VUV and X-ray Free-Electron Lasers

Introduction, Electron Motions in an Undulator, Undulator Radiation & FEL

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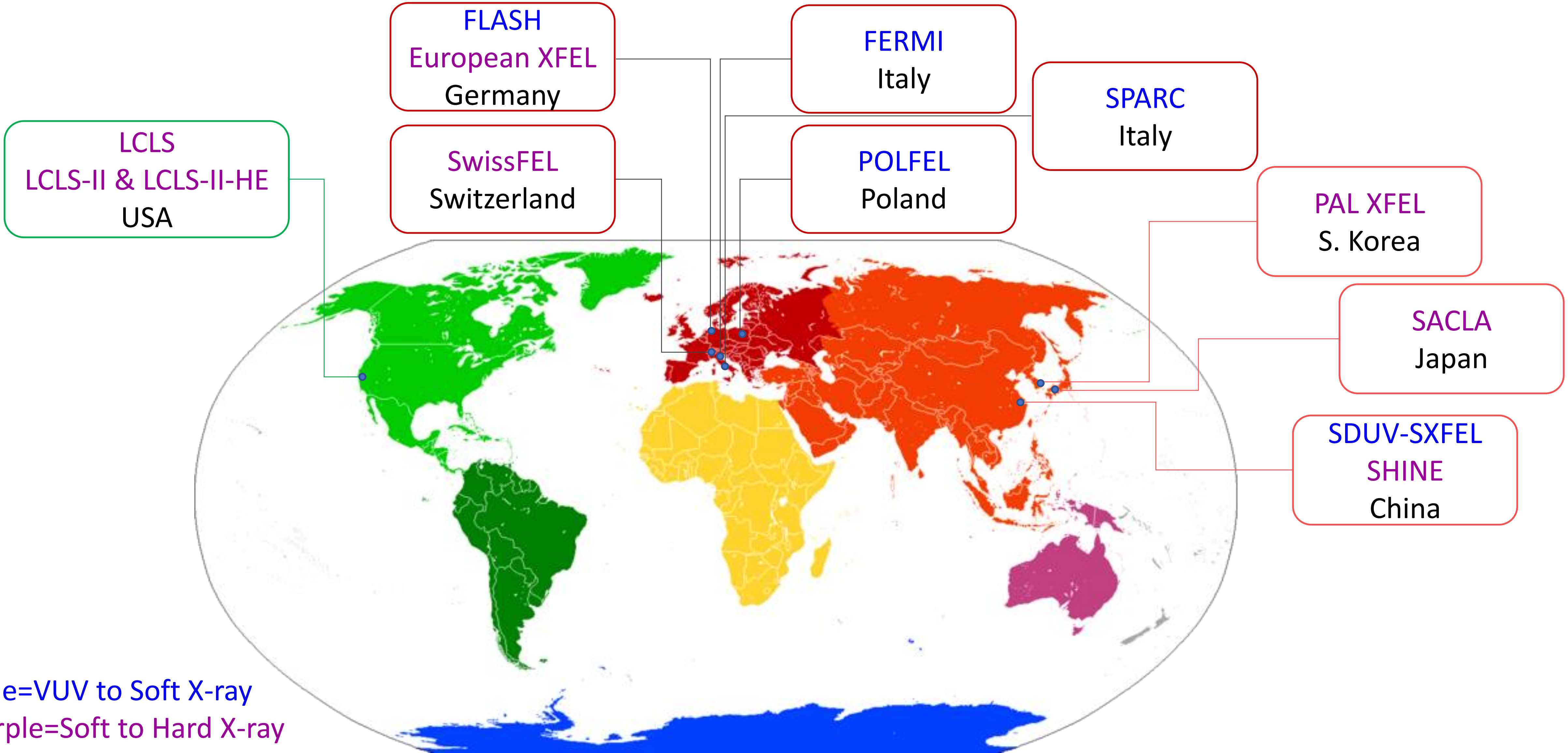
Monday (Jan 25) Lecture Outline

	Time
• VUV and X-ray FELs in the World	10:00 – 10:30
• Properties of electromagnetic radiation	10:30 – 10:50
• Break	10:50 – 11:00
• Electron motions in an undulator	11:00 – 11:20
• Undulator radiation	11:20 – 11:40
• Introduction to FELs	11:40 – Noon

VUV and X-ray FELs in the World

World Map of VUV and X-ray FELs

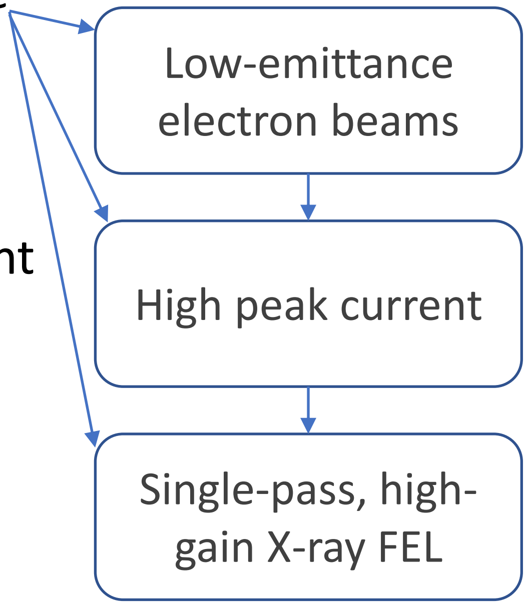


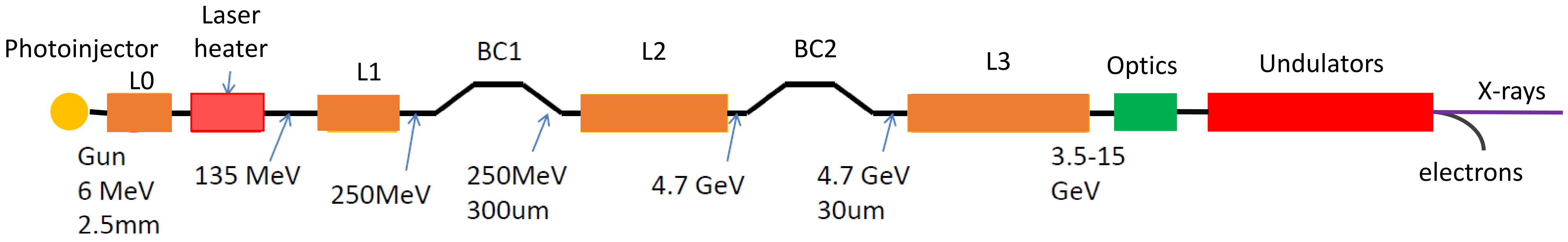
Blue=VUV to Soft X-ray
Purple=Soft to Hard X-ray

Sub-systems of an RF Linac Driven X-ray FEL

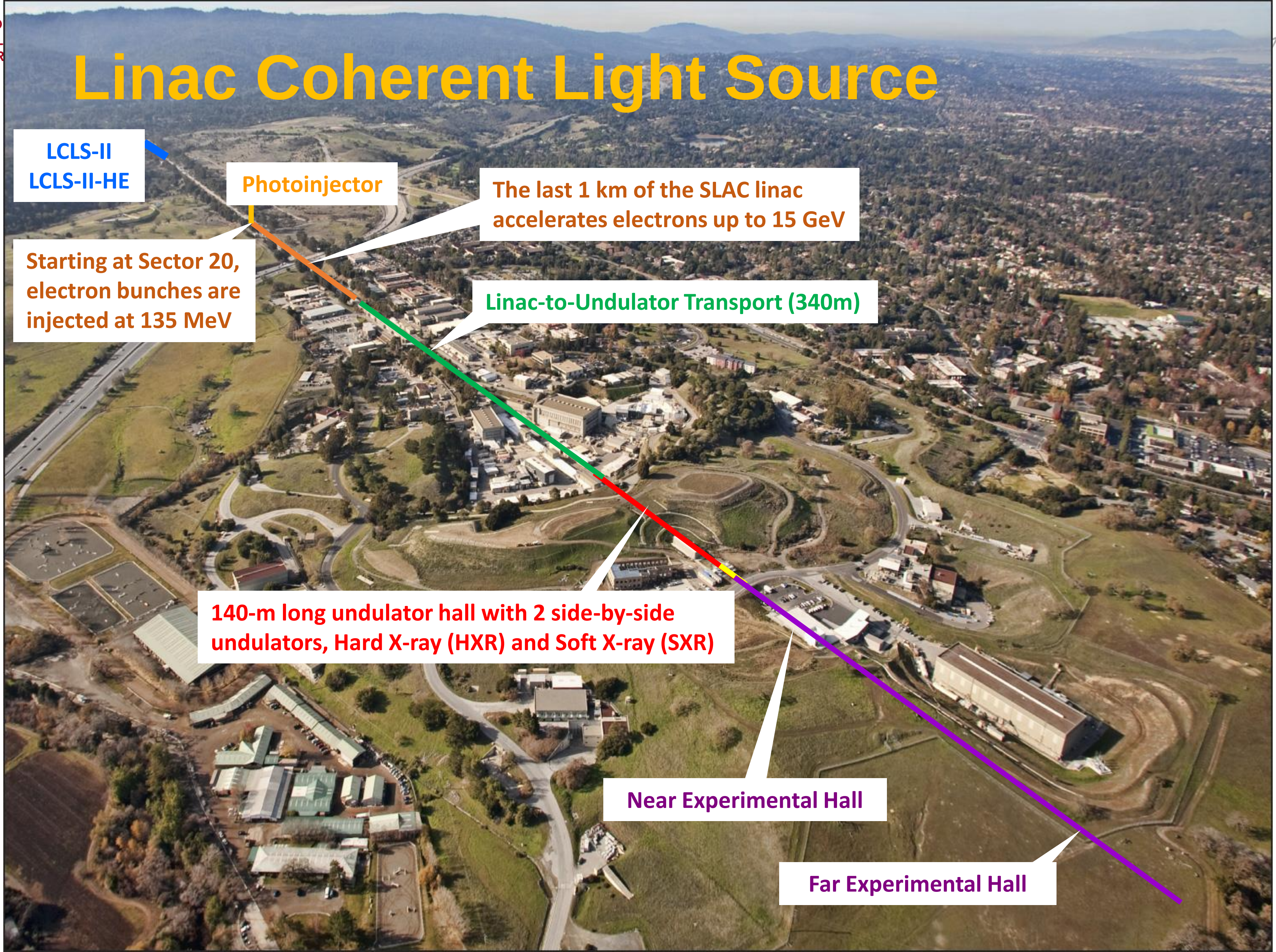
An RF-linac driven XFEL has the following sub-systems in order to produce

- **PHOTOINJECTOR** to generate low-emittance electrons in ps bunches
- **RF LINAC** to accelerate the electron beams to GeV energy
- **BUNCH COMPRESSORS** to shorten the bunches and produce kA current
- **LASER HEATER** to reduce the microbunching instabilities
- **BEAM OPTICS** to transport the electron beams to the undulators
- **UNDULATORS** to generate and amplify the radiation in a single pass
- **DIAGNOSTICS** to characterize the electron & FEL beams



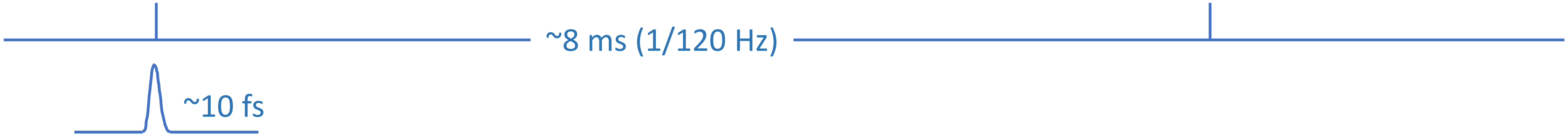


Layout of the sub-systems of the LCLS first X-ray FEL

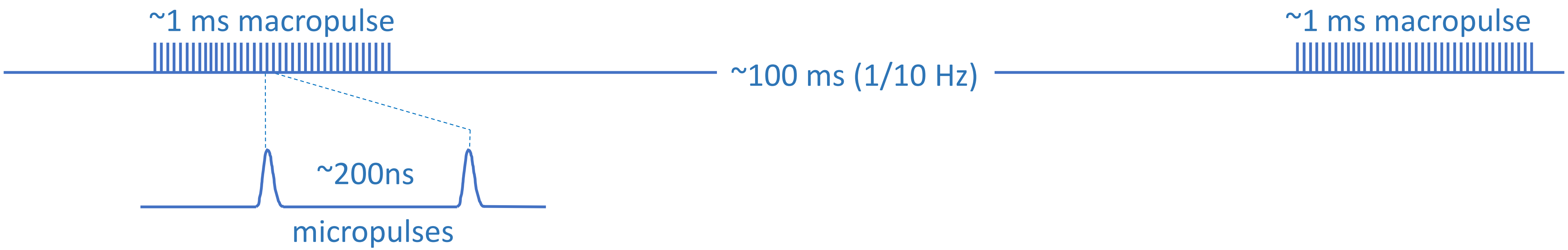

Linac Coherent Light Source
LCLS-II
LCLS-II-HE
Photoinjector
The last 1 km of the SLAC linac accelerates electrons up to 15 GeV
Starting at Sector 20, electron bunches are injected at 135 MeV
Linac-to-Undulator Transport (340m)
140-m long undulator hall with 2 side-by-side undulators, Hard X-ray (HXR) and Soft X-ray (SXR)
Near Experimental Hall
Far Experimental Hall

RF-linac Driven FEL Pulse Format

Low-repetition-rate Mode (e.g., LCLS CuRF)

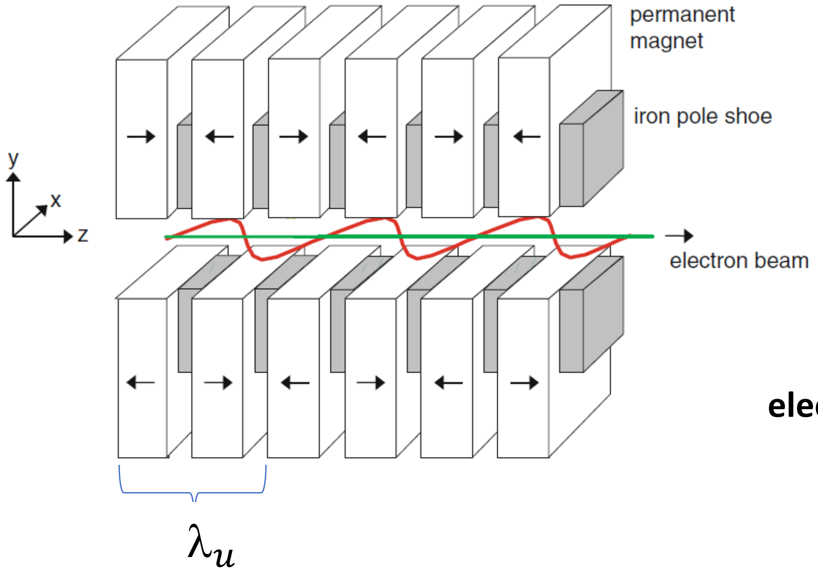


Burst Mode (e.g., Eu-XFEL)



Continuous-Wave Mode (e.g., LCLS-II/HE, SHINE)

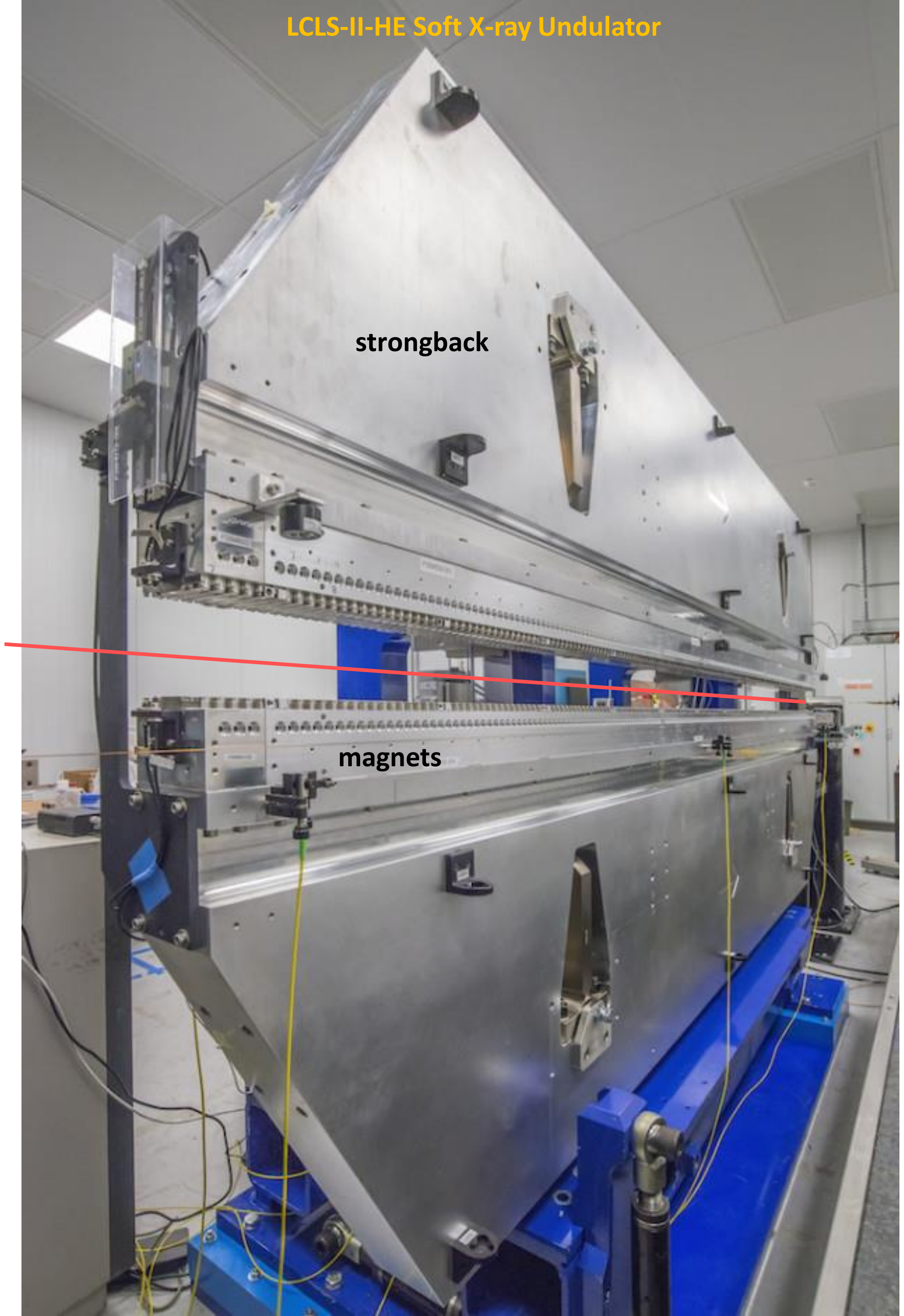
Planar Undulators



Undulator magnetic field varies sinusoidally with z and points in the y direction

$$\mathbf{B} = B_0 sin(k_u z)\,\hat{\mathbf{y}}$$

Planar undulators produce linearly (plane) polarized radiation at the fundamental frequency and also at harmonic frequencies.



magnetic field

X-ray polarization (same direction as electron oscillations)

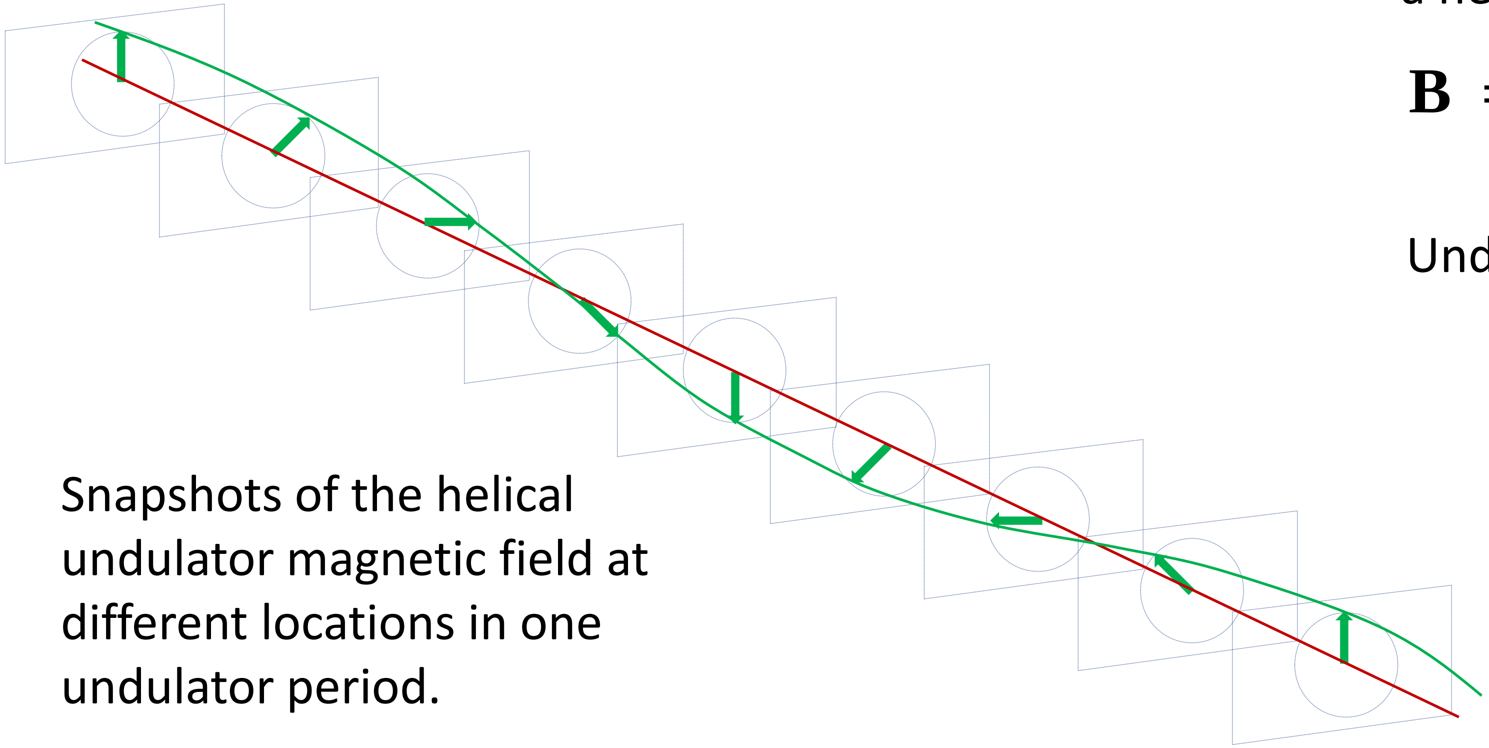
Helical Superconducting Undulators

Superconducting coils are wound around the electron beam pipe in a helical pattern.

