# Bubbles of Nothing in de Sitter

**Collapsing extra dimensions destroy spacetime**

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*Based on work with* **Patrick Draper**
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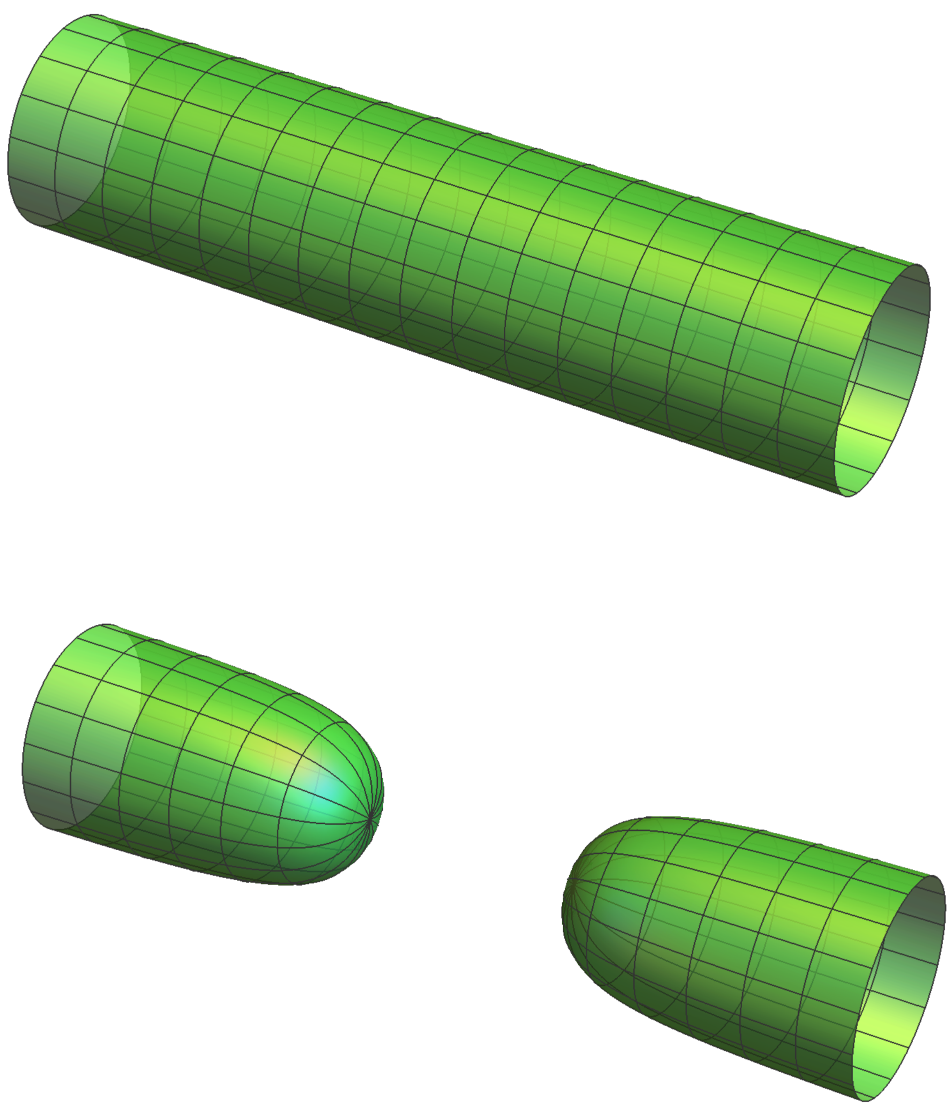
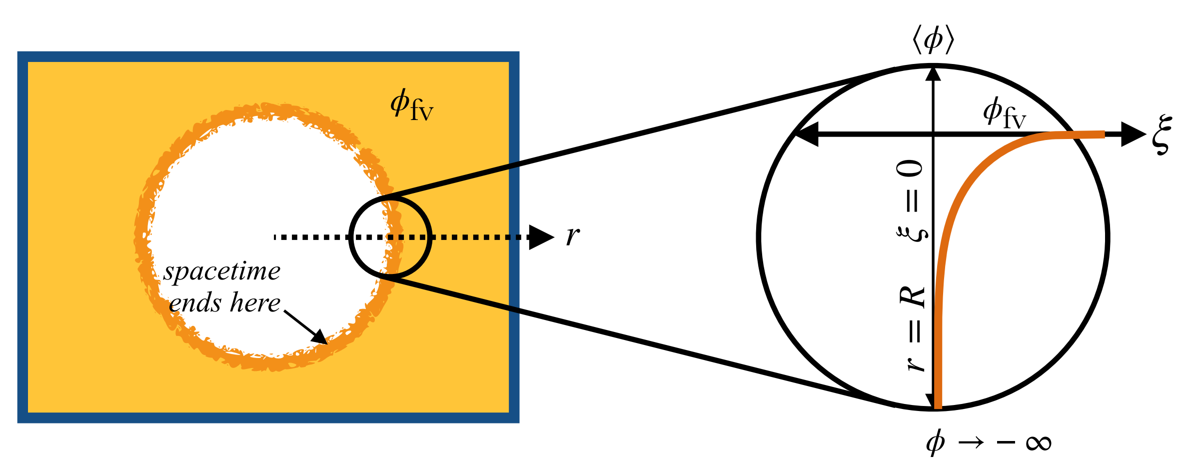




