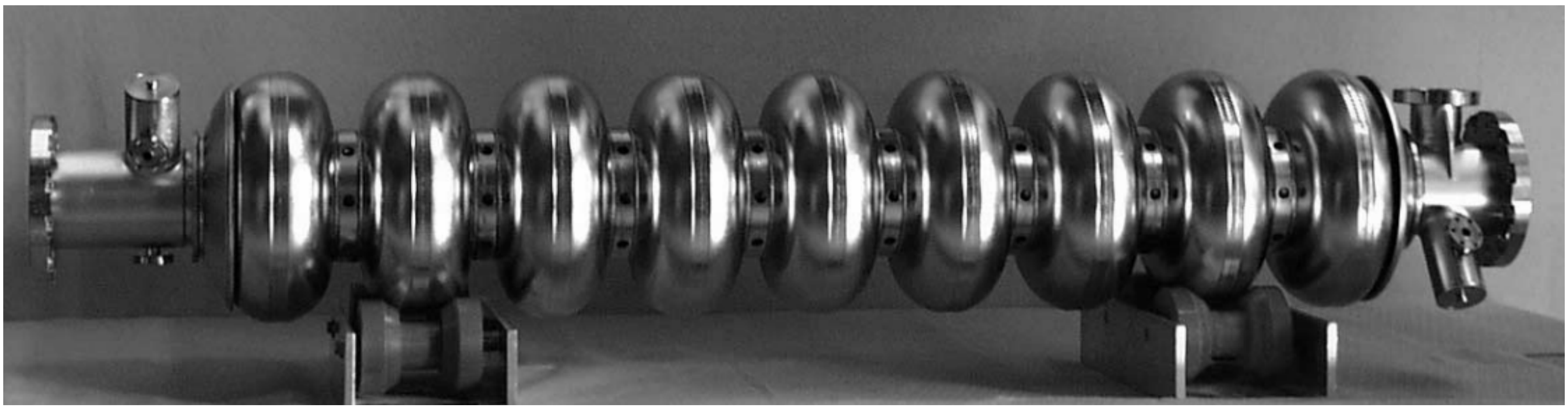
$$\boxed{\beta_s = \frac{\beta_n + \beta_{n-1}}{2}}$$

Cell length: $\Delta z_s = k\,\beta_s\lambda$ ($k$ = 1 for 0 mode; $k$ = 1/2 for $\pi$ - mode)

Drift tube length $l = \Delta z_s - g$

# π – Structures with Constant Cell Length

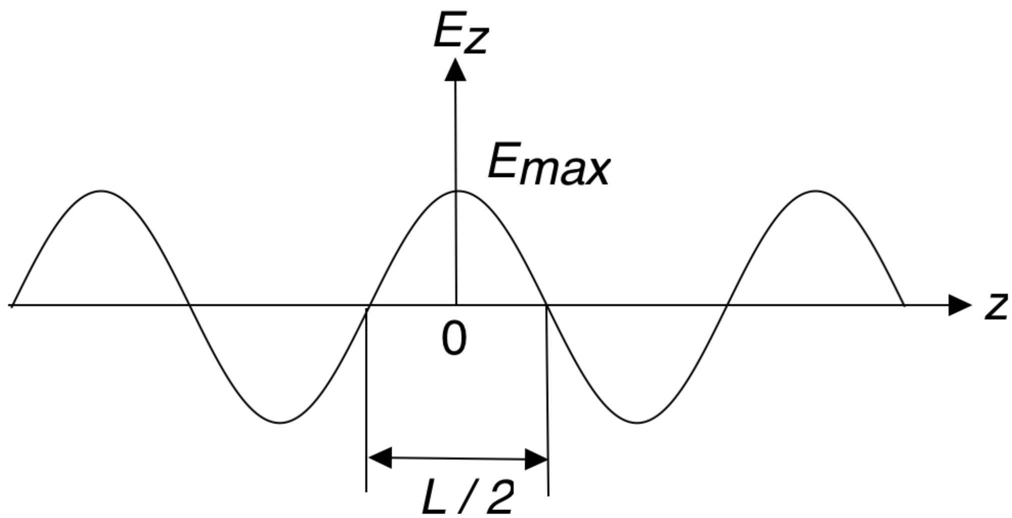
Many $\pi$ - type accelerating structures are based on combination of identical cells of length $\beta_g \lambda / 2$, where $\beta_g$ is a constant value of geometrical particle velocity. Structure containing $N$ cells has total length of $L_s = N \beta_g \lambda / 2$ .