Undulator magnetic field varies sinusoidally with z and points in both x and y directions in a helical fashion.

$$\mathbf{B} = B_0(cos(k_u z)\,\hat{\boldsymbol{x}} + sin(k_u z)\,\hat{\boldsymbol{y}})$$

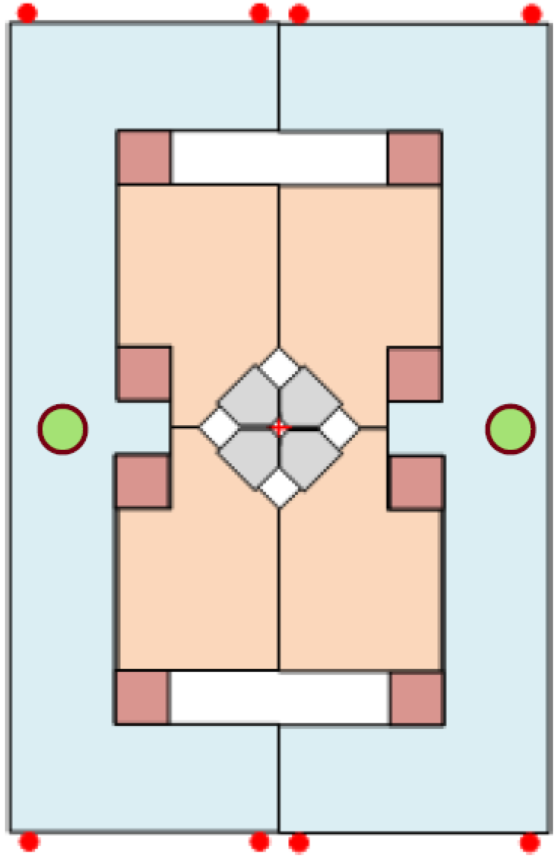
Undulator wavenumber $$k_u = \frac{2\pi}{\lambda_u}$$



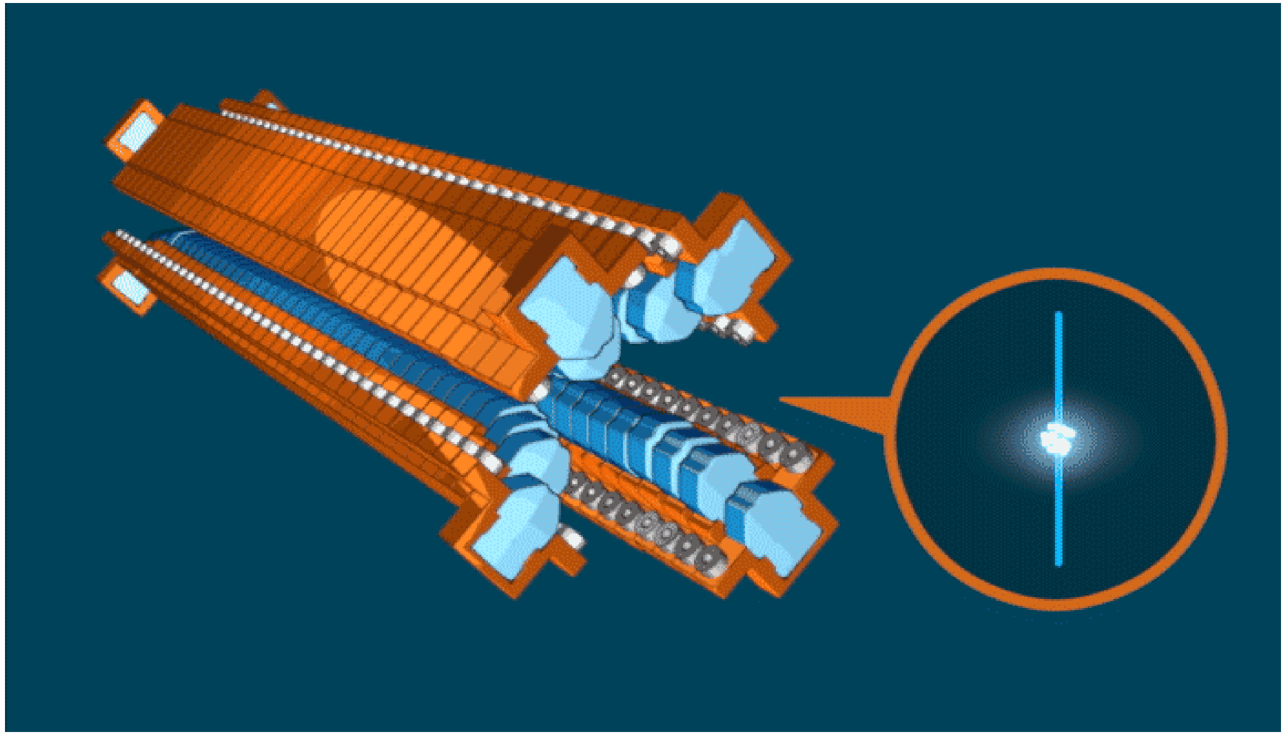
Snapshots of the helical undulator magnetic field at different locations in one undulator period.

Helical undulators produce circularly polarized radiation at the fundamental frequency. Helical undulators do not produce undulator radiation at the harmonic frequencies.

Delta Undulators (Variable Polarization)



Delta undulator cross-section



Varying the linear positions of the magnet jaws changes radiation polarization from linear to circular

X-ray FEL Wavelength for a Planar Undulator

FEL resonant wavelength

Undulator period

Undulator parameter

$$\boxed{\lambda = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2}\right)}$$

Lorentz relativistic factor
(Dimensionless beam energy)

$$\boxed{\gamma = \frac{E_{total}}{m_e c^2}}$$

$$K = \frac{eB_0}{k_u m_e c} = \frac{e\lambda_u B_0}{2\pi m_e c}$$

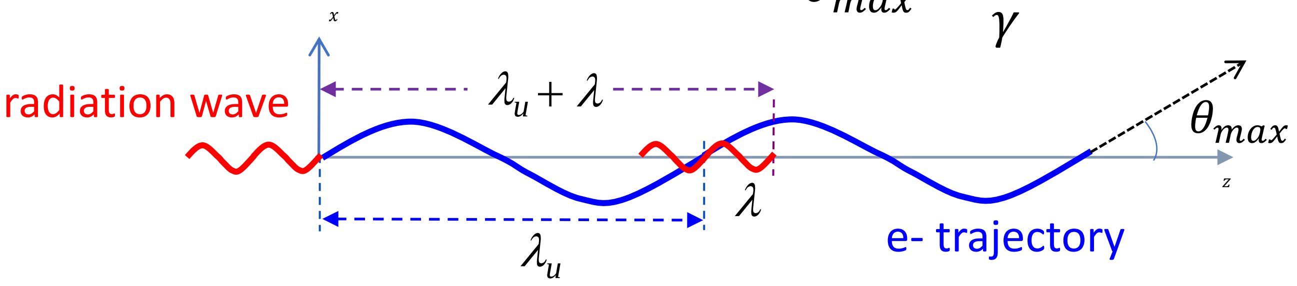
Dimensionless K parameter is a measure of how much the electron beam is deflected transversely in the undulator

$$\boxed{K = 0.934\, B_0 \lambda_u}$$

B_0 in tesla

λ_u in cm

$$\theta_{max} = \frac{K}{\gamma}$$



Electron Beam Kinematics

Dimensionless electron beam energy

$$\gamma = \frac{E_{total}}{m_e c^2}$$

$$E_{total} = E_k + m_e c^2$$

Kinetic energy

Rest mass energy
0.511 MeV

Approximate γ for GeV electrons

$$E_{total} \approx E_k = E_b$$

$$\gamma \approx 1957\, E_b[GeV]$$

Electron velocity relative to the speed of light

$$\beta = \frac{v}{c}$$

Using the γ–β relation, we can calculate β

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}}$$

Approximate β for GeV electrons

$$\beta \approx 1 - \frac{1}{2\gamma^2}$$

We shall see later the longitudinal component of the velocity, v_z is reduced when the electrons undergo transverse oscillations in the undulator.

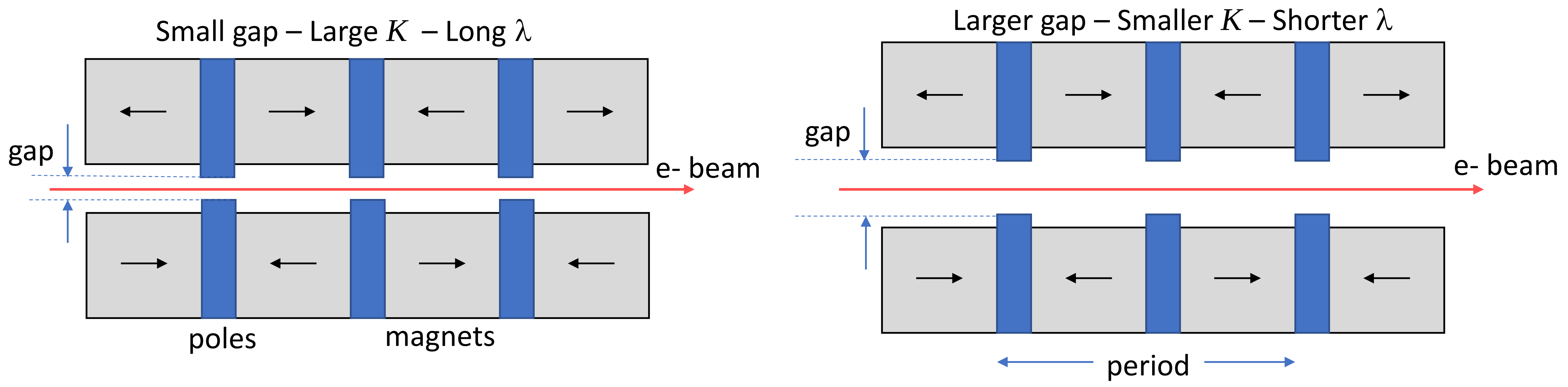
X-ray FEL Wavelength Tuning

The **FEL x-ray wavelength** can be tuned by one of the following methods

1. varying the electron **beam energy**, E_b and thus the beam γ
2. varying the **gap** by moving the magnet jaws symmetrically in and out, thus changing the K value

The on-axis magnetic field amplitude depends on the gap-to-period ratio. The remanence magnetic field, B_r of NdFeB is about 1.2 tesla.

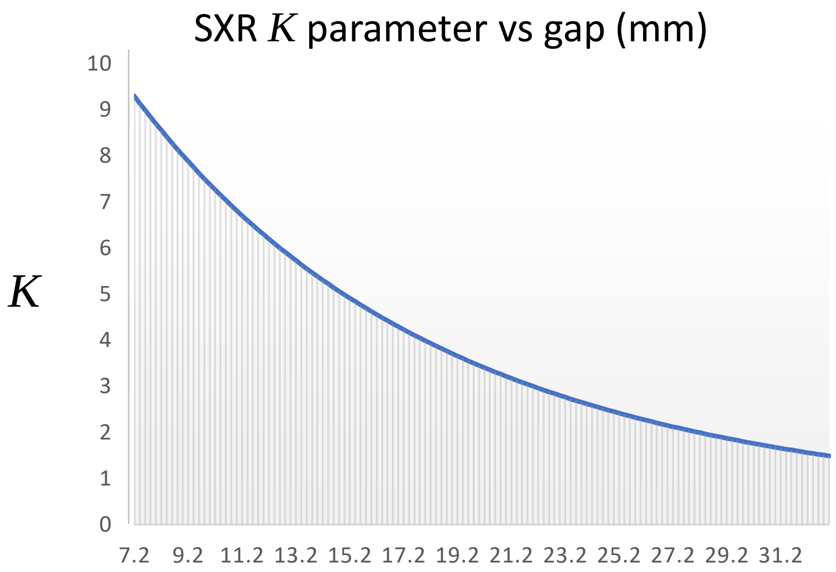
$$B_0(g,\ \lambda_u) = 3.13 B_r\ exp\left[-5.08\left(\frac{g}{\lambda_u}\right) + 1.54\left(\frac{g}{\lambda_u}\right)^2\right]$$

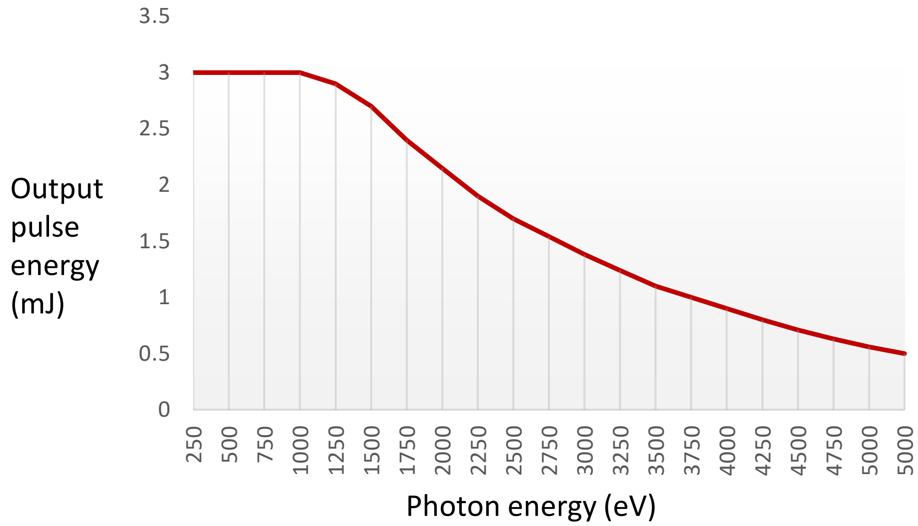


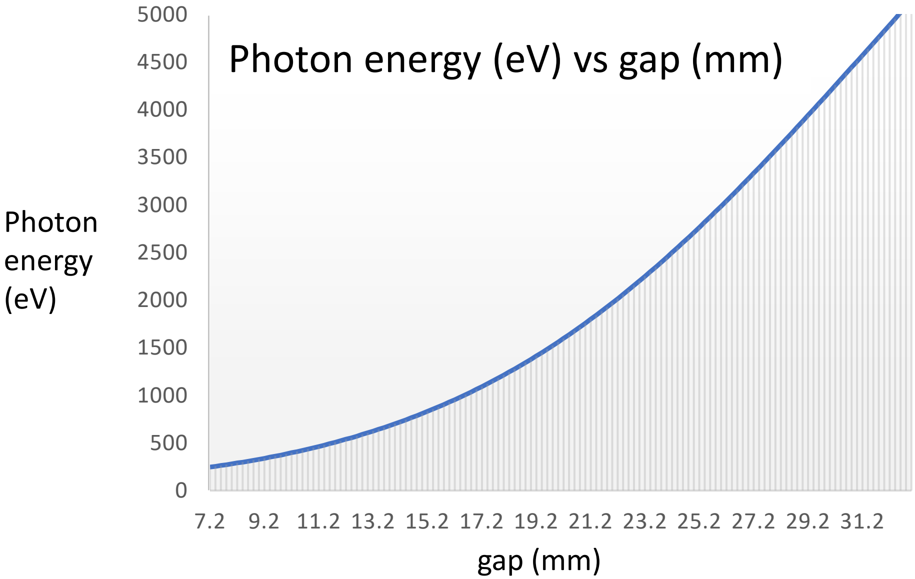
LCLS-II-HE Variable-Gap Soft X-ray Undulator

This example illustrates how we change the LCLS X-ray FEL wavelength by varying the magnet gap of the Soft X-ray (SXR) undulator which has an undulator period of 56 mm.

The undulator K parameter decreases and photon energy increases as we increase the gap (right figures), using the LCLS-II-HE 8-GeV electron beams as an example. However, opening the gap to increase the photon energy also reduces the FEL output pulse energy (below).