# Bubbles of Nothing (BON)

A collapsing extra dimension can destroy spacetime

This can happen even if the extra dimension is stabilized by some potential

- Given some potential $U(\phi)$, is there a BON instability?
- What does the bubble look like? What is the tunneling probability?

**This talk:**

- Review: Coleman–De Luccia tunneling, Witten BON
- Analytic bounce solutions for BON with $U(\phi)$
- Numeric methods and results

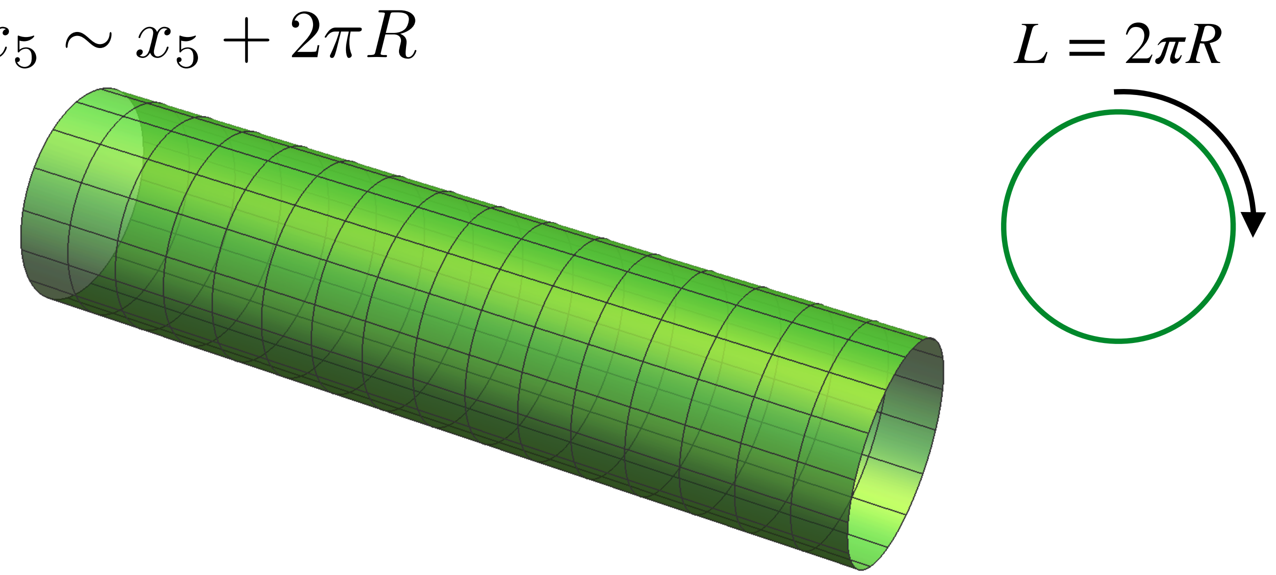
Focus on $(4+1)d$ with compact $S^1$, non-SUSY
Our methods can be applied to $(4+n)$ dimensions, e.g. with compact $S^n$