Superconducting 1.3 GHz 9-cell cavity (B. Aune et al, PRSTAB, Vol. 3, 092001 (2000).

# Transit Time Factor in Large – Bore Radius π - Structure

Axial field distribution in π – structure with equal cells




Field distribution at the axis

$$E_g(z) = E_{\max} \cos 2\pi \frac{z}{L}$$

Effective voltage applied to the RF gap

$$U = \int_{-L/4}^{L/4} E_g(z)dz = \frac{E_{\max} L}{\pi}$$

Increment of energy per RF gap:

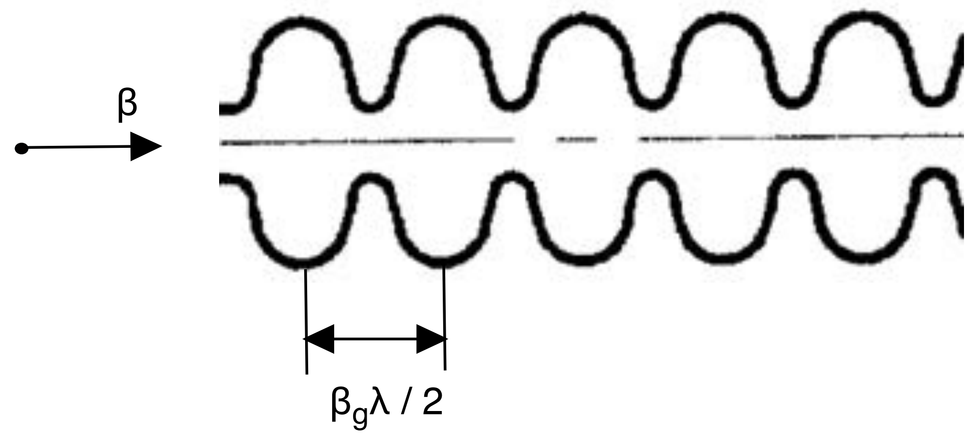
$$\Delta W_s \approx q\cos\varphi_s \int_{-L/4}^{L/4} E_g(z)\ \cos(k_s z)dz = \frac{1}{4} qE_{\max} L\cos\varphi_s = qUT\cos\varphi_s$$

Transit time factor

$$\boxed{T = \frac{\pi}{4}}$$

# Transit Time Factor of $\pi$ – Structure with Identical Cells

Acceleration of particles with velocity different from geometrical one, $\beta \neq \beta_g$, can be treated as that in a structure with modified value of transit time factor.




Accelerating structure with constant cell length.

Let us multiply and divide Transit Time Factor by $\int_{-L_s/2}^{L_s/2} E_g(z)\cos(\frac{2\pi z}{\beta_g \lambda})dz$

$$T_\pi = \frac{\int_{-L_s/2}^{L_s/2} E_g(z)\cos(\frac{2\pi z}{\beta\lambda})dz}{\int_{-L_s/2}^{L_s/2} E_g(z)dz} = \left[\frac{\int_{-L_s/2}^{L_s/2} E_g(z)\cos(\frac{2\pi z}{\beta_g\lambda})dz}{\int_{-L_s/2}^{L_s/2} E_g(z)dz}\right] \left[\frac{\int_{-L_s/2}^{L_s/2} E_g(z)\cos(\frac{2\pi z}{\beta\lambda})dz}{\int_{-L_s/2}^{L_s/2} E_g(z)\ \cos(\frac{2\pi z}{\beta_g\lambda})dz}\right]$$

# Transit Time Factor of $\pi$ – Structure with Identical Cells

Transit Time Factor in $\pi$ – structure with identical cells can be represented as a product of two terms:

$$T_\pi = T \cdot T_s(N, \beta / \beta_g)$$

Transit time factor for structure with $\beta = \beta_g$

$$T = \frac{\int_{-L_s/2}^{L_s/2} E_g(z) \cos\left(\frac{2\pi z}{\beta_g \lambda}\right) dz}{\int_{-L_s/2}^{L_s/2} E_g(z) dz}$$

Normalized factor, which represents reduction of transit time factor because of difference in design and actual particle velocities $\beta \neq \beta_g$