Properties of Electromagnetic Radiation

EM Radiation in Free Space

$$h = 4.1357 \cdot 10^{-15}\, eV \cdot s$$

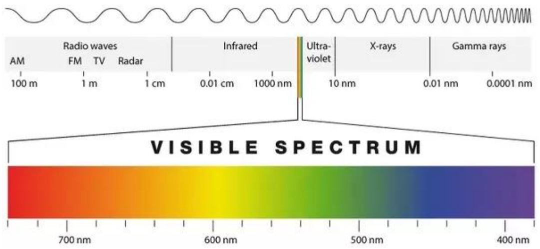
$$c = 2.9979 \cdot 10^{8}\, m/s$$

$$\lambda \cdot h\nu = hc = 1.23984 \cdot 10^{-6} eV\text{–}m$$

Photon energy $\boxed{h\nu = \dfrac{1239.84\ eV}{\lambda[nm]}}$

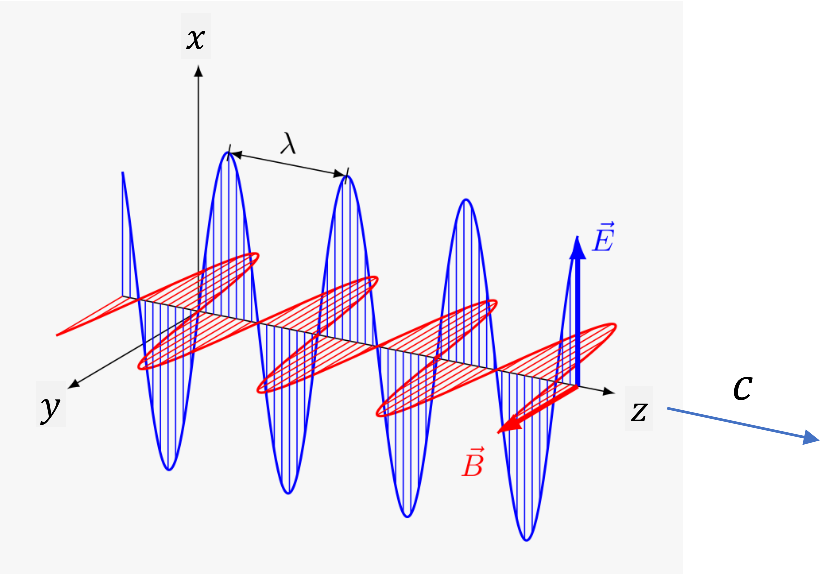
Low frequency

Long wavelength

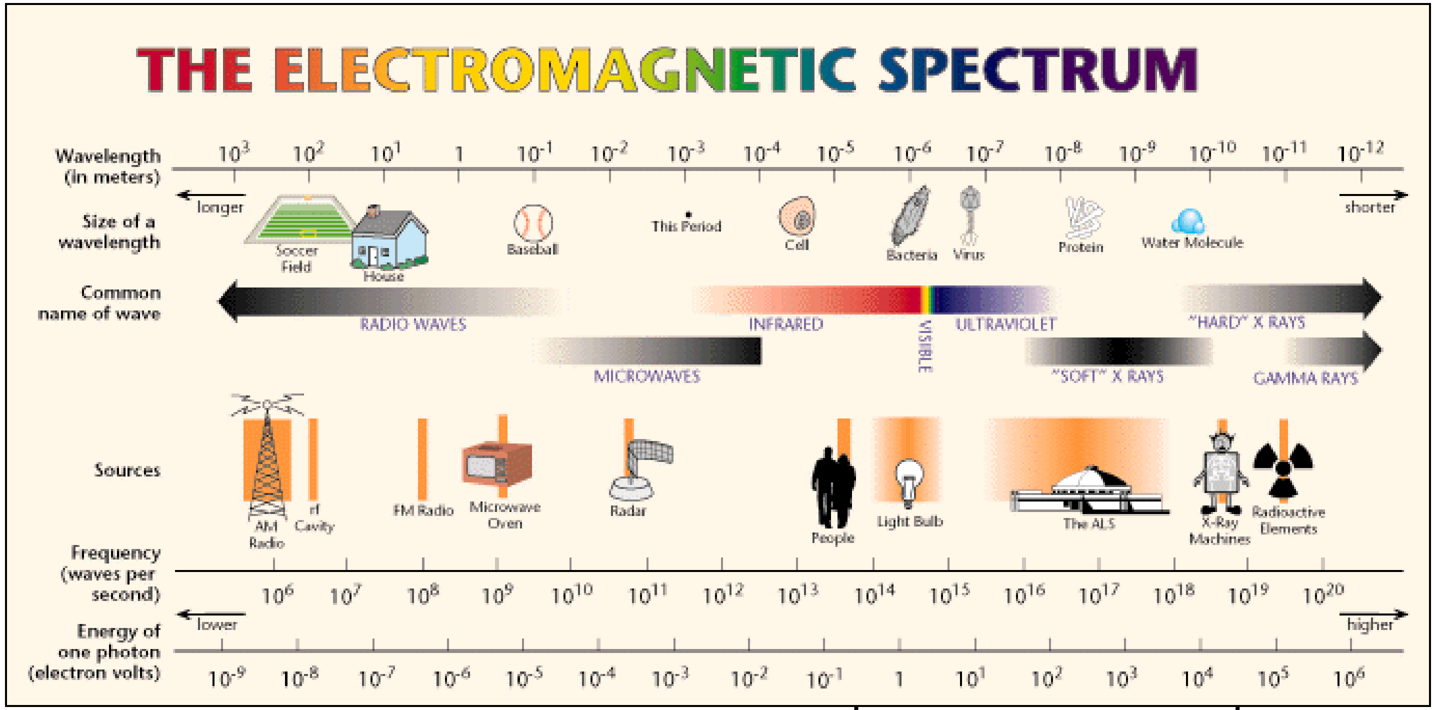


High frequency

Short wavelength

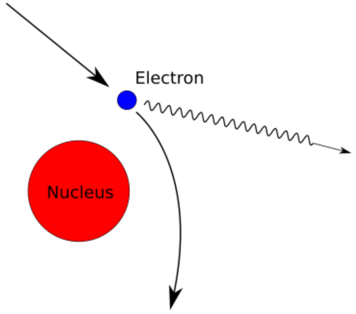


Accelerated charged particles emit EM radiation

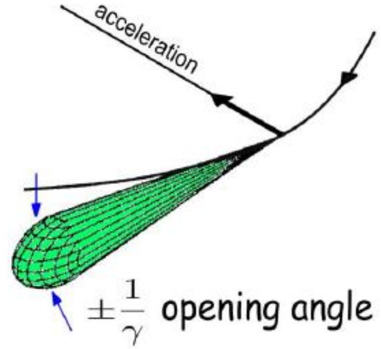


Radiation from circular motions

Bremsstrahlung

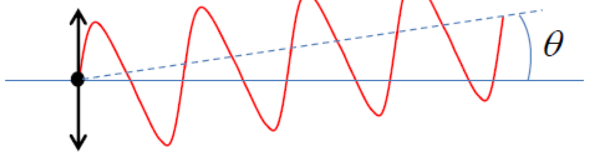


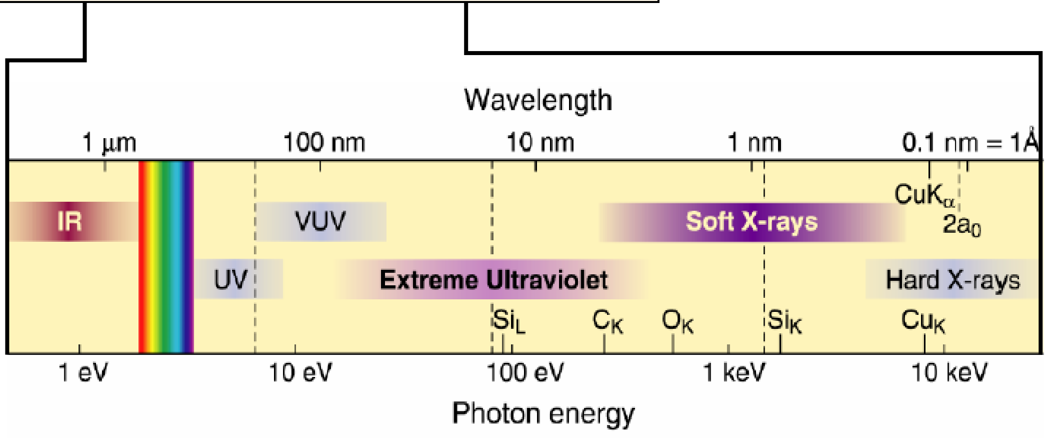
Synchrotron radiation



Radiation from oscillatory motions

$a = \dot{v}_{\perp}$





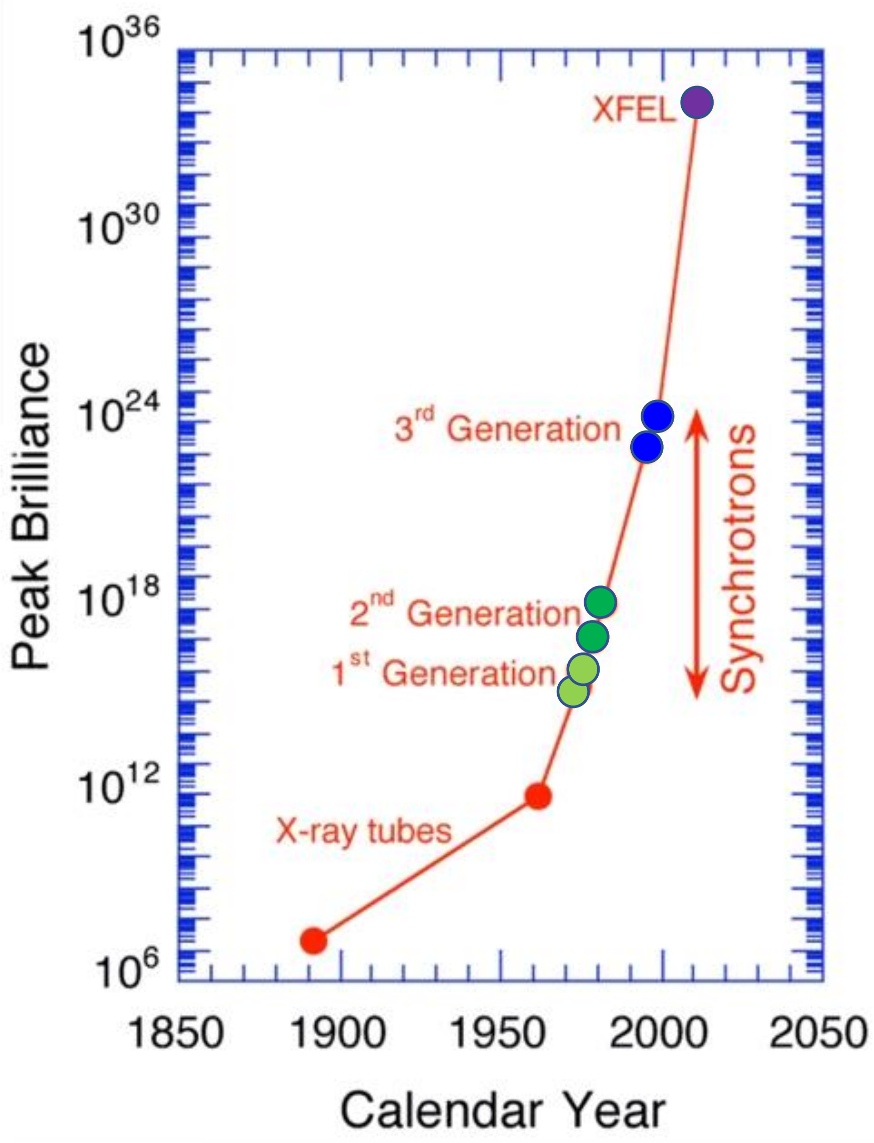
Generations of Beam-based Radiation Sources

- **X-ray Tubes**
  - X-ray tubes emit **Bremsstrahlung** and **characteristic peaks**
  - Characteristic peaks are narrow-line atomic transitions

- **Synchrotron Radiation (1st and 2nd Generation Light Sources)**
  - Electrons going around bends produce **synchrotron radiation**
  - **First generation SR** operated as parasitic radiation devices
  - **Second generation SR** are dedicated to radiation production

- **Synchrotron Radiation (3rd Generation Light Sources)**
  - **3rd Gen SR** have low-emittance lattice with straight sections
  - **Bending magnet** and **wiggler radiation** is broadband
  - **Undulator radiation** has narrow spectral lines

- **X-ray Free-Electron Lasers (4th Generation Light Sources)**
  - **XFEL** produce coherent, tunable, narrow-band x-rays
  - X-ray pulses are typically a few femtoseconds long
  - **FEL peak brilliance** is typically 10 orders of magnitude brighter than **third-generation SR.**



Electric Field of a Gaussian Wave-Packet

Complex electric field of a Gaussian wave-packet

$$E(z,t) = E_0 e^{i(kz-\omega t+\psi)} e^{-\frac{(t-t_0)^2}{2\sigma_t^2}} + \text{c.c.}$$

Amplitude

Phase

Gaussian envelope rms temporal width

Wavenumber

$$k = \frac{2\pi}{\lambda}$$

Angular frequency

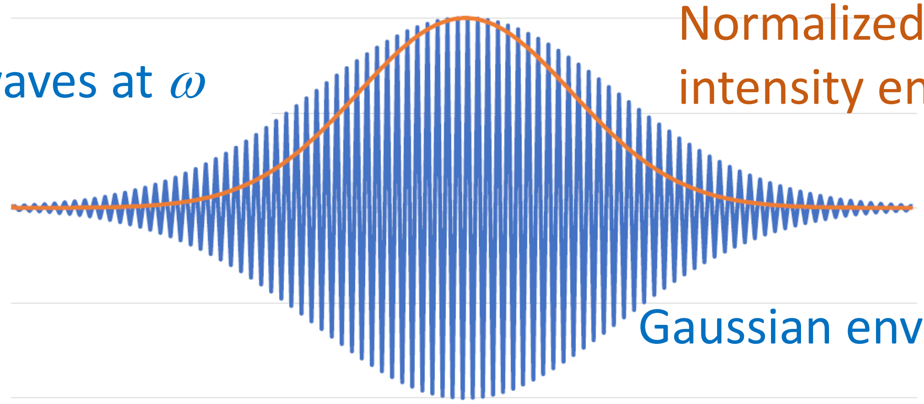
$$\omega = \frac{2\pi c}{\lambda}$$

Radiation intensity

$$I = \frac{1}{2Z_0}|E(z,t)|^2$$

$$I_0 = \frac{1}{2Z_0}|E_0|^2$$

Impedance of free space = $120\pi\ \Omega$



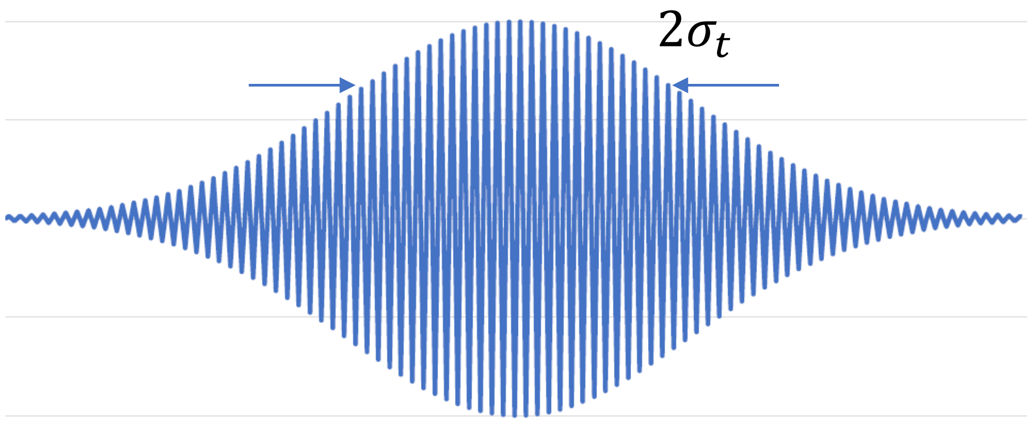
Fourier Transform a Gaussian Pulse

Consider only the time-dependent part of a Gaussian wave-packet. Its Fourier Transform is a Gaussian spectrum centered at ± ω_r

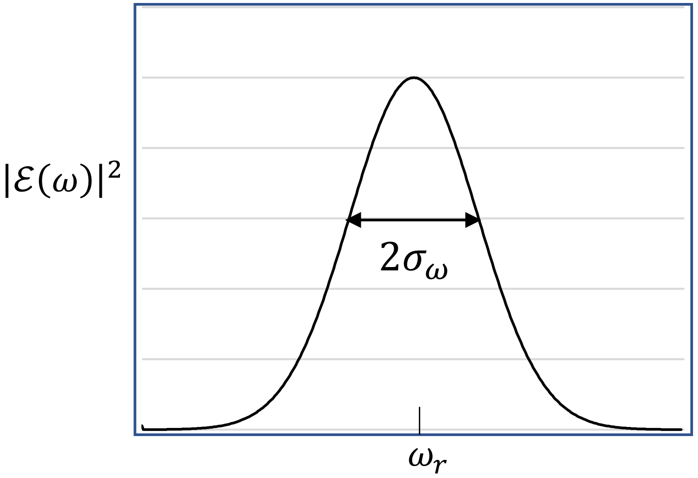
$$E(t) = E_0 e^{-i\omega_r t} e^{-\frac{t^2}{2\sigma_t^2}}$$

$$\mathcal{E}(\omega) = \frac{E_0}{\sqrt{2\pi}} \int e^{-i(\omega-\omega_r)t} e^{-\frac{t^2}{2\sigma_t^2}} dt$$

$$\mathcal{E}(\omega) = \frac{E_0}{2\sigma_\omega} e^{-\frac{(\omega-\omega_r)^2}{2\sigma_\omega^2}}$$



Time domain



Frequency domain

Fourier Transform limit of time-bandwidth product (rms widths)

$$\sigma_\omega \sigma_t = ½$$

In general, TBW is larger than 1/2

Radiation Pulse & Time-Bandwidth Product

Full-width-at-half-maximum (FWHM) in time δt and linear frequency domain $\delta \nu$

- Time-bandwidth product for a Gaussian pulse

$$\delta\nu \cdot \delta t = \frac{4ln2}{\pi}\sigma_\omega\sigma_t = 0.44$$

- Multiply both sides by the Planck's constant in eV-s

$$h = 4.136 \cdot 10^{-15}\ e\text{V-s}$$

$$h\delta\nu \cdot \delta t = 1.82\ eV \cdot fs$$

Energy (eV) – time FWHM product

$$\delta\varepsilon \cdot \delta t \geq 1.82\ eV \cdot fs$$

Linear frequency

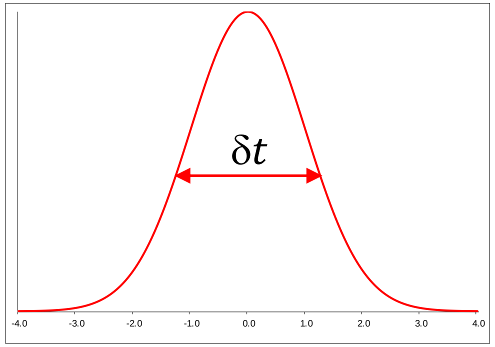
$$\nu = \frac{\omega}{2\pi}$$

$$\delta t = 2\sqrt{2ln2}\,\sigma_t$$

$$\delta\nu = 2\sqrt{2ln2}\,\frac{\sigma_\omega}{2\pi}$$

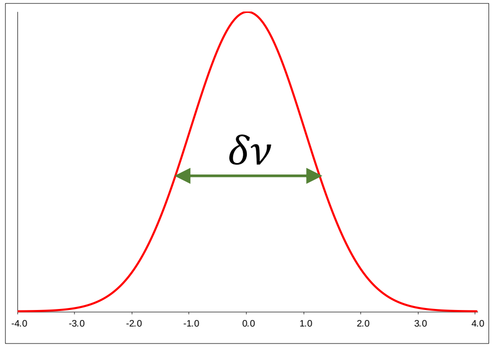
FWHM

$|E(t)|^2$



$t - t_0$

$|\mathcal{E}(\nu)|^2$



$\nu - \nu_r$

Wave Equation & Helmholtz Equation

- Wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) u(\boldsymbol{r}, t) = 0$$

- Solution to the wave equation

$$u(\boldsymbol{r}, t) = \mathrm{Re}\{\underbrace{\psi(\boldsymbol{r})}_{\text{Time-independent wave amplitude}}\ \underbrace{e^{-i\omega_r t}}_{\text{Time-dependent oscillatory term}}\}$$

- Helmholtz equation for the time-independent wave amplitude

$$(\nabla^2 + k^2)\,\psi(\boldsymbol{r}) = 0$$

$$\psi(\boldsymbol{r}) = A(\boldsymbol{r})\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$$

Paraxial Approximation

Paraxial wave equation for a wave propagating in the z direction

$$\left(\nabla_T^2 - 2ik\frac{\partial}{\partial z}\right)A(x,y,z) = 0$$

∇_T denotes transverse spreading due to optical diffraction

$$\nabla_T^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

and $k = \sqrt{k_x^2 + k_y^2 + k_z^2}$

Paraxial approximation $k_x^2 + k_y^2 \ll k_z^2$

For axisymmetric Gaussian beams, $k_x = k_y$

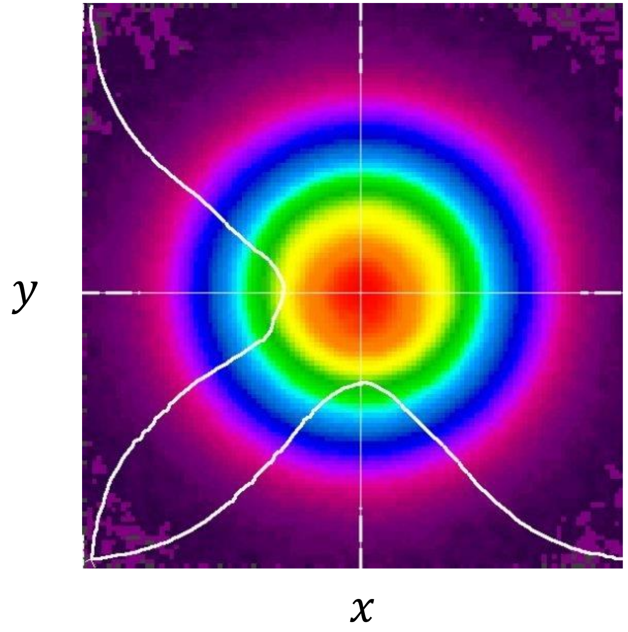
$$E(r,z) = E_0\frac{w_0}{w(z)}e^{-\frac{r^2}{w^2(z)}}e^{i(kz-\omega t+\psi(z))}e^{i\left(k\frac{r^2}{2R(z)}\right)}$$

Gouy phase shift

Radius of curvature of the Gaussian beam wavefront

w_0 is the radius where the field decays to 1/e of E_0 at the beam waist
$w(z)$ is the 1/e radius at location z

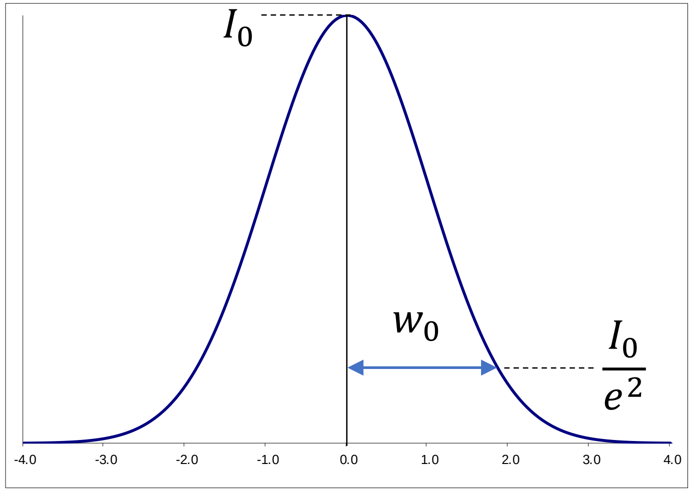
Gaussian beam transverse amplitude (beam propagating in the z direction)



Gaussian Beam Intensity & Diffraction

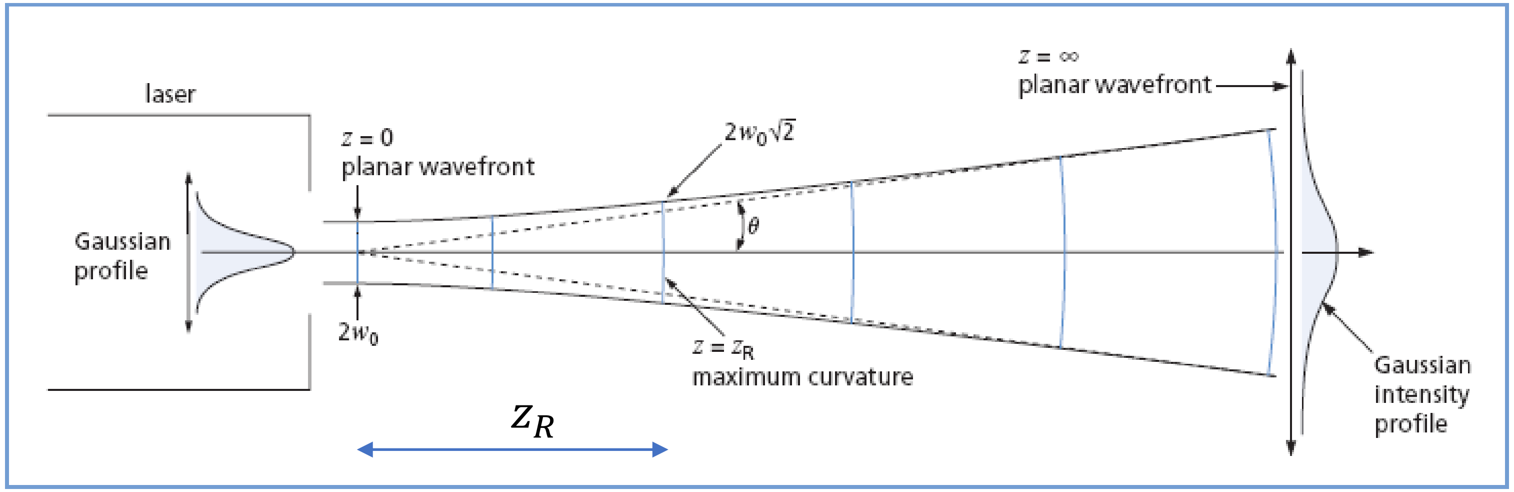
- Optical intensity $$I(r,z) = \frac{1}{2Z_0}|E(r,z)|^2$$

- Gaussian beam $$I(r,z) = I_0\left(\frac{w_0}{w}\right)^2 e^{-\frac{2r^2}{w^2}}$$
$$w = w_0\sqrt{1+\frac{z^2}{z_R^2}}$$



Beam profile at beam waist

- Gaussian beam diffracting from the beam waist



Far-field divergence half-angle

$$\theta = \frac{w_0}{z_R} = \frac{\lambda}{\pi w_0}$$

$$\theta w_0 = \frac{\lambda}{\pi}$$

Radiation Beam FWHM, Radius and Emittance

Gaussian beam radial FWHM

$$\delta r_{FWHM} = \sqrt{2ln2}\, w_0 \qquad (\, w_0 = 1/e^2 \text{ radius})$$

Gaussian beam angular divergence FWHM

$$\delta r'_{FWHM} = \sqrt{2ln2}\, \theta \qquad (\theta = 1/e^2 \text{ half-angle})$$