# Extra Dimensions:

- Simplest example: one compact (periodic) extra dimension

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 + dx_5^2$$

$$x_5 \sim x_5 + 2\pi R$$

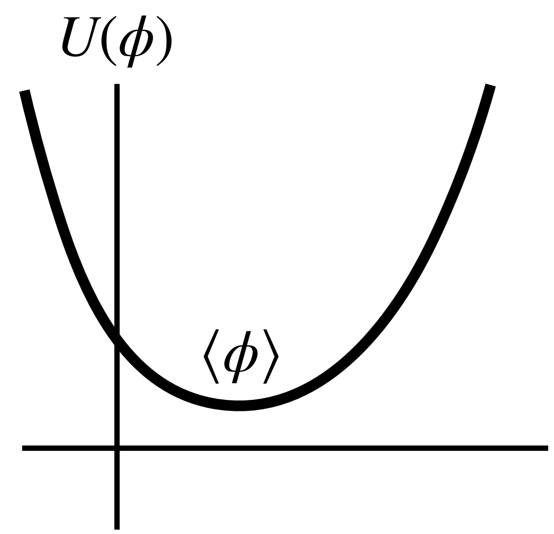



- Setting the size of $L$ dynamically: $L \longrightarrow L(\phi)$, for some modulus $\phi$.
- **Radius stabilization:** Give $\phi$ some scalar potential, $U(\phi)$, with a minimum at some $L(\langle\phi\rangle) = 2\pi R$.

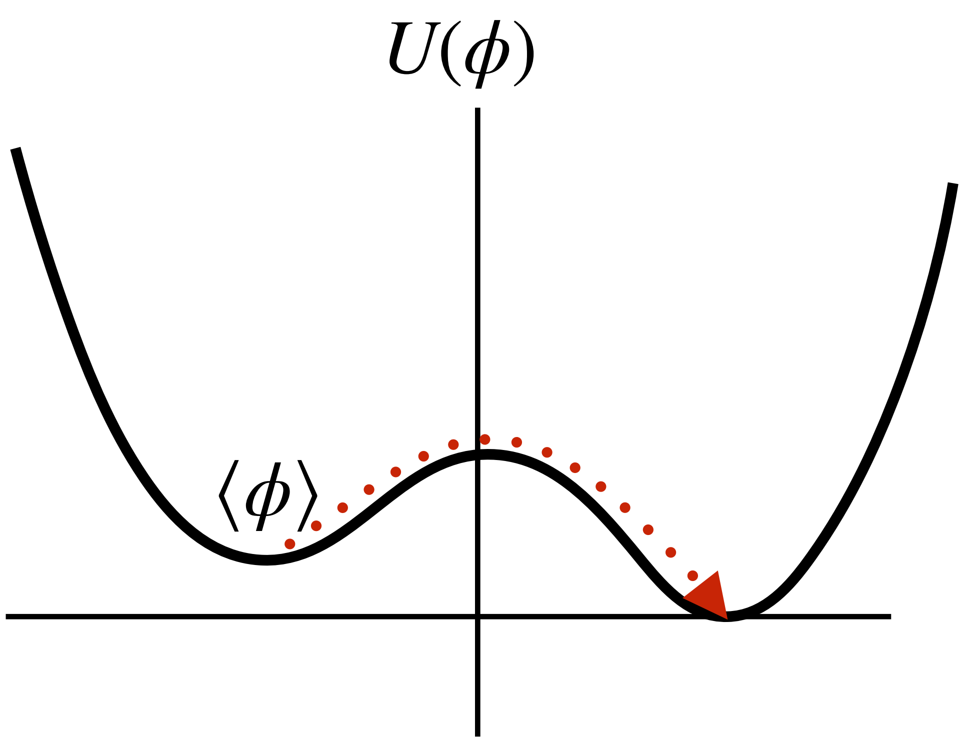
# Extra Dimensions: Connection to de Sitter