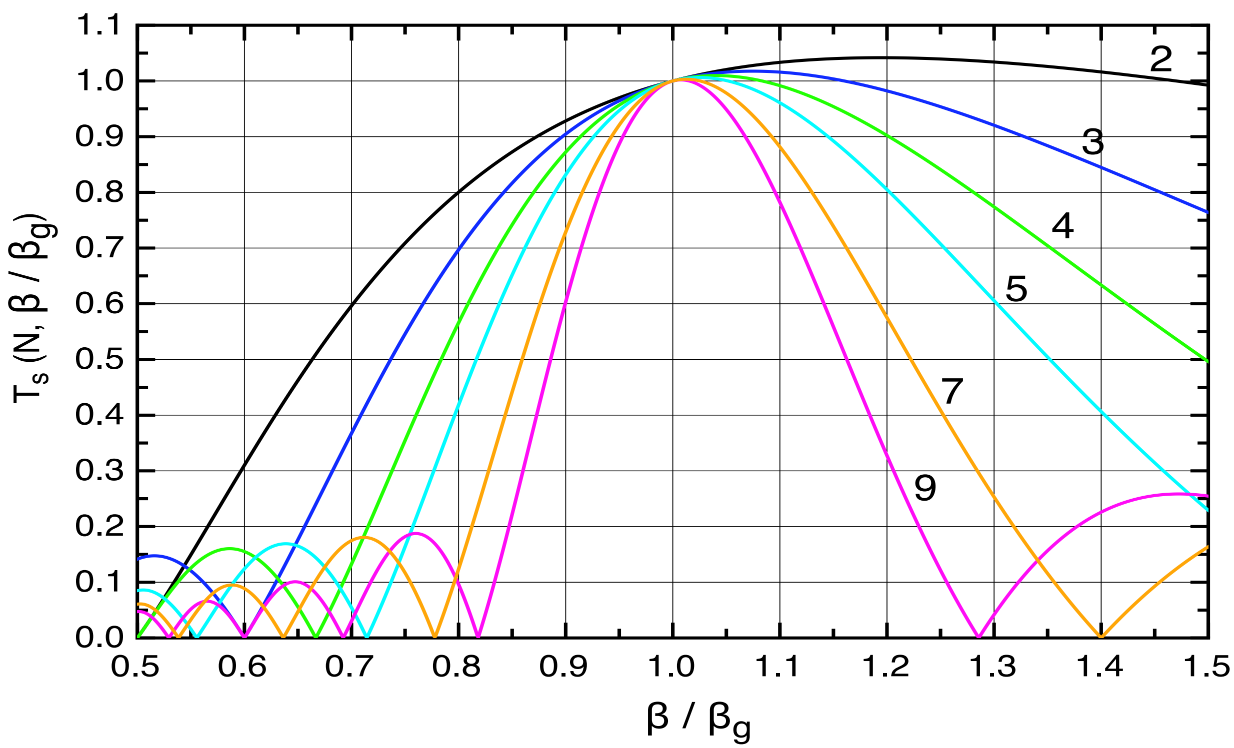
$$T_s(N, \beta / \beta_g) = \frac{\int_{-L_s/2}^{L_s/2} E_g(z) \cos\left(\frac{2\pi z}{\beta \lambda}\right) dz}{\int_{-L_s/2}^{L_s/2} E_g(z) \cos\left(\frac{2\pi z}{\beta_g \lambda}\right) dz}$$

# Normalized Transit Time Factor in $\pi$ – Structure with Identical Cells

Assuming particle velocity $\beta$ is constant along structure, the calculation of normalized factor in a structure with arbitrary number of cells gives [J.-F.Ostiguy, "Transit Time Factor of a Multi-Cell Standing Wave Cavity", Fermilab Report, 2017]:

$$T_s(N,\beta/\beta_g) = \frac{\sin\frac{\pi N}{2}(\frac{\beta_g}{\beta}-1)}{\frac{\pi N}{2}(\frac{\beta_g}{\beta}-1)} - (-1)^N \frac{\sin\frac{\pi N}{2}(\frac{\beta_g}{\beta}+1)}{\frac{\pi N}{2}(\frac{\beta_g}{\beta}+1)}$$

Normalized transit time factor $T_s$ for π – structure with constant geometrical phase velocity $\beta_g$ for different values of of cells $N$.



# Transit Time Factor in $\pi$ – Structure with Identical Cells (cont.)

Introducing small variable $x = \beta / \beta_g - 1$, and taking into account that $x \ll 1$, the normalized transit time factor $T_s$ can be approximated as

$$T_s(x) = 1 - \frac{x}{2} + \frac{x^2}{4}\left(1 - \frac{\pi^2 N^2}{6}\right) + \frac{x^3}{8}\left(\frac{\pi^2 N^2}{6} - 1\right)$$

The optimal value $\beta_{opt}$ where normalized transit time factor reaches maximum, is given by

$$\boxed{\frac{\beta_{opt}}{\beta_g} \approx 1 + \frac{6}{\pi^2 N^2}}$$