$I(r)$

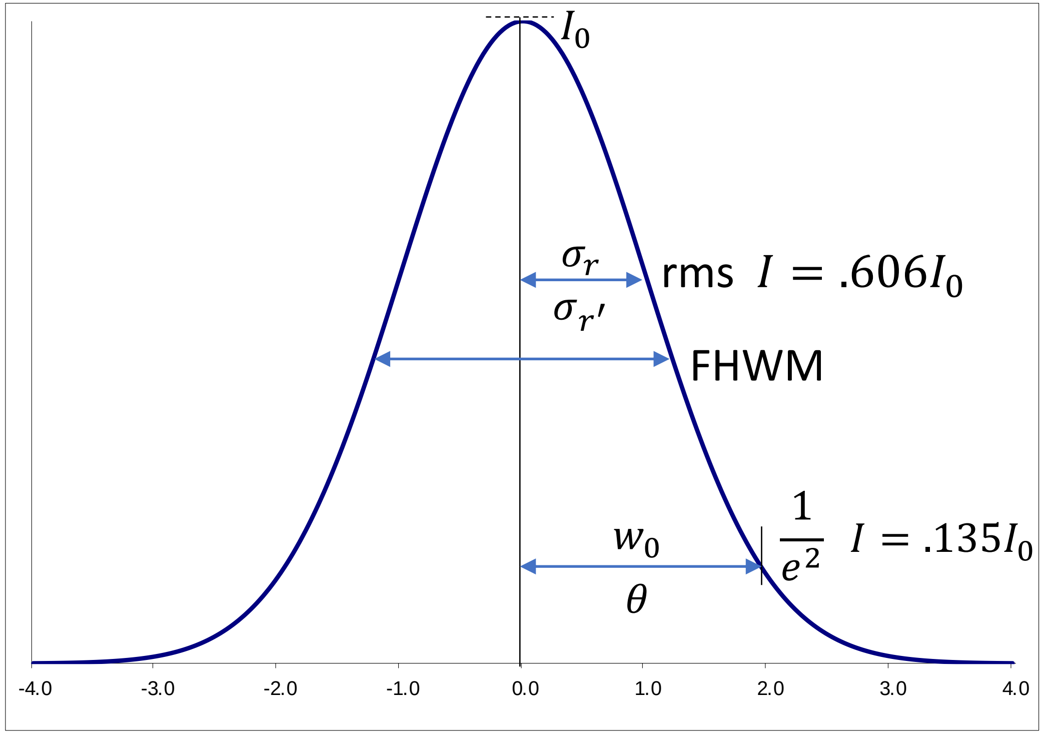
$I(r')$

rms beam radius $\quad \sigma_r = \dfrac{w_0}{2}$

rms angular divergence $\quad \sigma_{r'} = \dfrac{\theta}{2}$

Gaussian beam emittance $\quad \epsilon_r = \sigma_r \sigma_r'$

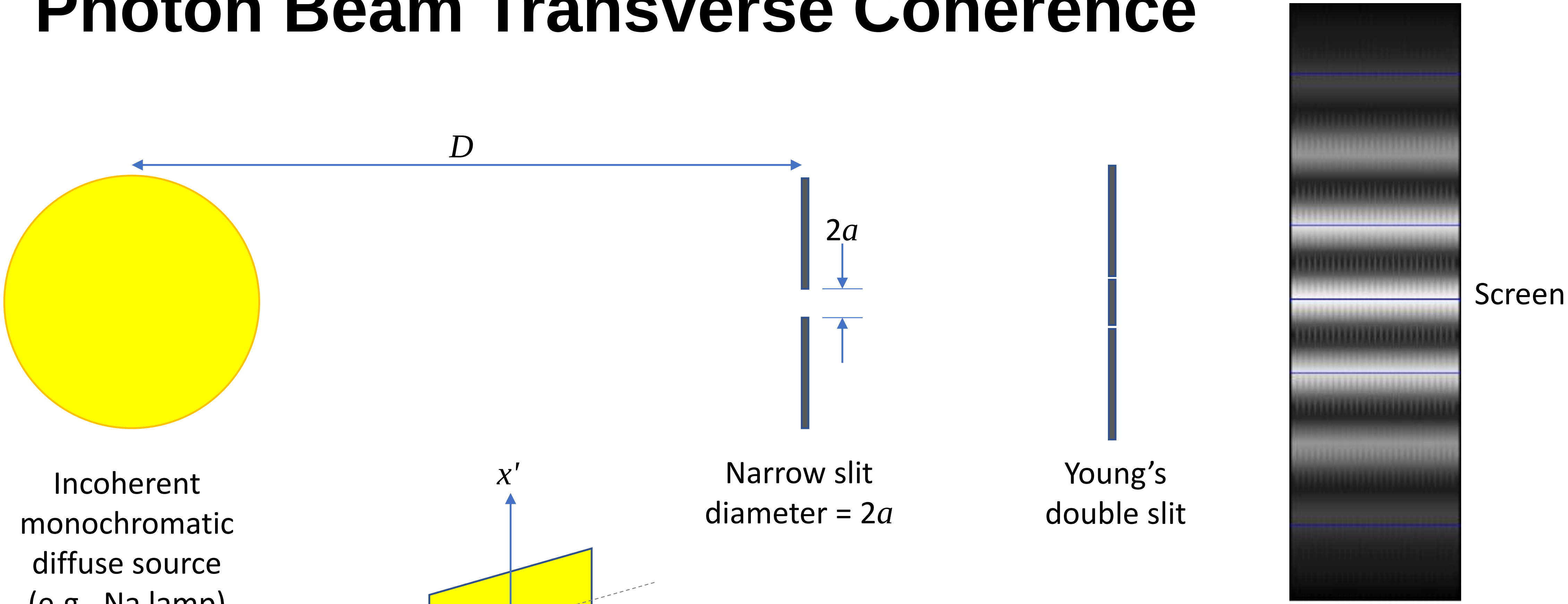
$$\boxed{\epsilon_r = \frac{\lambda}{4\pi}}$$

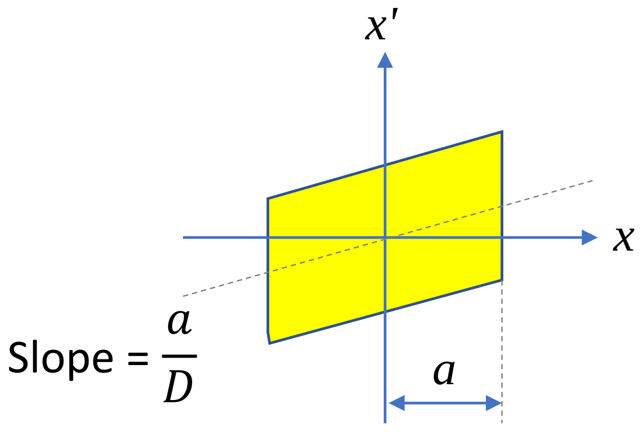


Radial dimension (or angle)

Photon beam emittance for transversely incoherent (not diffraction limited) radiation $\quad \epsilon_r = M^2 \dfrac{\lambda}{4\pi} \qquad M^2 > 1$

Photon Beam Transverse Coherence





Radiation phase-space

Condition for interference pattern to appear on screen

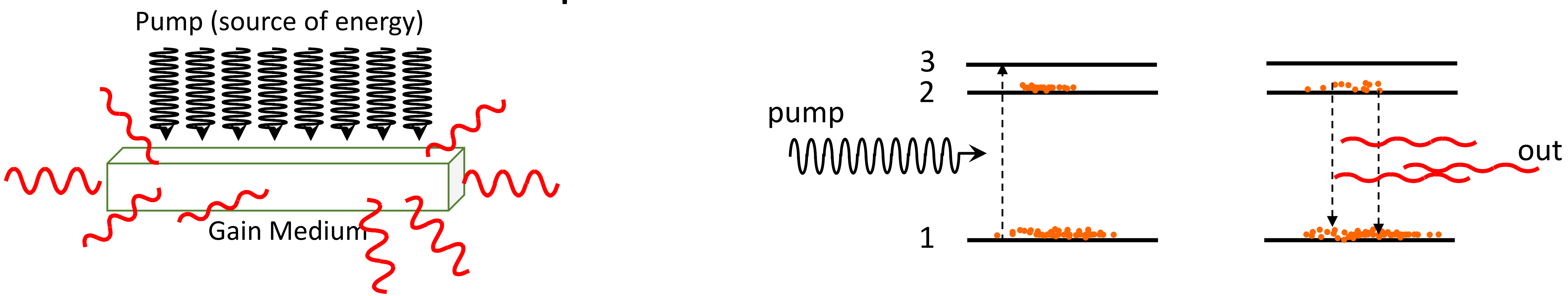
$$A \leq \frac{\lambda}{2}$$

Coherent phase-space area

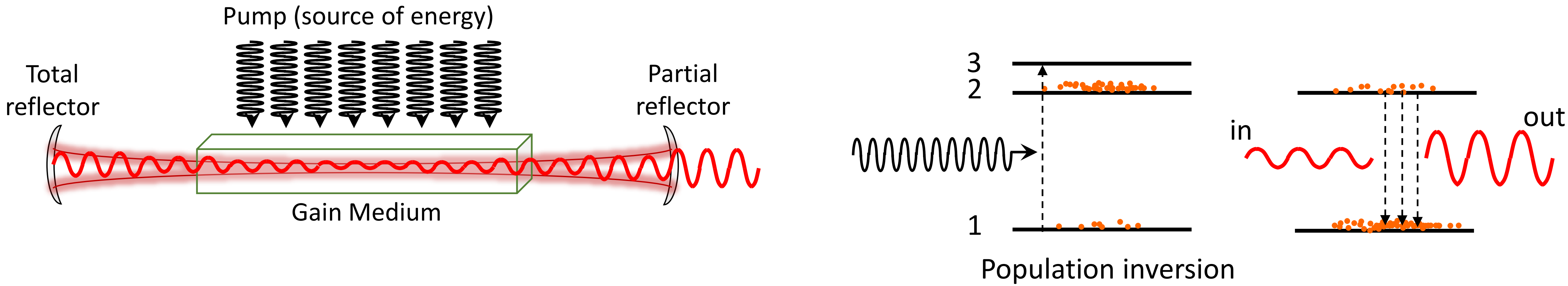
$$\boxed{A = 2\pi\epsilon_r}$$

Spontaneous vs Stimulated Emission

Spontaneous emission



Stimulated emission



SLAC NATIONAL ACCELERATOR LABORATORY

Los Alamos NATIONAL LABORATORY EST. 1943

How does an FEL differ from a laser?

- FEL gain medium is **free electrons in vacuum** (lasers have bound electrons)
- FEL have **broad wavelength tunability** (lasers have no or limited tunability)
- FEL beams are **distortion-free** (laser gain media have optical distortions)
- FEL work at **x-ray wavelengths** (x-ray laser upper state lifetimes are too short)
- The **coherence length** of a SASE FEL is much shorter than that of a typical laser.

Electron Motions in an Undulator

Lorentz Force

Lorentz force governs the rate of change in the electron beam energy and momentum

$$\mathbf{F} = -e(\mathbf{E} + \boldsymbol{\upsilon} \times \mathbf{B})$$

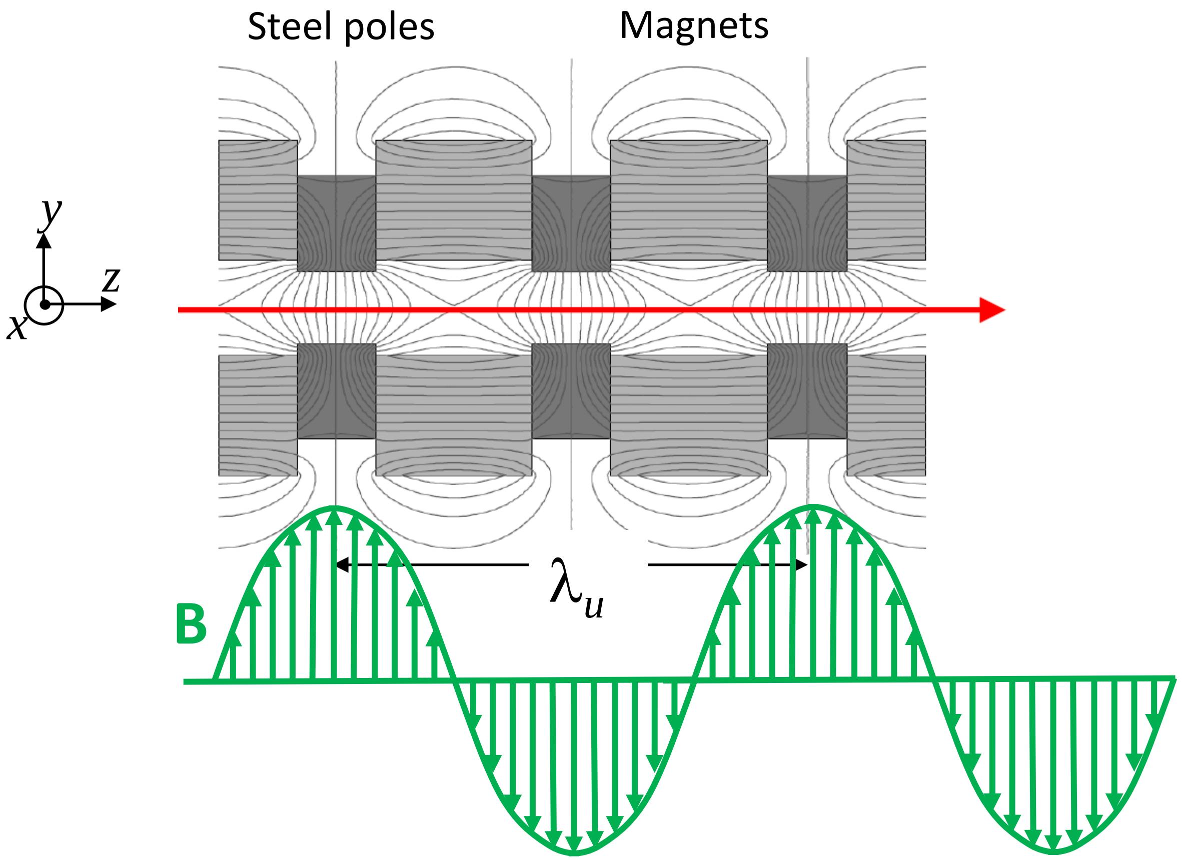
Force caused by an electric field acts along the electron beam propagation direction, thus changing the beam energy

$$\Delta W = \int \mathbf{F} \cdot d\mathbf{s} = -\int e\mathbf{E} \cdot d\mathbf{s}$$

Force caused by a magnetic field is perpendicular to the beam propagation direction, thus changing the beam momentum by Δp and the beam direction by $\Delta p/p_0$. Magnetic force does not change the beam energy.

$$\Delta \mathbf{p} = \int \mathbf{F}\, dt = -\int e(\boldsymbol{\upsilon} \times \mathbf{B})dt$$

Planar Hybrid Permanent Magnet Undulators



Electrons travel mainly in the z direction

Electrons also have a small initial velocity in x

The on-axis ($y = 0$) magnetic field is sinusoidal with z and points in the y direction

$$\mathbf{B} = B_0 sin(k_u z)\,\hat{y}$$

Undulator wavenumber $$k_u = \frac{2\pi}{\lambda_u}$$

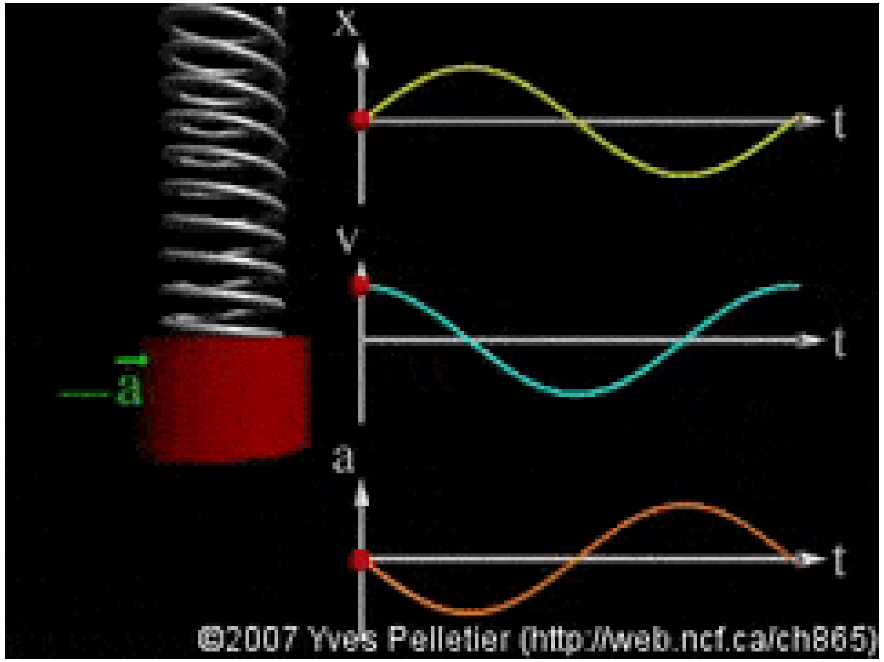
Lorentz force

$$\mathbf{F} = -e\,\mathbf{v} \times \mathbf{B}$$

The Lorentz force imparts a force in the x direction that is sinusoidal with z and opposes the electrons' motion (electrons going into the page experience a force pointing out of the page, and vice versa).

Transverse Motion in a Planar Undulator

Electrons enter the undulator with a small initial velocity v_x. Lorentz force is the restoring force that brings them back to the equilibrium position, similar to an oscillating mass on a spring.



Transverse acceleration

$$\frac{d\boldsymbol{p}_x}{dt} = \gamma m_e \frac{d\mathbf{v}_x}{dt} = -e v_z B_0 sin(k_u z)\,\hat{x}$$

$$\frac{dv_x}{v_z dt} = \frac{dv_x}{dz} = -\frac{eB_0}{\gamma m_e}\sin(k_u z)$$

Integrate with respect to z

$$v_x = c\frac{eB_0}{\gamma m_e k_u c}\cos(k_u z)$$

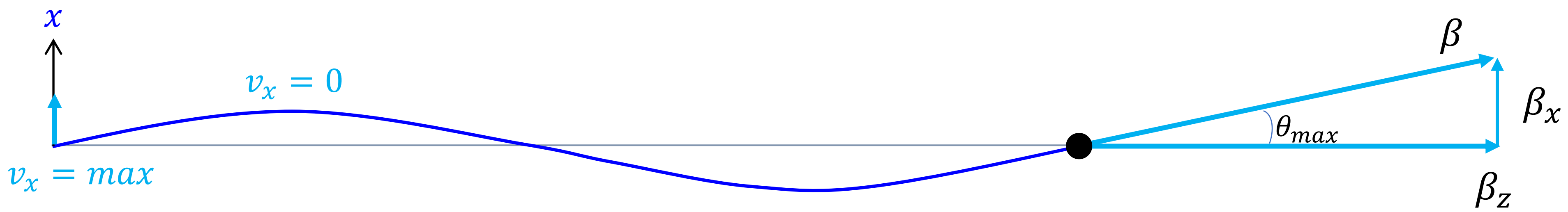
Transverse velocity

$$v_x = c\frac{K}{\gamma}\cos(k_u z)$$

Transverse displacement

$$x = \frac{K}{\gamma k_u}\sin(k_u z)$$

Longitudinal Motion in a Planar Undulator



Transverse velocity in x relative to c

$$\beta_x = \beta sin\theta$$

Average transverse velocity squared

$$\overline{\beta_x^2} \approx \beta^2 \frac{\theta_{max}^2}{2}$$

$$\theta_{max} = \frac{K}{\gamma}$$

$$\overline{\beta_x^2} \approx \beta^2 \frac{K^2}{2\gamma^2}$$

Longitudinal velocity in z relative to c

$$\beta_z = \sqrt{\beta^2 - \beta_x^2}$$

$$\beta_z = \beta\sqrt{1 - \frac{\beta_x^2}{\beta^2}}$$

Average longitudinal velocity along the undulator

$$\bar{\beta}_z \approx 1 - \frac{1}{2\gamma^2} - \frac{K^2}{4\gamma^2}$$

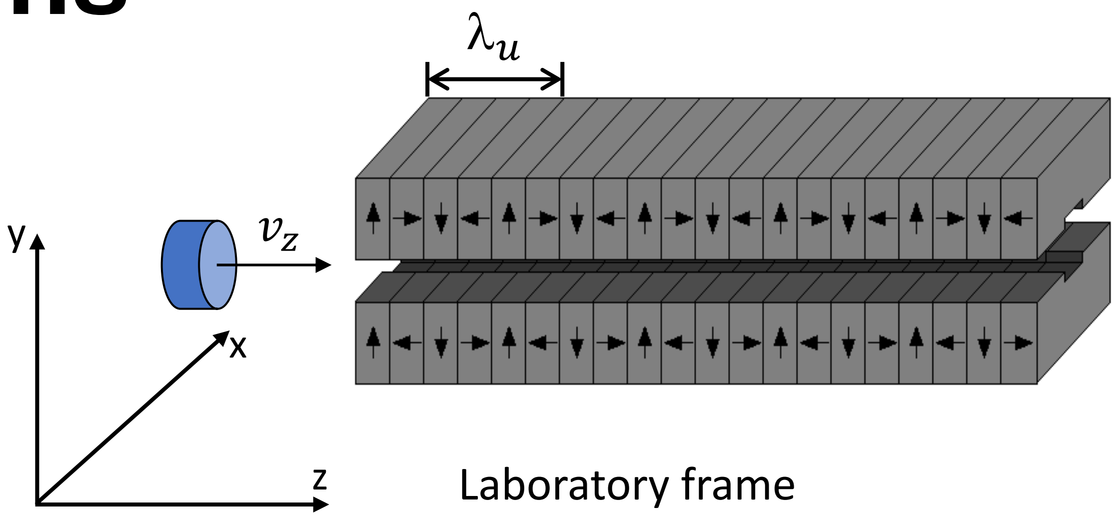
SLAC NATIONAL ACCELERATOR LABORATORY

Los Alamos NATIONAL LABORATORY EST. 1943

Moving with the Electrons

- In the laboratory frame, the electrons are moving in the z direction at the average longitudinal velocity

$$\bar{v}_z = c\left(1 - \frac{1}{\gamma^2}\left[1 + \frac{K^2}{2}\right]\right)^{\frac{1}{2}}$$

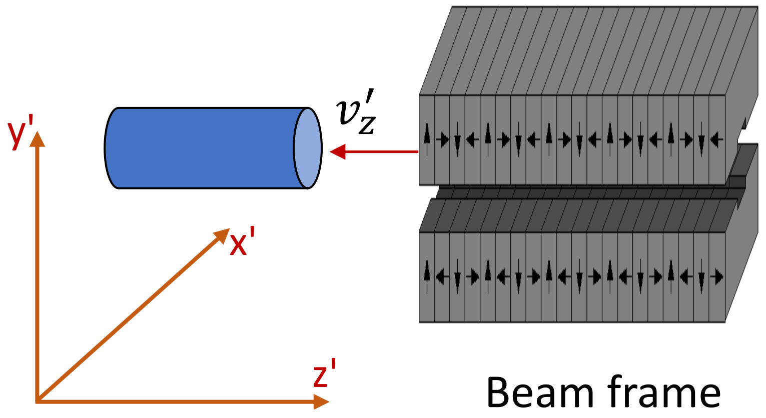


Laboratory frame

- In the frame moving with the electrons, the "beam frame," the undulator period is shortened by γ^*

$$\gamma^* = \frac{1}{\sqrt{1 - \frac{\bar{v}_z^2}{c^2}}}$$

$$\gamma^* = \frac{\gamma}{\sqrt{1 + \frac{K^2}{2}}}$$

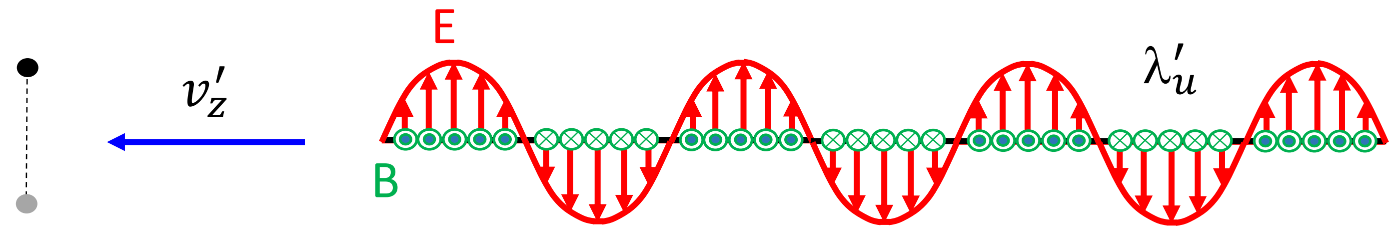


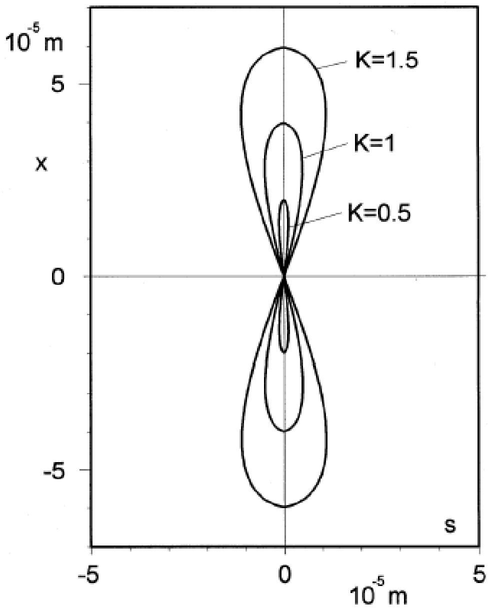
$$\lambda_u' = \frac{\lambda_u}{\gamma^*}$$

Beam frame

Oscillations in Electron Beam Frame

In the beam frame, the undulator is a traveling EM wave with the Lorentz contracted period λ_u'
The electron oscillates at a wavelength equal to the contracted undulator period, $\lambda' = \lambda_u'$





FEL wavelength in beam frame

$$\lambda' = \frac{\lambda_u}{\gamma^*} = \frac{\lambda_u}{\gamma}\sqrt{1 + \frac{K^2}{2}}$$

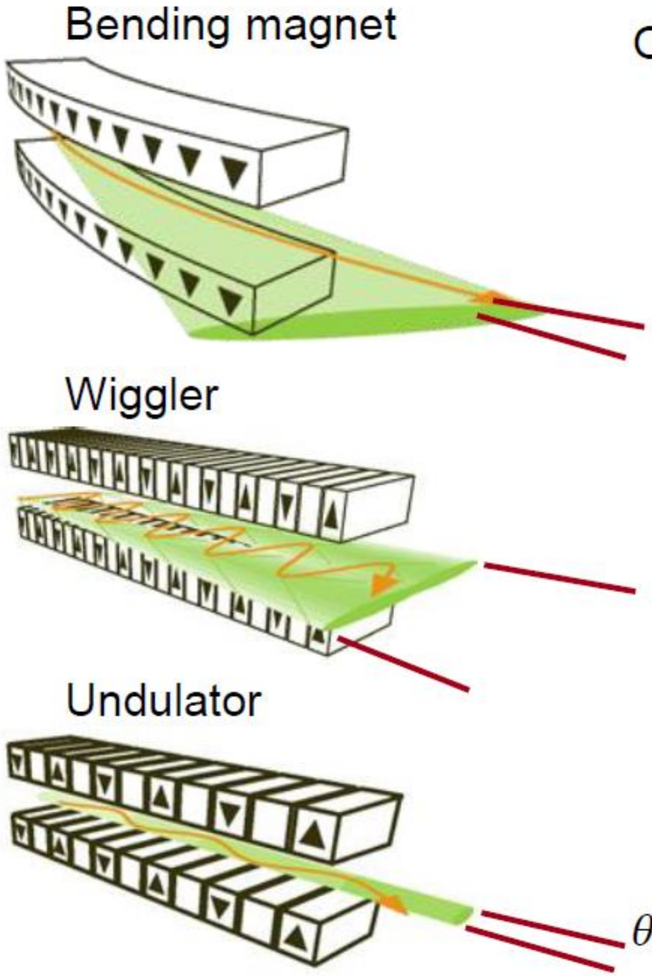
In the beam frame, the electron oscillates transversely (along the x' axis) and also longitudinally (along the z' axis) at twice the frequency of the transverse oscillations. This figure-8 motion gives rise to radiation at the harmonics of the fundamental frequency.

Fast and Slow Electron Motions

Motion	What causes the motion
▪ Fast transverse motion in x ▪ Once every undulator period	Lorentz force due to v_z cross B_y
▪ Fast longitudinal motion in z ▪ Twice every undulator period	Modulations of v_x in a planar undulator (Helical undulators do not have this motion)
▪ Slow transverse motion ▪ Occurring over many undulator periods	Weak focusing due to transverse field gradient Strong focusing due to external quadrupoles
▪ Slow longitudinal motion ▪ Occurring over the entire undulator length	Microbunching due to FEL interaction

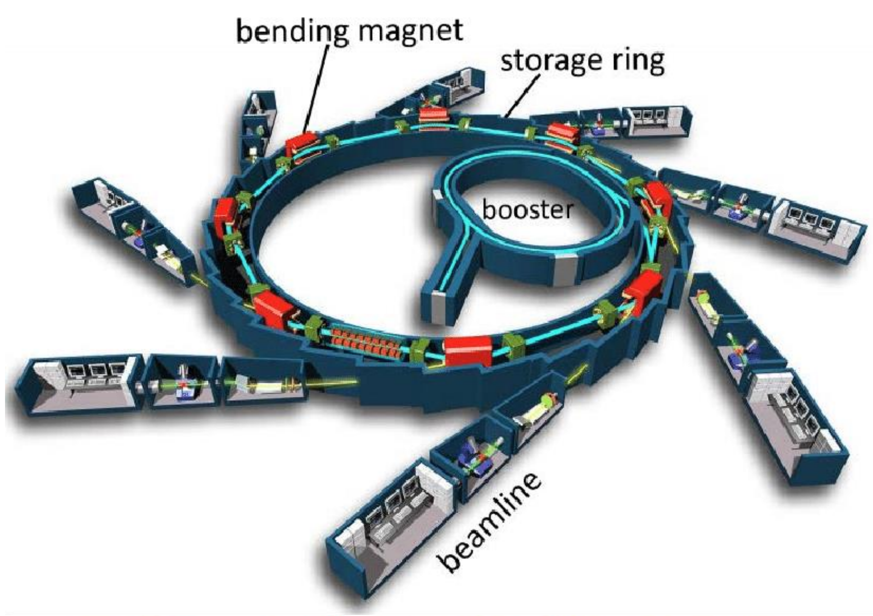
Undulator Radiation

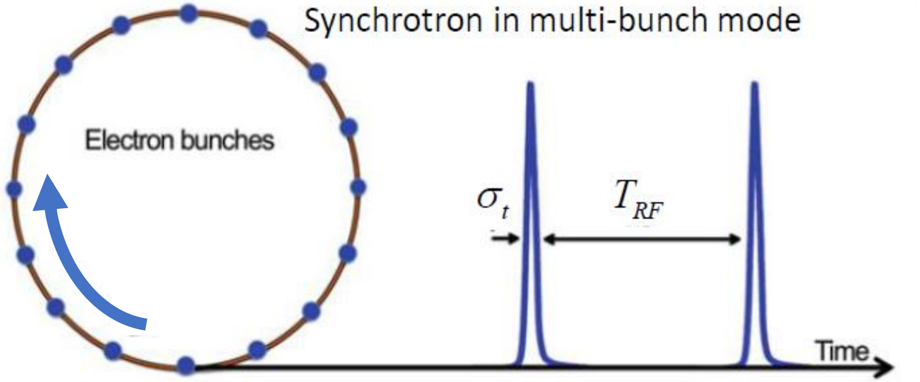
Synchrotron Radiation (SR)



	Opening angle	Spectral property
Bending magnet	$\theta \approx \frac{1}{\gamma}$	Broadband
Wiggler	$\theta \approx \frac{K}{\gamma}$	Broadband
Undulator	$\theta \approx \frac{1}{\sqrt{N}\gamma}$	Sharp lines

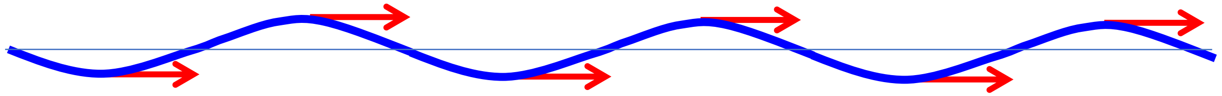
Synchrotron Radiation Facility



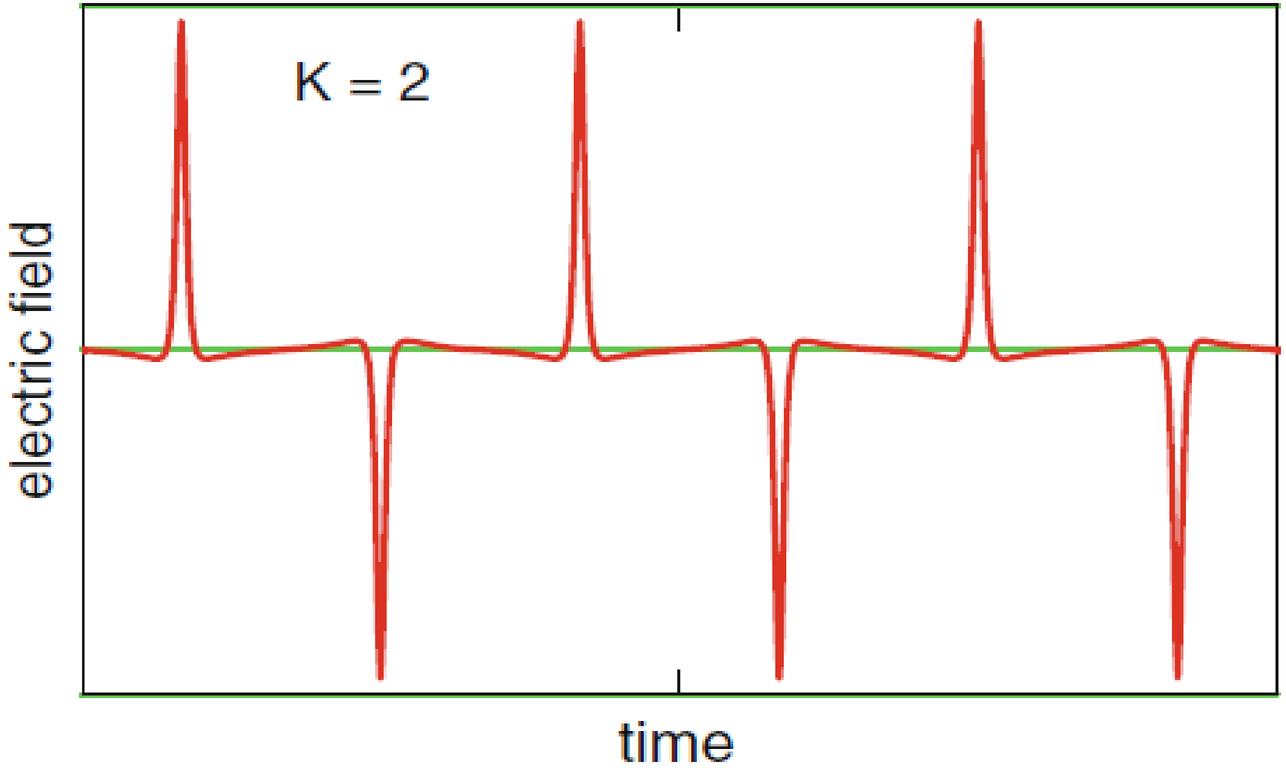


Electrons radiate when they are accelerated

Electron trajectory



In the electron rest frame, their motions are non-relativistic. We can calculate the total power radiated by an oscillating dipole.

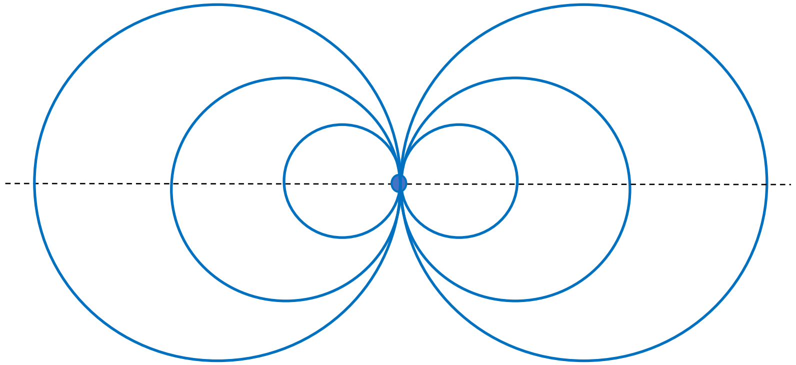


Larmor formula

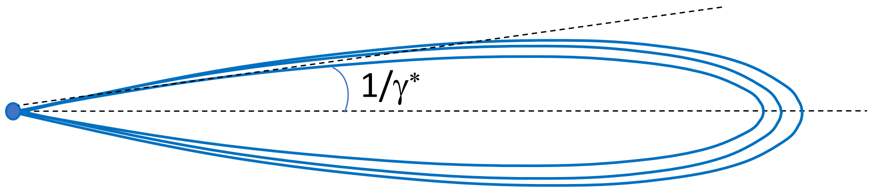
$$P = \frac{\gamma^6}{6\pi\varepsilon_0}\frac{e^2}{c^3}\left[\dot{\mathbf{v}}^2 - \frac{(\mathbf{v}\ \times\ \dot{\mathbf{v}})^2}{c^2}\right]$$

In the Lab frame, electrons' motions are relativistic and we need to use the relativistic Larmor formula. The electrons emit the highest radiation power where they experience the greatest acceleration.

Relativistic Doppler Shift



In the beam frame, the electron oscillates like a dipole antenna and emits radiation in a dipole radiation pattern with two spherical lobes propagating in opposite directions.



Transforming back to the laboratory frame, the lobes get Doppler shifted and turn into a narrow x-ray beam with a half-cone angle of $1/\gamma^*$

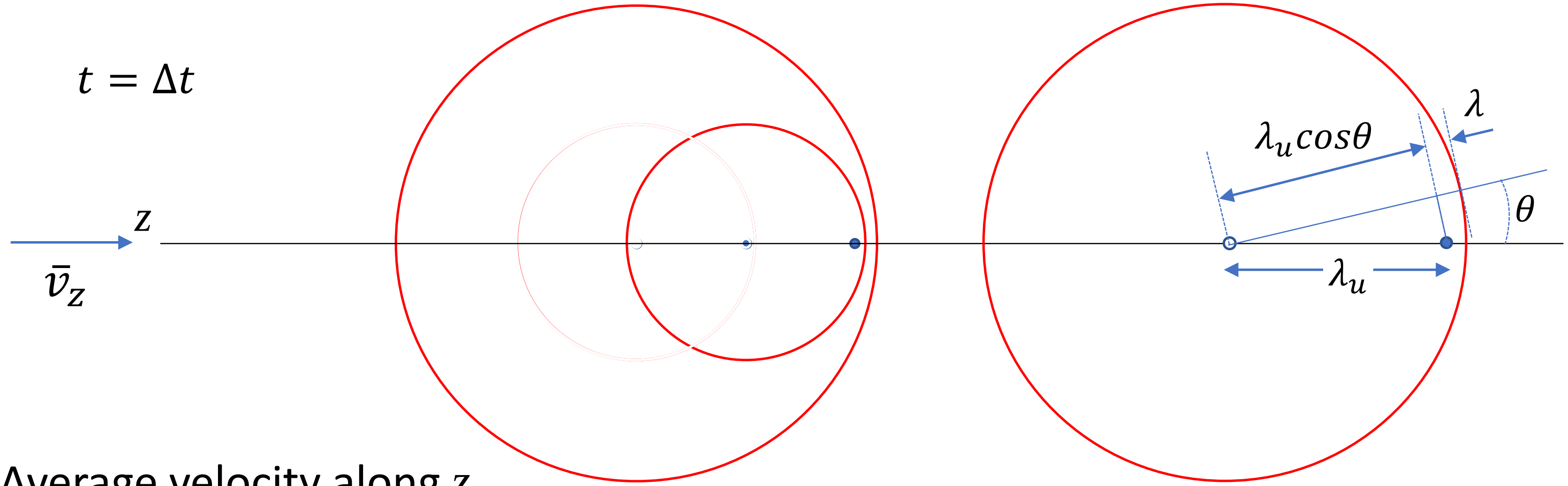
$$\lambda = \frac{\lambda'}{2\gamma^*}$$

Length Contraction & Doppler Shift

$$\frac{1}{\gamma^*} \quad \text{x} \quad \frac{1}{2\gamma^*} \quad = \quad \frac{1}{2\gamma^{*2}}$$

$$\lambda = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2}\right)$$

Undulator Radiation Wavelength



Consider an observer looking at the electron at an angle θ w.r.t. the z axis.

In the time Δt the electron travels one period, λ_u

$$\frac{\lambda_u}{\bar{v}_z} = \frac{\lambda_u cos\theta + \lambda}{c}$$

Average velocity along z

$$\bar{v}_z = c\left(1 - \frac{1}{2\gamma^2}\left(1 + \frac{K^2}{2}\right)\right)$$

$$\frac{\lambda_u}{\bar{v}_z} - \frac{\lambda_u cos\theta}{c} = \lambda$$

Fundamental undulator wavelength

Small-angle approximation

$$\lambda \approx \frac{\lambda_u}{\left(1 - \frac{1}{2\gamma^2}\left(1 + \frac{K^2}{2}\right)\right)} - \lambda_u\left(1 - \frac{\theta^2}{2}\right)$$

$$\lambda = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2} + \gamma^2\theta^2\right)$$

Undulator Radiation Harmonics

Harmonics number

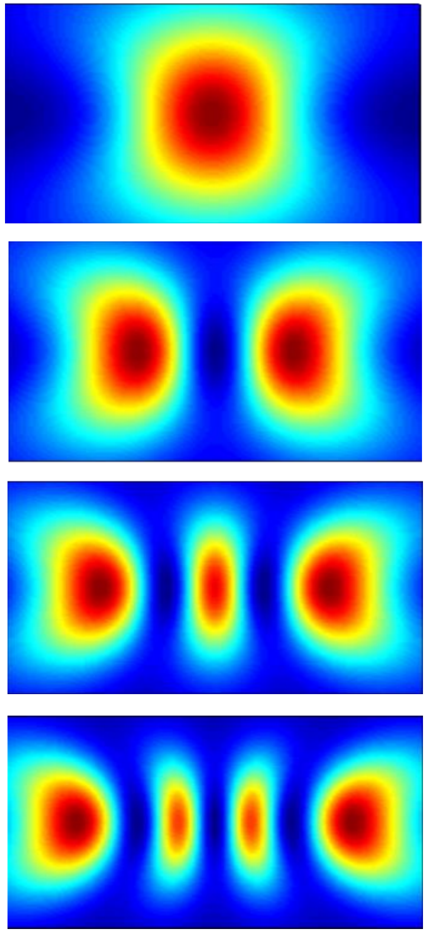
Radiation beam transverse profile

m = 1

m = 2

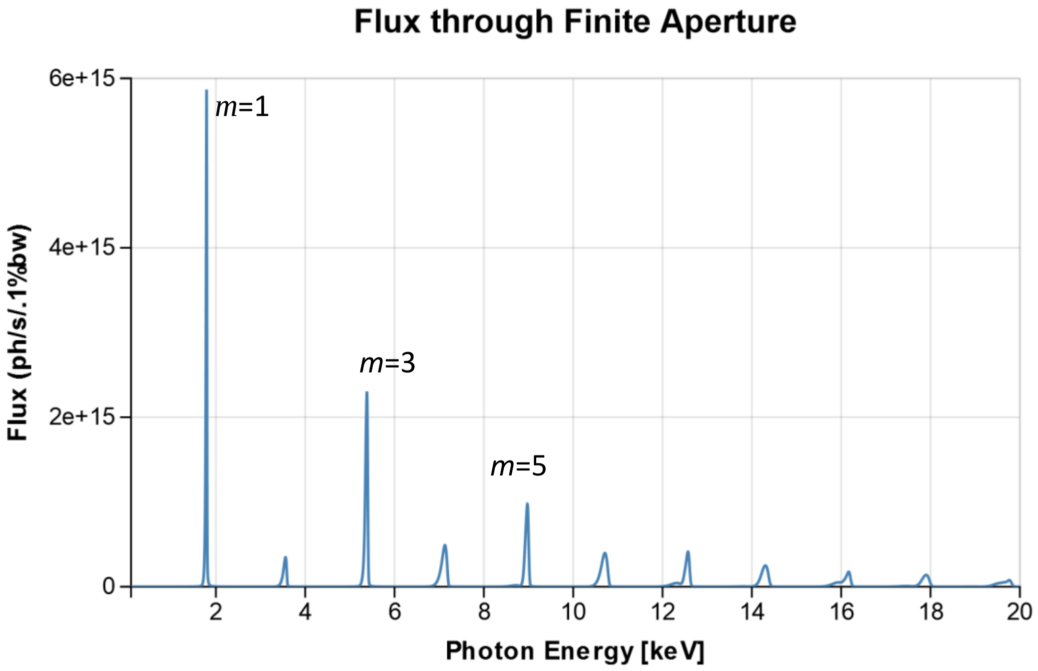
m = 3

m = 4

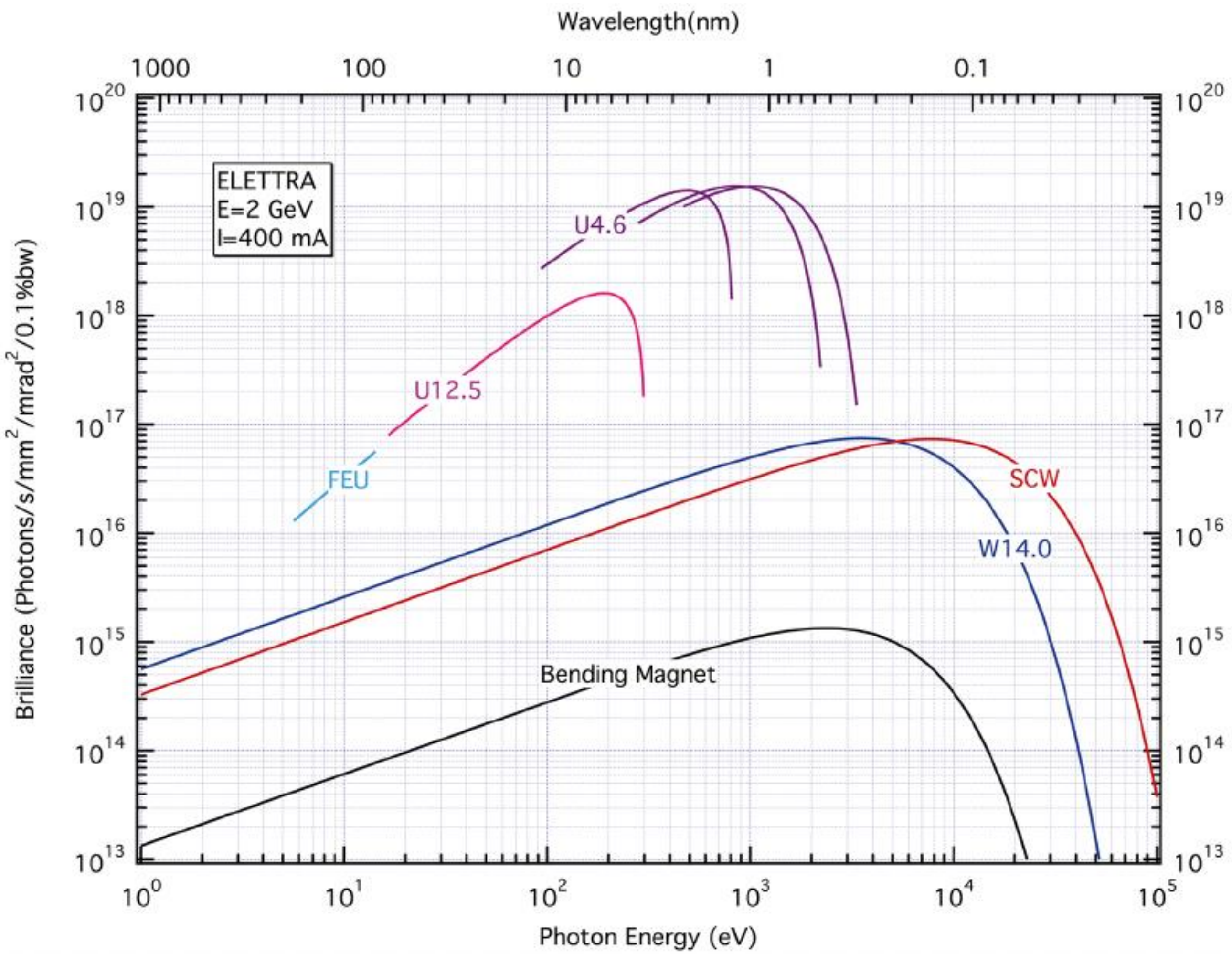


Harmonics wavelength

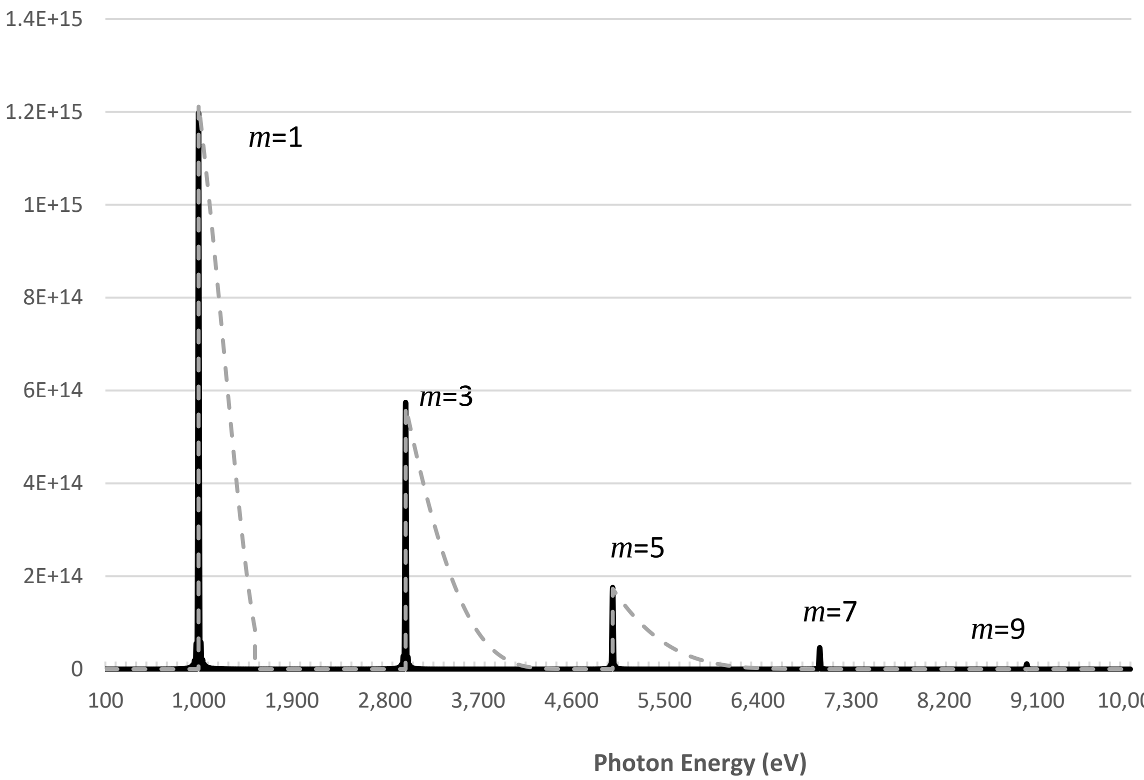
$$\lambda_m = \frac{\lambda_u}{m\,2\gamma^2}\left(1 + \frac{K^2}{2} + \gamma^2\theta^2\right)$$



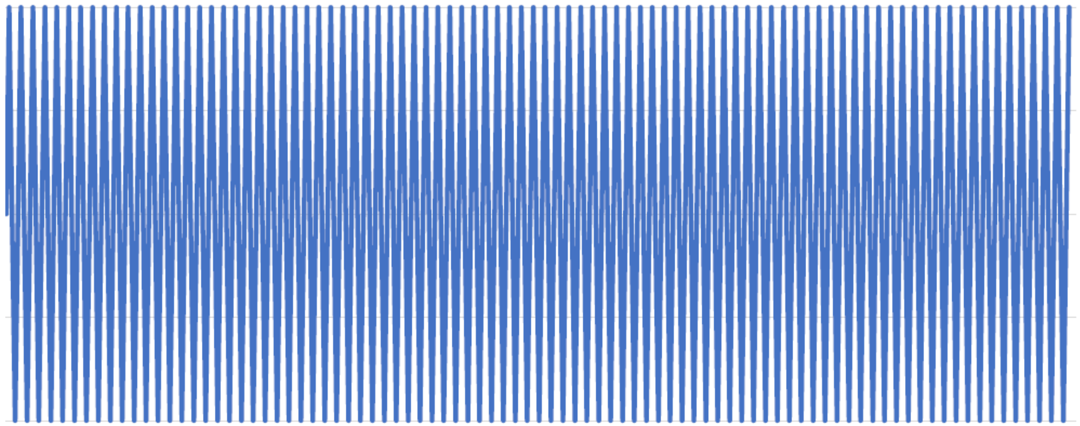
SR Brilliance and Photon Energy Spectra

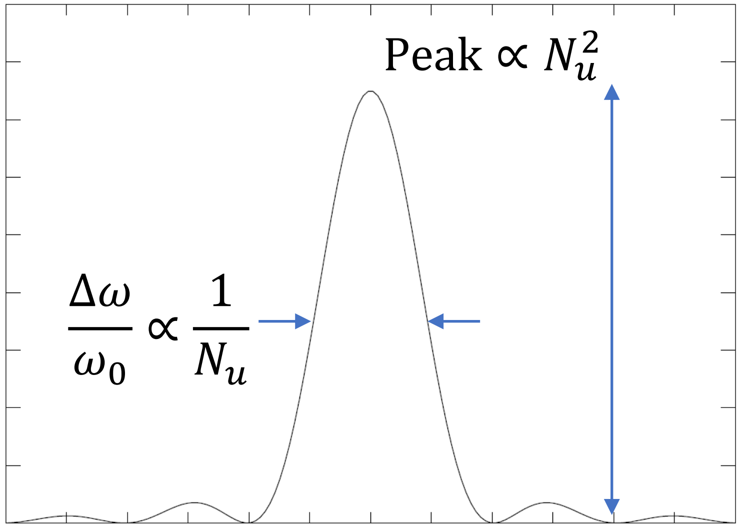


Undulator radiation spectrum with odd harmonics. The gray lines denote energy tuning curves toward higher energy by opening up the magnet gap.



Undulator Radiation Spectral Property





A single electron traversing an undulator with N_u periods will produce a constant amplitude train of electromagnetic waves with N_u wavelengths.

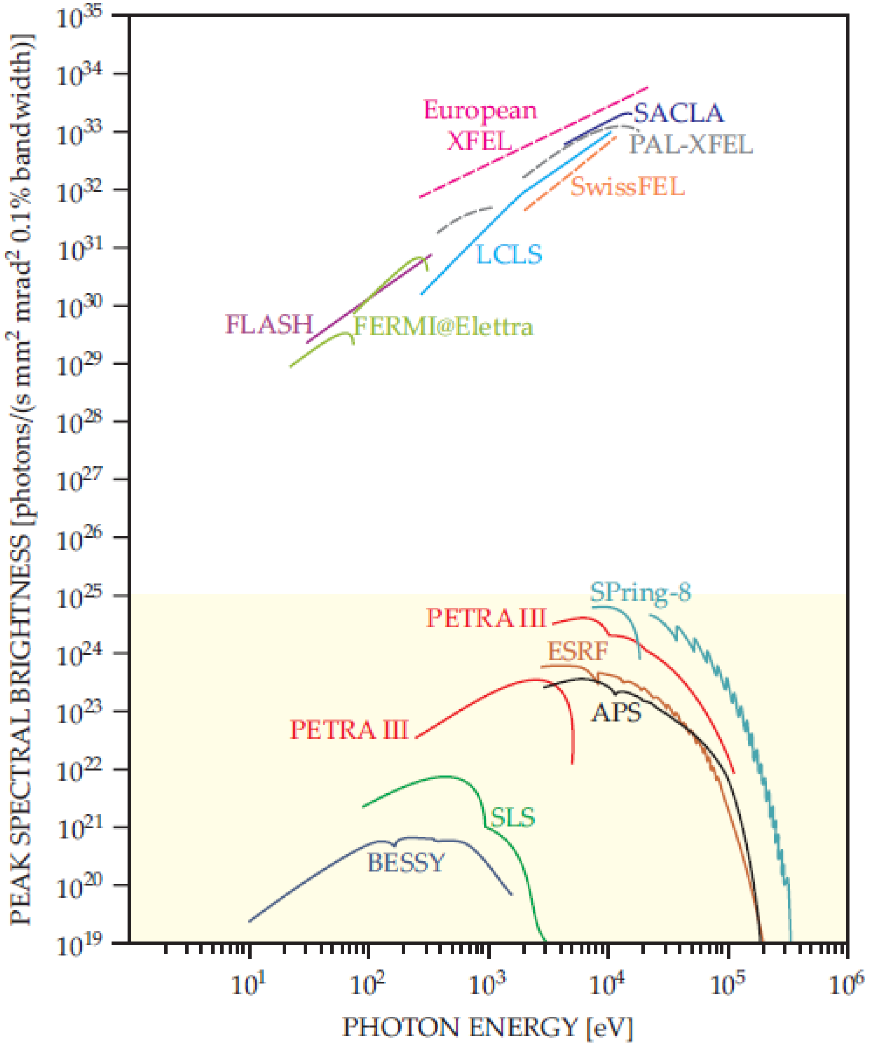
The Fourier Transform of a constant amplitude wave train with N_u wavelengths is a $sinc^2$ function with a spectral FWHM of approximately $1/N_u$

The number of photons within the coherent angular and spectral bandwidths is proportional to the number of electrons and the fine-structure constant, $\alpha = 1/137$

$$\frac{d^2U}{d\Omega d\omega} \propto N_u^2 E_b^2 \left[\frac{sin\left(\pi N_u \left(\frac{\Delta\omega}{\omega_0}\right)\right)}{\pi N_u \left(\frac{\Delta\omega}{\omega_0}\right)}\right]^2$$

$$N_{coherent} = \pi\alpha N_b \left(\frac{K}{1+K^2}\right)^2$$

Undulator Radiation and FEL Brilliance



Peak Brilliance
(Spectral Brightness)

$$B_p = \frac{\#\, photons}{A_x A_y \left(\frac{\Delta E}{E}\right) \tau}$$

Phase-space areas in x and y

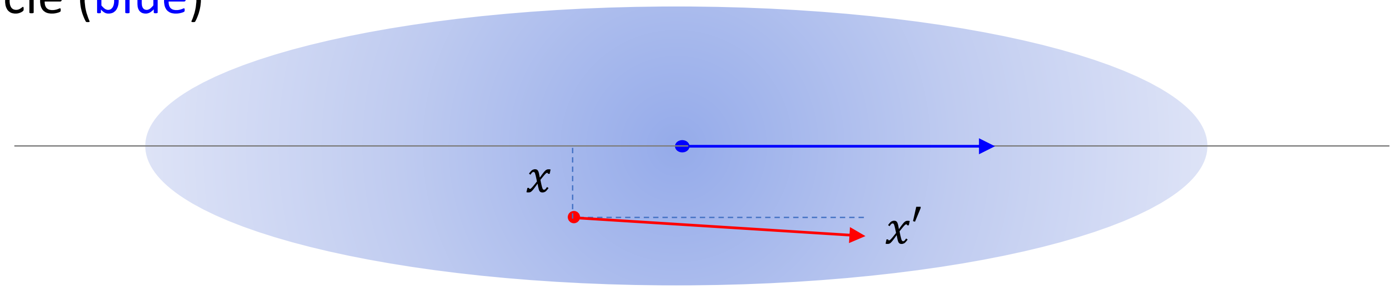
Relative x-ray energy bandwidth

Pulse length

	Und. Radiation	FEL
# photons/pulse	~ 10^8	~ 10^{12}
Phase space area	$A_x > 100\ A_{hv}$	A_{hv}
Relative BW	~ 1%	~ 0.1%
Pulse length	~ ps	10s of fs
Total increase	1	10^8 - 10^{10}

Particle Transverse Positions and Angles

Consider a single electron (red) in an ensemble of billions of particles co-traveling with the reference particle (blue)



x is the transverse position of the particle relative to the reference particle

x' is the angle the particle makes with respect to the reference particle's trajectory

$$x' = \frac{dx}{dz} = \frac{p_x}{p_z} \approx \frac{v_x}{c}$$

Similarly, the particle is also described by its transverse position y and angle y'

Paraxial approximation: transverse velocities are much smaller than c so the angles x' and $y' \ll 1$

Ensemble Averages and rms Values

Ensemble average value of x^2

$$\langle x^2 \rangle = \int x^2 f(x, x', y, y')\, dx\, dx' dy\, dy'$$

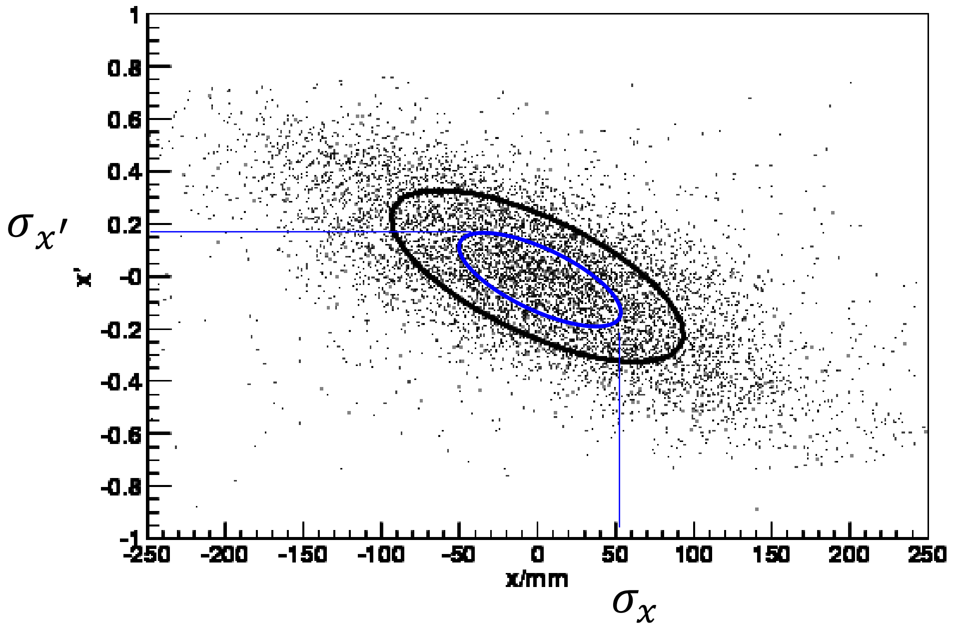
Ensemble average value of x'^2

$$\langle x'^2 \rangle = \int x'^2 f(x, x', y, y')\, dx\, dx' dy\, dy'$$

Ensemble average value of xx'

$$\langle xx' \rangle = \int xx' f(x, x', y, y')\, dx\, dx' dy\, dy'$$

xx' is the correlation between the particle's position and angle (xx' is also known as the flow).



For cases where f is a Gaussian distribution, the ellipse corresponding to rms values, σ_x and $\sigma_{x'}$ (blue) with a phase-space area equal to $\pi\varepsilon_{x,rms}$ encompasses only 39% of the particles. Going to $2\sigma_x$ and $2\sigma_{x'}$ increases the phase-space area to $4\pi\varepsilon_{x,rms}$ (black) and raises the fraction of particles enclosed in the ellipse to 87%.

Normalized and Un-normalized Emittance

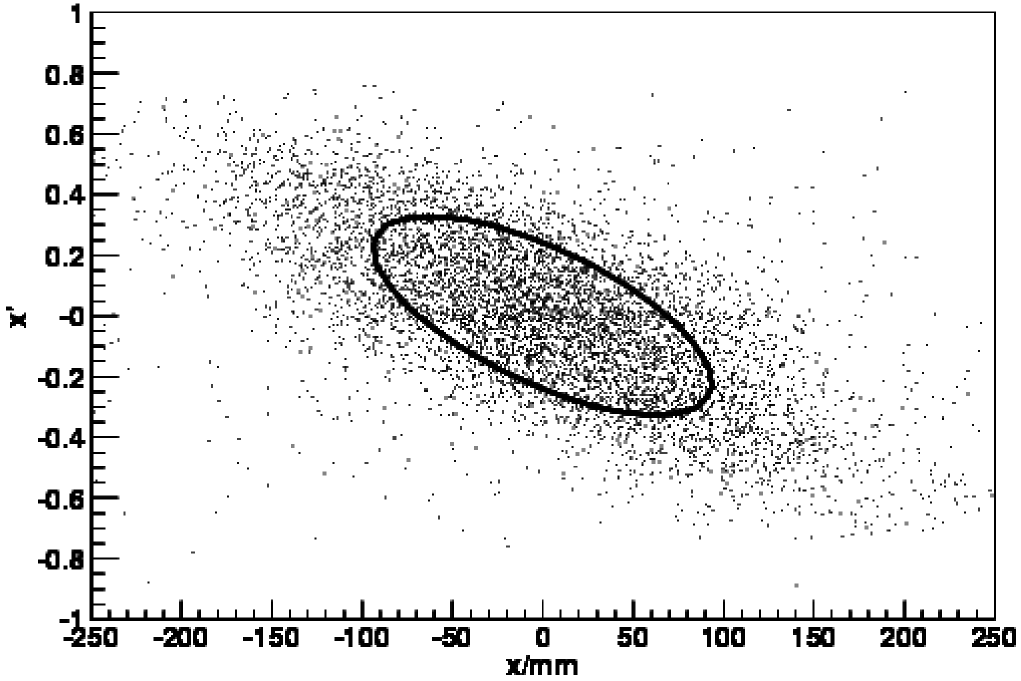
Un-normalized rms emittance in x

$$\varepsilon_{x,rms} = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2}$$

Un-normalized emittance is larger at low energy

Un-normalized emittance decreases as the beams are accelerated to higher energy (adiabatic damping)

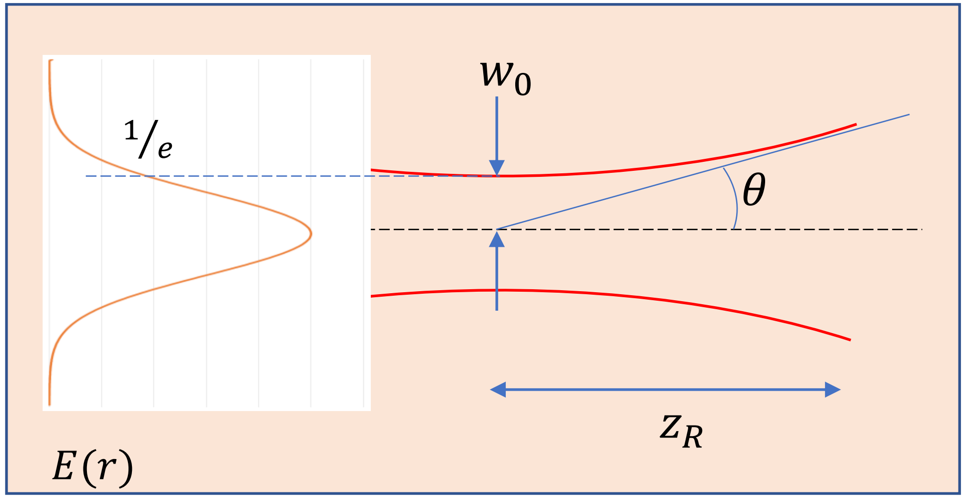
To compare emittance of particle beams with different energy, we "normalize" the emittance by multiplying it by $\beta\gamma$ (or γ since $\beta \sim 1$). The normalized emittance is conserved in the absence of non-linear forces.



$$\varepsilon_{n,rms} = \beta\gamma \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2}$$

Photon Beam Emittance

- Consider the TEM_{00} of a Gaussian beam at the beam waist



$$E(r) = E_0 exp\left(-\frac{r^2}{w_0^2}\right) = E_0 exp\left(-\frac{r^2}{4\sigma_r^2}\right)$$

rms beam radius $\sigma_r = \frac{w_0}{2}$

$$z_R = \frac{\pi w_0^2}{\lambda} \qquad \theta w_0 = \frac{\lambda}{\pi}$$

- We also write the electric field as a function of beam divergence

$$\mathcal{E}(r') = \mathcal{E}_0 exp\left(-\frac{r'^2}{\theta^2}\right) = \mathcal{E}_0 exp\left(-\frac{r'^2}{4\sigma_{r'}^2}\right)$$

rms angular divergence $\sigma_{r'} = \frac{\theta}{2}$

Photon beam rms emittance

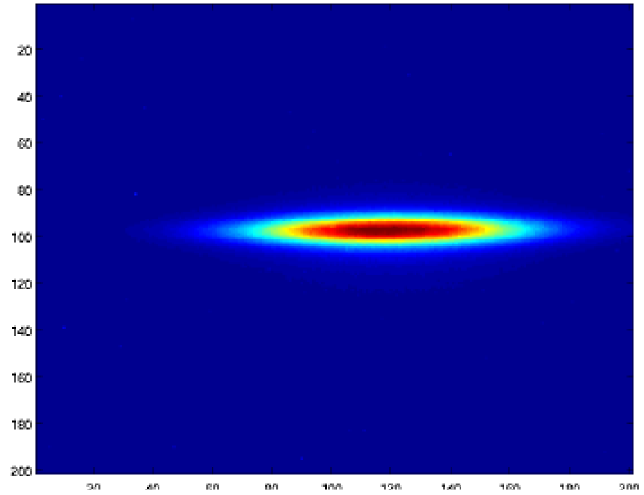
$$\boxed{\varepsilon_r = \sigma_r \sigma_{r'} = \frac{\lambda}{4\pi}}$$

Beam-limited Radiation Brightness

Phase space area in x, y $\quad A_x = 2\pi\Sigma_x\Sigma_{x'} \qquad A_y = 2\pi\Sigma_y\Sigma_{y'}$

Source size $\quad \Sigma_x = \sqrt{\sigma_x^2 + \sigma_r^2}$

Angular divergence $\quad \Sigma_{x'} = \sqrt{\sigma_{x'}^2 + \sigma_{r'}^2}$



Example: typical geometric emittance at ALS

$$\varepsilon_x = 2nm - rad$$

$$\varepsilon_y \approx 0.04nm - rad$$

Third generation undulator radiation is ***e*-beam emittance dominated**

$$\sigma_x \gg \sigma_r \qquad \sigma_{x'} \gg \sigma_{r'} \qquad \varepsilon_x = \sigma_x\sigma_{x'} \gg \varepsilon_r$$

Undulator radiation brightness

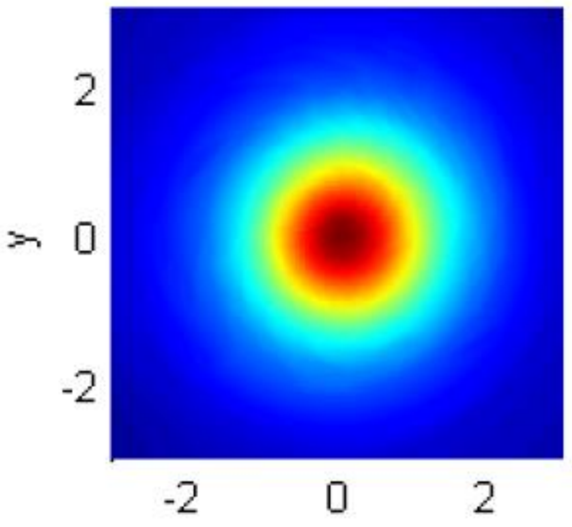
$$\boxed{\mathcal{B}_{UR} = \frac{\mathcal{F}}{4\pi^2\varepsilon_x\varepsilon_y}}$$

$\mathcal{F} \equiv$ Spectral photon flux $\left(\dfrac{\#\,photons}{0.1\%\,BW \cdot s}\right)$

Diffraction-limited Radiation Brightness

If the electron beam emittance is less than or equal to the radiation beam emittance, the output radiation is considered **diffraction limited**

$$\varepsilon_{x,y} \leq \frac{\lambda}{4\pi}$$



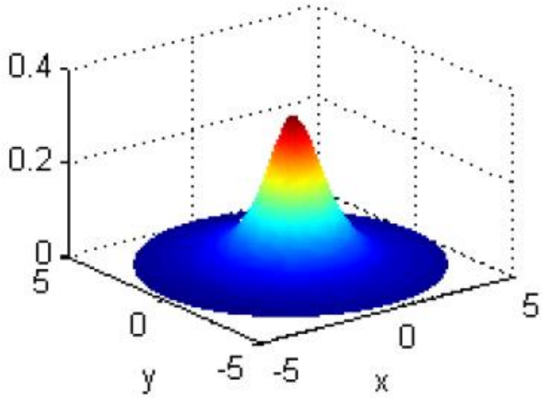
Diffraction-limited phase-space area

$$A_r = 2\pi\sigma_r\sigma_{r'} = 2\pi\varepsilon_r$$

$$A_r = \frac{\lambda}{2}$$

Diffraction-limited radiation brightness

$$\boxed{\mathcal{B}_{DL} = \frac{4\mathcal{F}}{\lambda^2}}$$



This is true for both FEL and diffraction-limited synchrotron radiation

Benefits of FEL over Undulator Radiation

- Small beam size
- Low angular divergence

⇨ Full transverse coherence
Diffraction limited radiation
Coherent diffractive imaging

- Femtosecond pulses ⇨ Time-resolved studies of physical, chemical, biological and materials science dynamics

- Narrow spectral BW ⇨ High-resolution spectroscopy

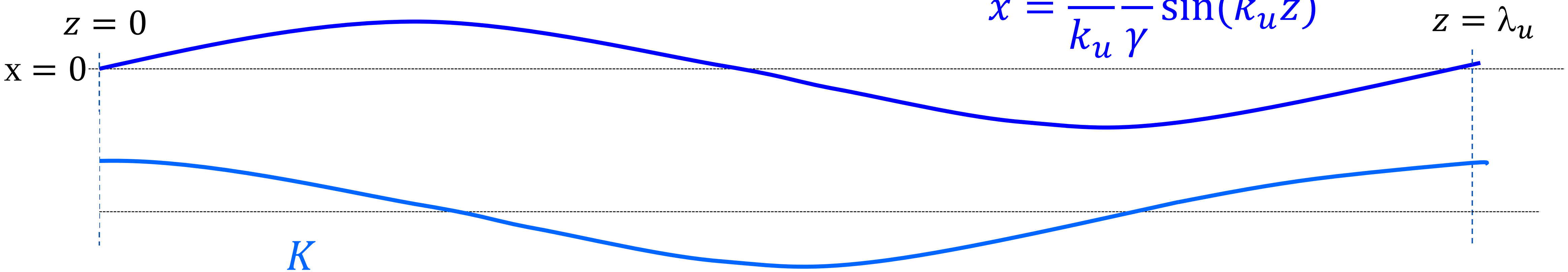
- Higher photon flux
  ⇨ X-ray diffraction of small crystals, single viruses, etc.
  ⇨ Better signal-to-noise ratios

Introduction to FEL

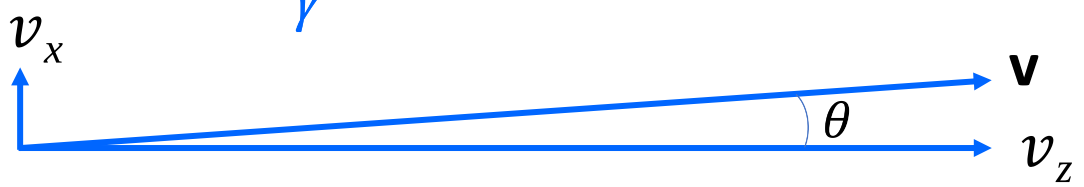
Electron Position & Velocity in the Undulator

Electron **position** and **velocity** in one undulator period

$$x = \frac{1}{k_u}\frac{K}{\gamma}\sin(k_u z)$$



$$v_x = c\frac{K}{\gamma}\cos(k_u z)$$



$$v_z = c\left(1 - \frac{1}{\gamma^2} - \frac{K^2}{\gamma^2}cos^2(k_u z)\right)^{\frac{1}{2}}$$

Example: $\lambda_u = 5.6$ cm, K = 3 and E_b = 8 GeV, the maximum transverse position and angle of the trajectory are:

$$x_{max} = 1.7\ \mu m$$

$$\theta_{max} = 192\ \mu rad$$

Lorentz Force in a Planar Undulator

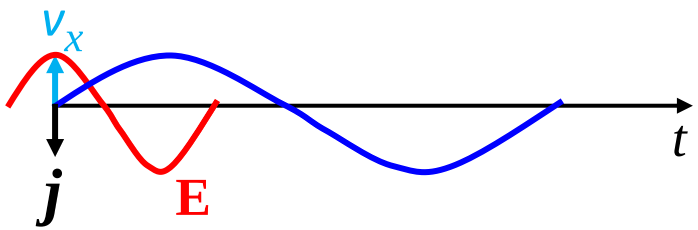
The **v x B** force produces the transverse acceleration, i.e., rate of change in the electrons' transverse momentum. By integrating the rate of change in the relativistic transverse momentum, we obtain the electrons' sinusoidal velocity along the *x* direction.

$$\boxed{\frac{d\boldsymbol{p}}{dt} = -e\mathbf{v} \times \mathbf{B}}$$

$$\frac{d(\gamma m_e v_x)}{dt} = -e v_z B_y$$

$$v_x = c\frac{eB_0}{\gamma m_e c k_u}\cos(k_u z)$$

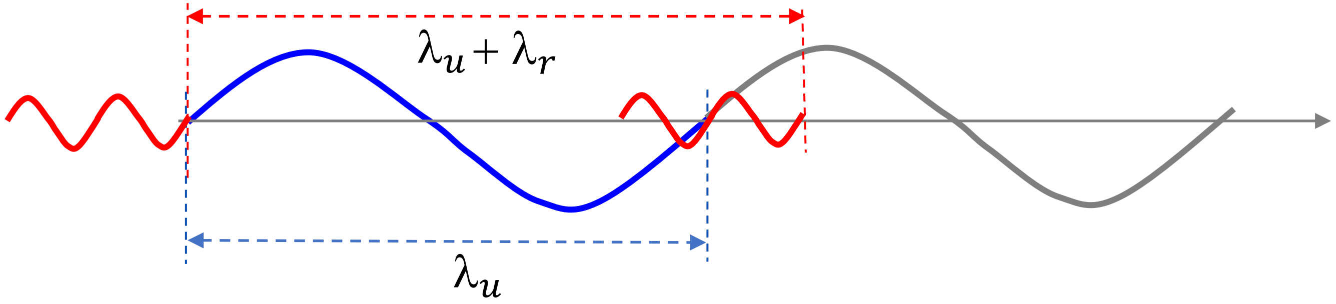
The rate of change in the electron energy (W) is proportional to the dot product of the transverse electron current (A-m) and radiation beam transverse electric field (V/m).



$$\boxed{\frac{dW}{dt} = \boldsymbol{j} \cdot \mathbf{E}}$$

$$m_e c^2 \frac{d\gamma}{dt} = -e v_x E_x$$

Resonant Wavelength



In the time the electrons travel one undulator period (blue), the optical wave (red) has traveled one undulator period plus one wavelength. The wave slips ahead of the electron one wavelength. This special wavelength is called the Resonant Wavelength.

$$\frac{\lambda_u}{\bar{v}_z} = \frac{\lambda_u + \lambda_r}{c} \quad \Rightarrow \quad \frac{\lambda_r}{\lambda_u} = \frac{c}{\bar{v}_z} - 1 \quad \Rightarrow \quad \frac{\lambda_r}{\lambda_u} = \frac{1}{1 - \frac{1}{2\gamma^2}\left[1 + \frac{K^2}{2}\right]} - 1$$

$$\boxed{\lambda_r = \frac{\lambda_u}{2\gamma^2}\left[1 + \frac{K^2}{2}\right]}$$

Electron-Wave Energy Exchange

Transverse electron current density

$$j_x = -ev_x$$

Rate of energy exchange

$$\boxed{\frac{dW}{dt} = \mathbf{j} \cdot \mathbf{E}}$$

Transverse electric field of the optical radiation

$$E_x = E_0 \cos(kz - \omega t + \varphi)$$

Energy exchange occurs via the vector dot product between the transverse electron current density and the transverse electric field of the radiation

$$j_x = -e\frac{cK}{\gamma} \cos(k_u z)$$

$$\frac{d}{dt}(\gamma m_e c^2) = -e\frac{cKE_0}{\gamma}\cos(k_u z)\cos(kz - \omega t + \varphi)$$

$$\frac{d\gamma}{dz} = -e\frac{cKE_0}{2\gamma m_e c^2}\left[\cos[\underbrace{(k_u + k)z + \varphi - \omega t}_{\psi}] + \cos[(k_u - k)z + \varphi - \omega t]\right]$$

This phase can be made constant with a judicious choice of k

Resonant Wavenumber

Ponderomotive phase

$$\psi = (k_u + k_r)z + \varphi - \omega_r t$$

Differentiate with respect to t and set it to zero

$$\frac{d\psi}{dt} = (k_u + k_r)\bar{v_z} - \omega_r = 0$$

Divide both sides by c

$$(k_u + k_r)\left(1 - \frac{1}{2\gamma_n^2}\left[1 + \frac{K^2}{2}\right]\right) - k_r = 0$$

$$k_u = \frac{k_r}{2\gamma_n^2}\left[1 + \frac{K^2}{2}\right]$$

The ponderomotive phase remains constant with time as the electrons travel along z for a special wavenumber called the **resonant wavenumber** k_r

Average axial velocity of the n^{th} electron

$$\bar{v}_{z_n} \approx c\left(1 - \frac{1}{2\gamma_n^2}\left[1 + \frac{K^2}{2}\right]\right)$$

Resonant wavenumber

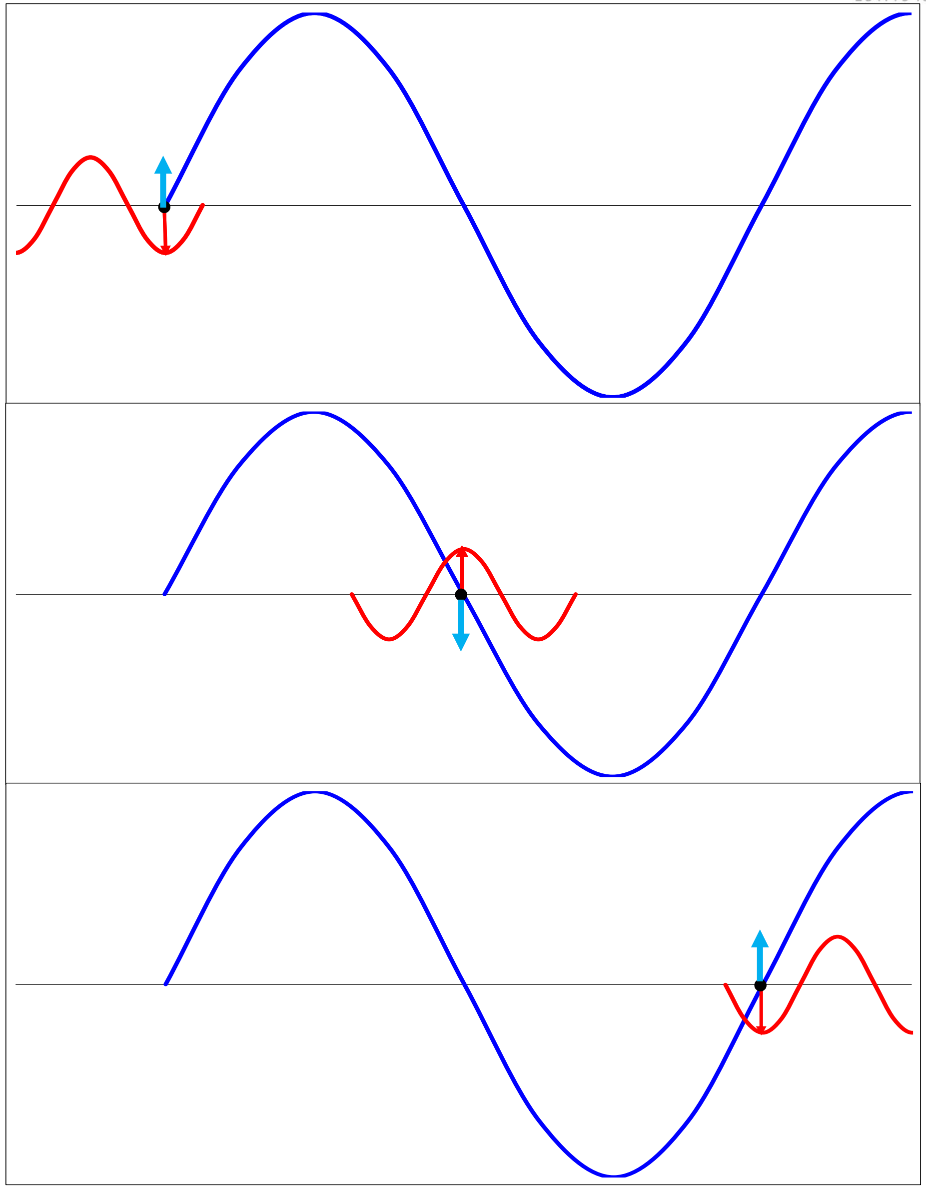
$$k_r = k_u \frac{2\gamma_r^2}{\left(1 + \frac{K^2}{2}\right)}$$

Resonance Condition Energy Gain

Snap shots of an optical wave (**red**) traveling collinearly with an electron (black circle) that follows a sinusoidal trajectory (**blue**) at three different points along an undulator period from top to bottom.

The ponderomotive phase is equal to **-π/2**. The **wave electric field vector** points in the opposite direction of the **transverse electron velocity**. The electron is accelerated by the optical electric field.

The rate of energy exchange is positive, i.e., the **electron gains energy** from the optical wave.

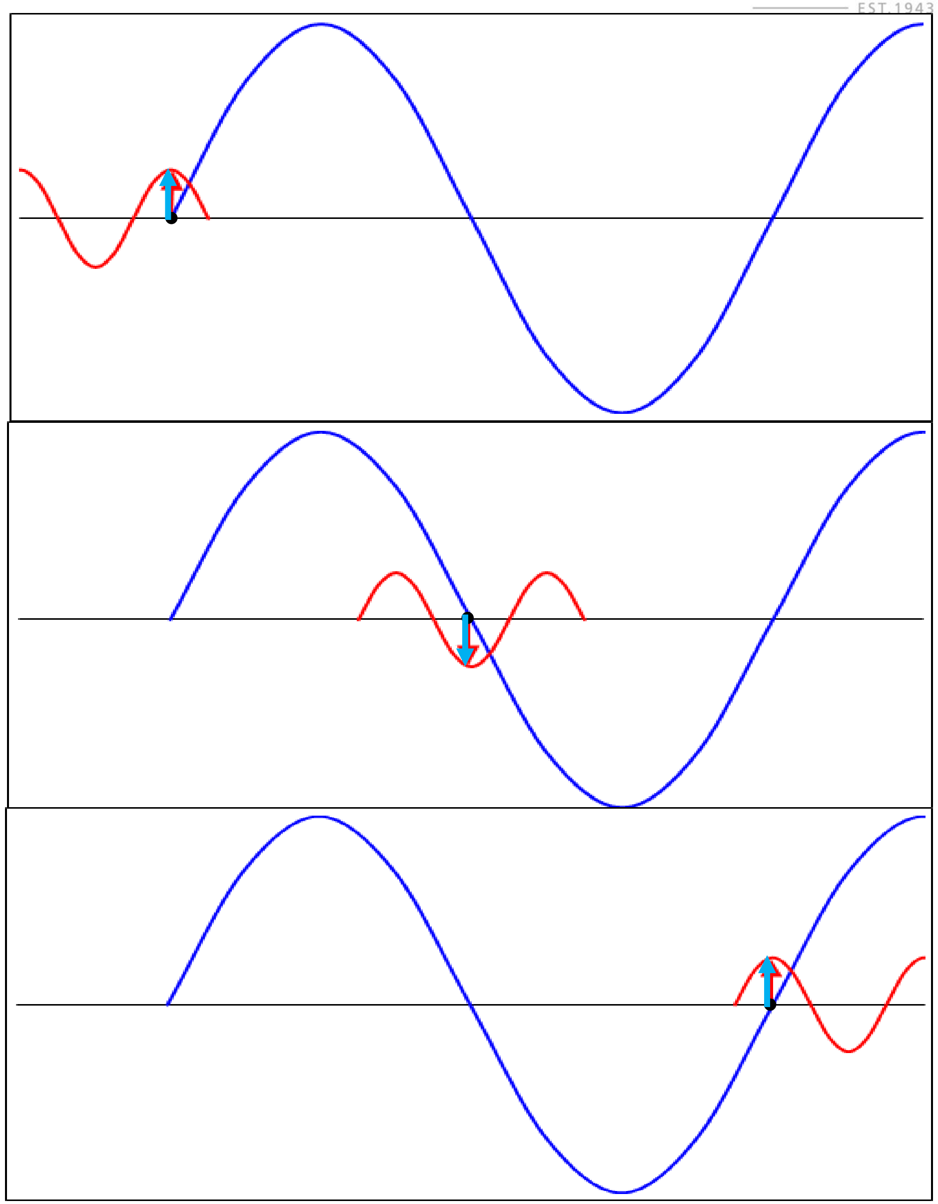


Resonance Condition Energy Loss

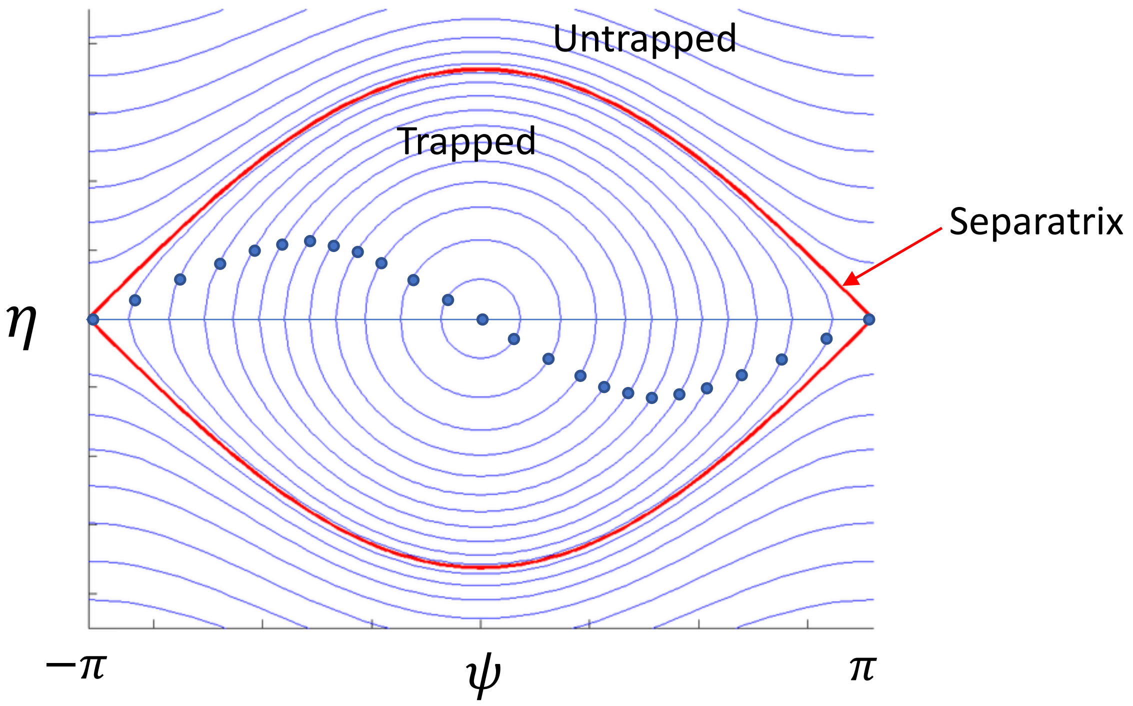
Snap shots of an optical wave (**red**) traveling collinearly with an electron (black circle) that follows a sinusoidal trajectory (**blue**) at three different points along an undulator period from top to bottom.

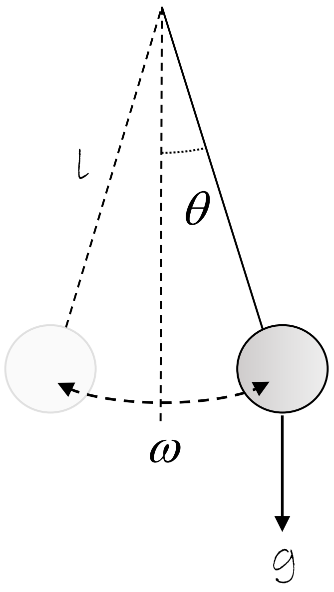
The ponderomotive phase is equal to **π/2**. The **wave electric field vector** points in the same direction with the **transverse electron velocity**. The electron is decelerated by the optical electric field.

The rate of energy exchange is negative, i.e., the **electron loses energy** to the optical wave.



FEL Energy-Phase & Pendulum Equations





FEL energy-phase equations

$$\frac{d\psi}{dz} = 2k_u\eta$$

$$a = \frac{eE_0\widehat{K}}{2m_ec^2\gamma_0^2} \qquad \frac{d\eta}{dz} = -|a|sin\psi$$

Pendulum equations

$$\frac{d\theta}{dt} = \omega$$

$$\frac{d\omega}{dt} = -\frac{g}{l}sin\theta$$

Electrons Gain/Lose Energy & Bunch Up

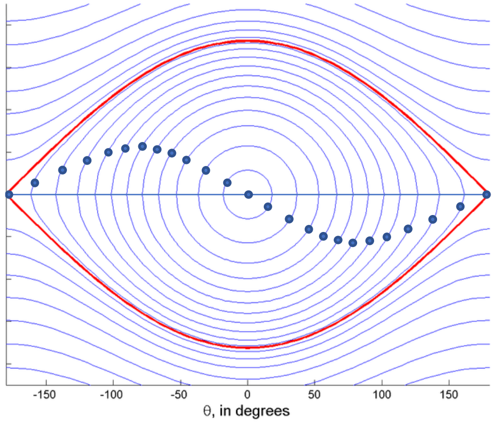
Hamiltonian (kinetic energy + potential energy) of the pendulum

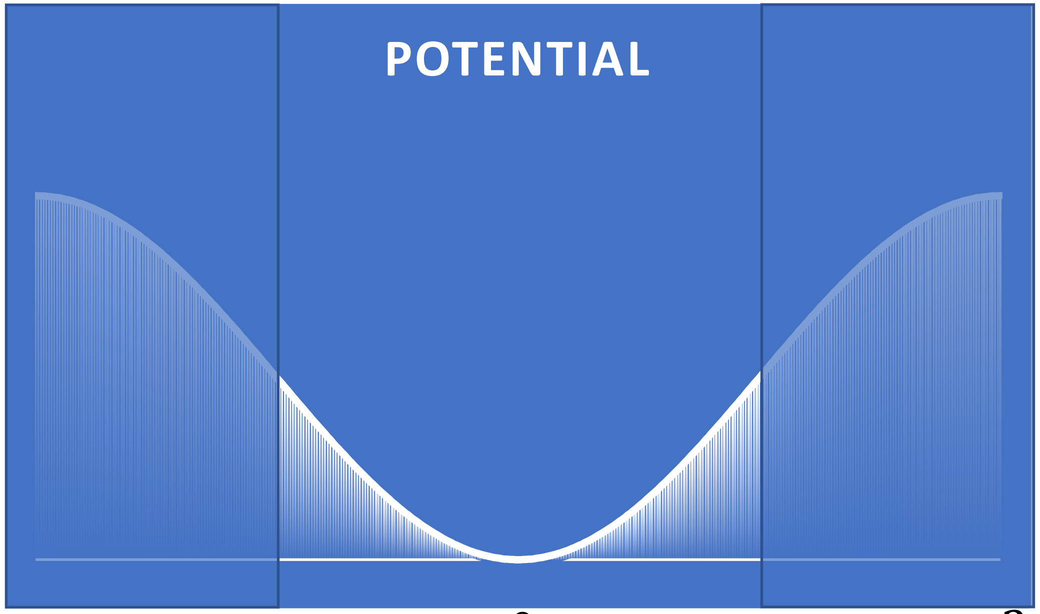
$$H = \frac{ml^2(\dot{\theta})^2}{2} - mgl(1 - \cos\theta)$$

Pendulum potential energy

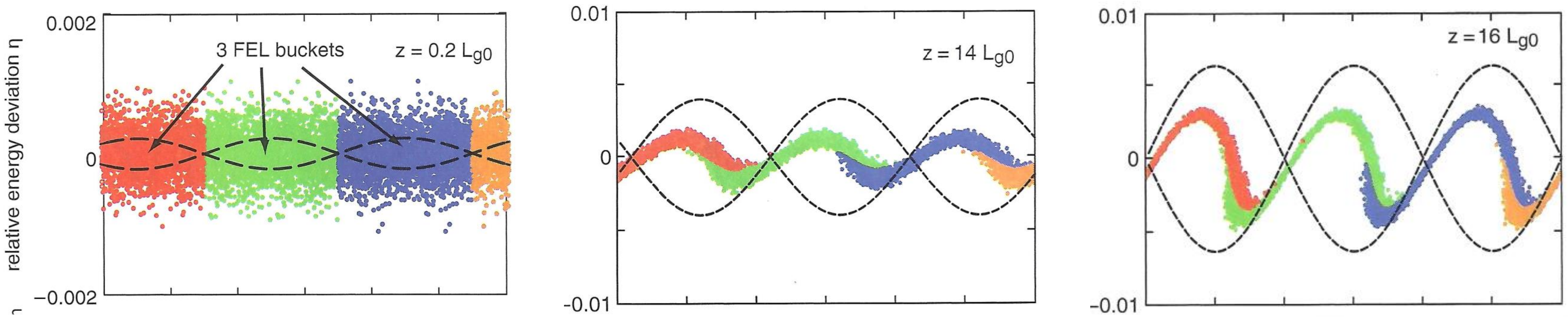
$$V = gl(1 - \cos\theta)$$

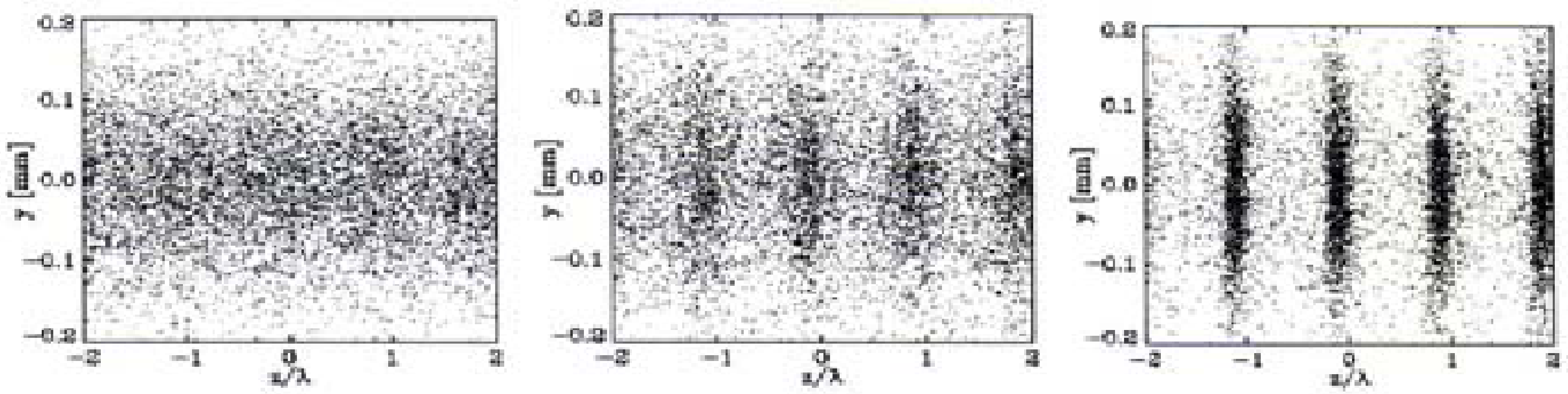
Half of the electrons (ponderomotive phase from -π to 0) gain energy and move up in the bucket. The other half of the electrons (phase from 0 to π) lose energy and move down in the bucket. In term of pendulum potential energy, they all fall down to the bottom of the potential well and bunch up near phase = 0.





Energy Modulations & Density Modulations





At entrance to the undulator

Exponential gain regime

Saturation(maximum bunching)

Electrons interacting with the ponderomotive waves develop energy and density modulations.

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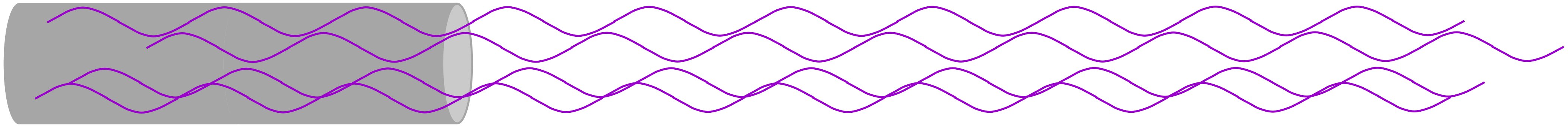
Radiation from a Bunch of Electrons

Electrons are randomly distributed along z

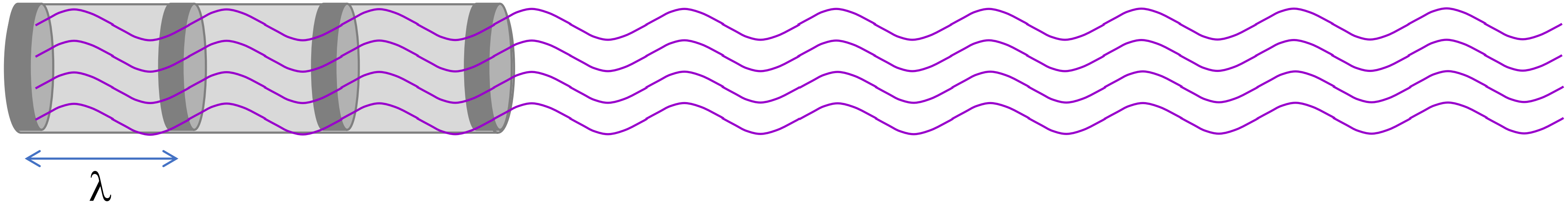
Incoherent undulator radiation



FEL interaction induces density modulations with period equal to the radiation wavelength. The emitted fields are in phase and add coherently. **Coherent intensity scales with N_λ^2**

Electrons are bunched with period of a radiation λ

Coherent FEL radiation



Ratio of coherent power to incoherent power is N_λ (the number of electrons in one λ)

Bunched Beams Emit Coherent FEL Radiation

Bunching factor

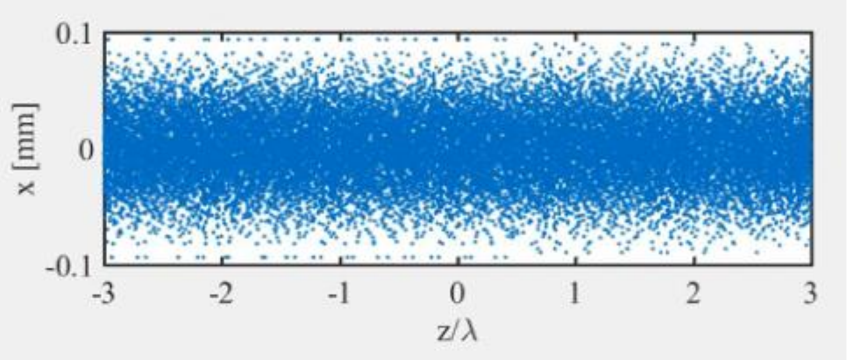
$$b = \frac{1}{N_\lambda}\sum_{n}^{N_\lambda} e^{i\psi_n(z)}$$

Radiation from an ensemble of N_λ electrons
N_λ is number of electrons in one wavelength

$$|E|^2 = |\epsilon|^2[N_\lambda + N_\lambda(N_\lambda - 1)b^2]$$

$|\epsilon|^2$ = power emitted by one electron

Unbunched beam



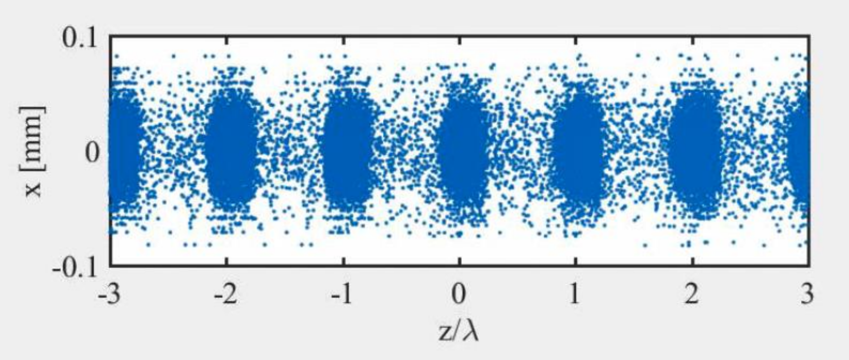
Bunching factor

$$b = 0$$

Incoherent undulator radiation

$$|E|^2_{UR} = |\epsilon|^2 N_\lambda$$

Bunched beam



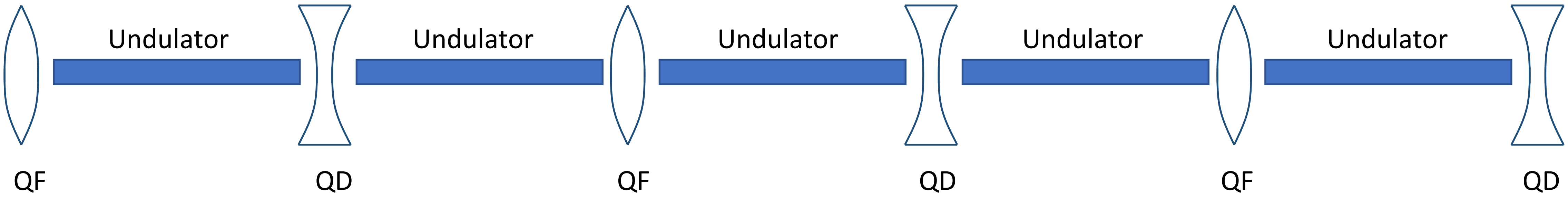
Bunching factor

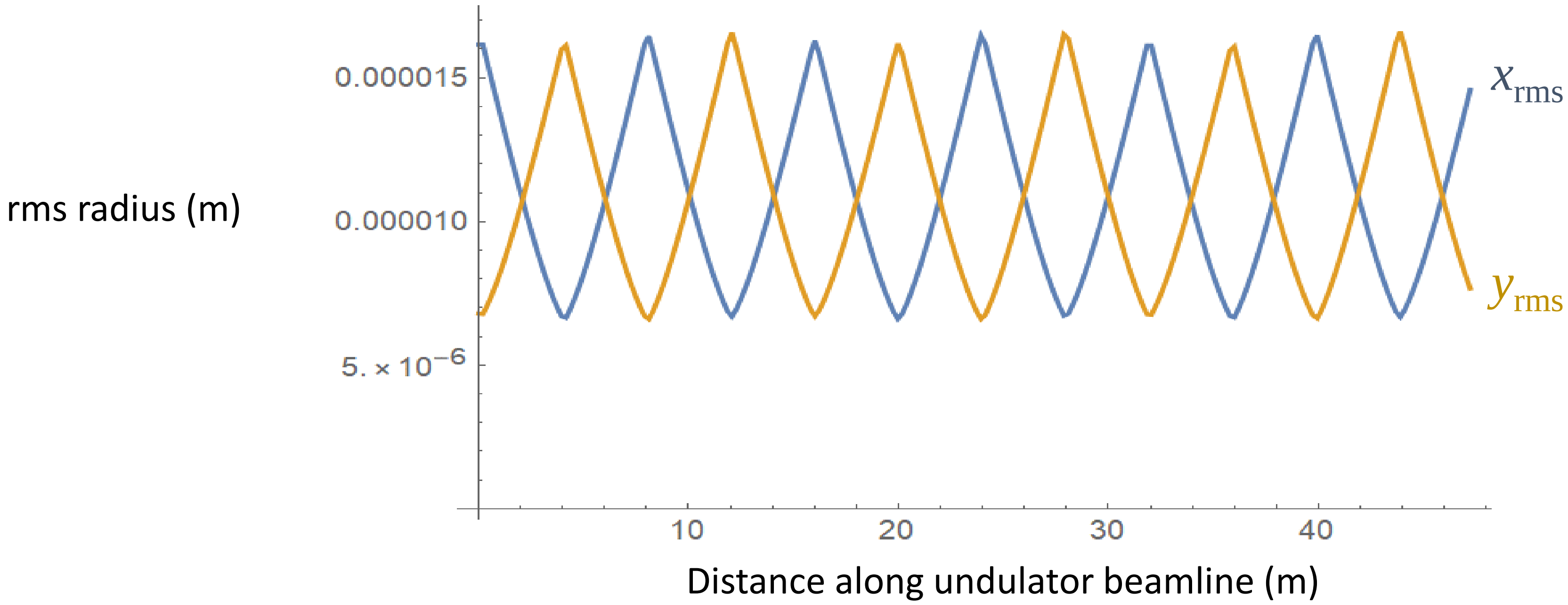
$$b \sim 1$$

Coherent FEL emission

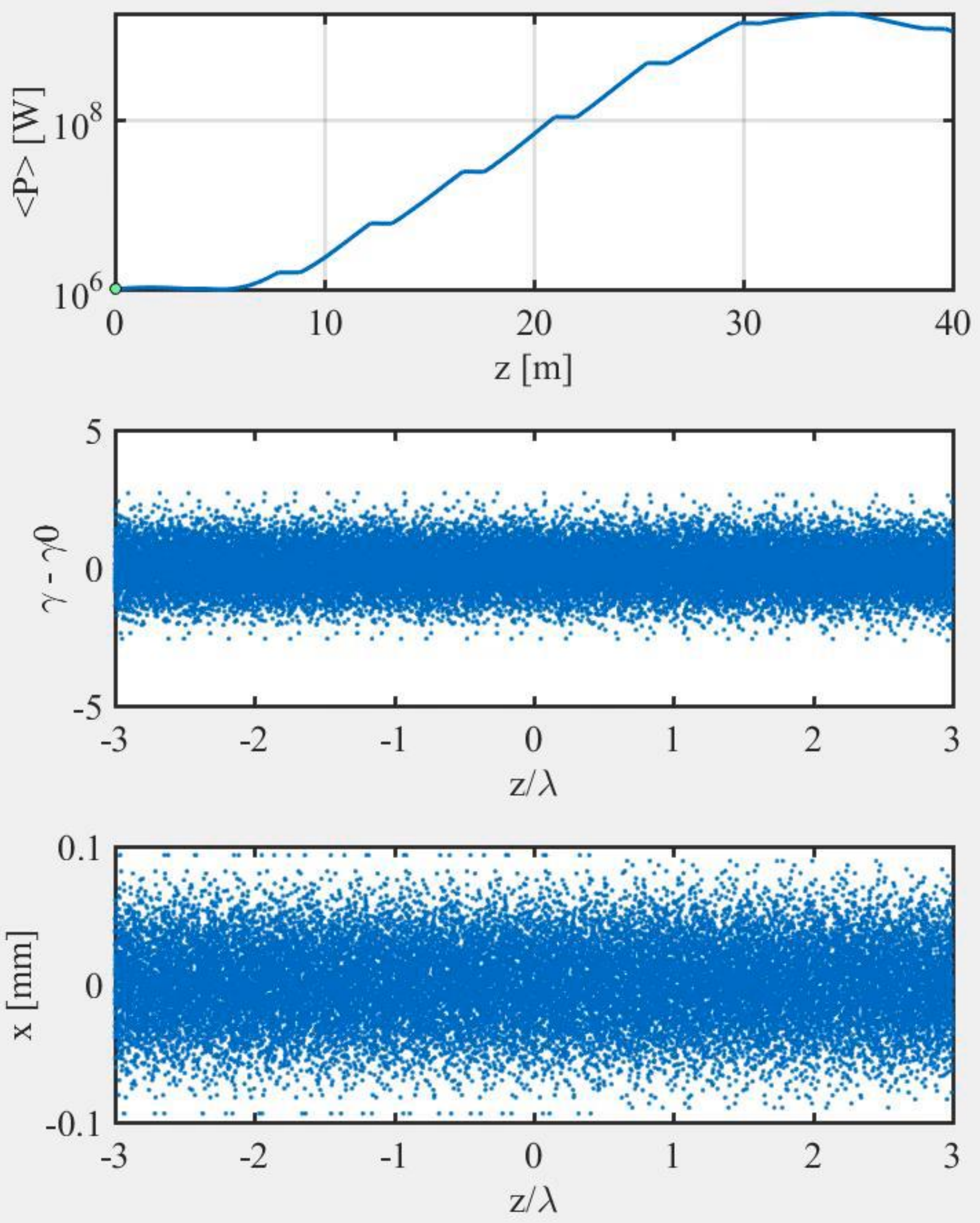
$$|E|^2_{FEL} = |\epsilon|^2\left[N_\lambda + N_\lambda^2\right]$$

Segmented Undulators in a FODO Lattice





FEL Animation



Courtesy of Gabriel Marcus

Summary of FEL Radiation Properties

- FELs are tunable sources of coherent radiation based on the same principle of operation, i.e., resonant wavelength, energy and density modulations followed by coherent bunched beam radiation, over the entire electromagnetic spectrum.
- FEL radiation, similar to undulator radiation, originates from the sinusoidal motions of electrons in undulators. However, the FEL beams have full transverse coherence, large numbers of photons per pulse and peak brightness several orders of magnitude above the peak brightness of undulator radiation.
- X-ray FELs produce nearly Gaussian coherent beams similar to a high-quality conventional laser beam but with very small angular divergence.
- The radiation generation process in an FEL is completely classical. The motions of electrons in energy-phase space can be described by two coupled differential equations similar to those describing the motions of a classical pendulum.

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