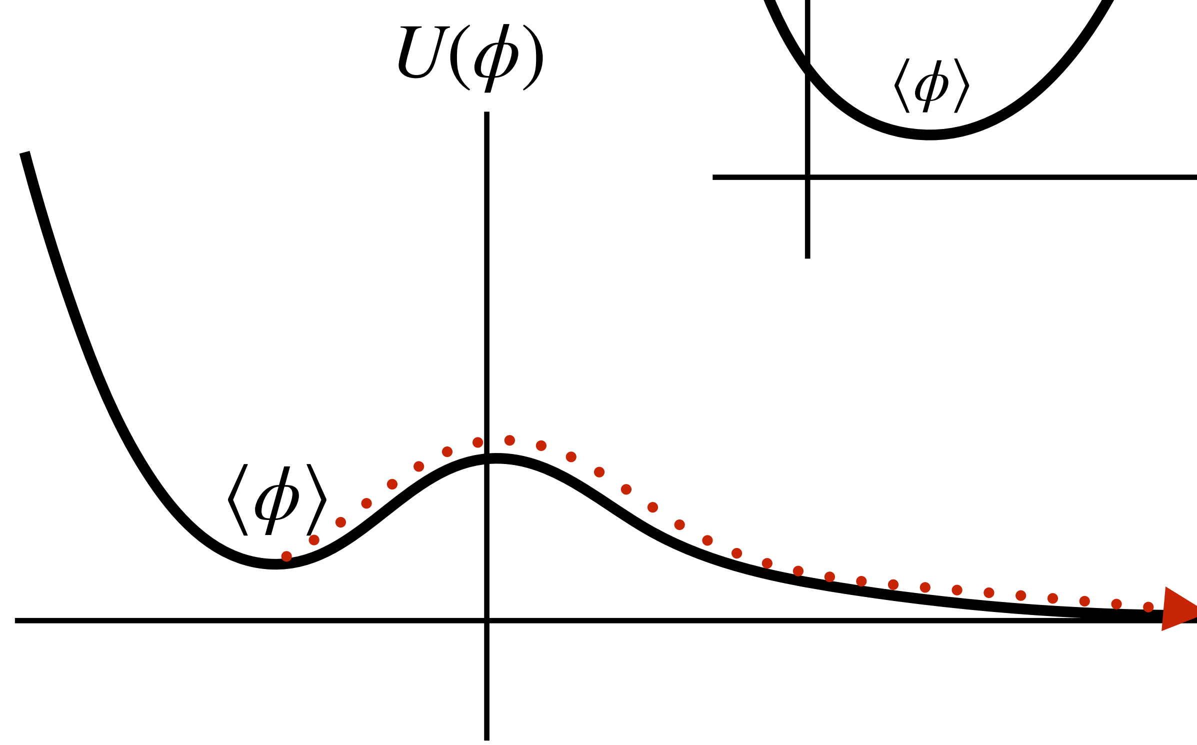
Speaking of potentials…

- The universe today appears to have a positive cosmological constant, $\Lambda \sim (\mathrm{meV})^4$. A constraint on viable potentials: $\Lambda \sim \langle U \rangle$




dS vacuum may be merely **metastable**:



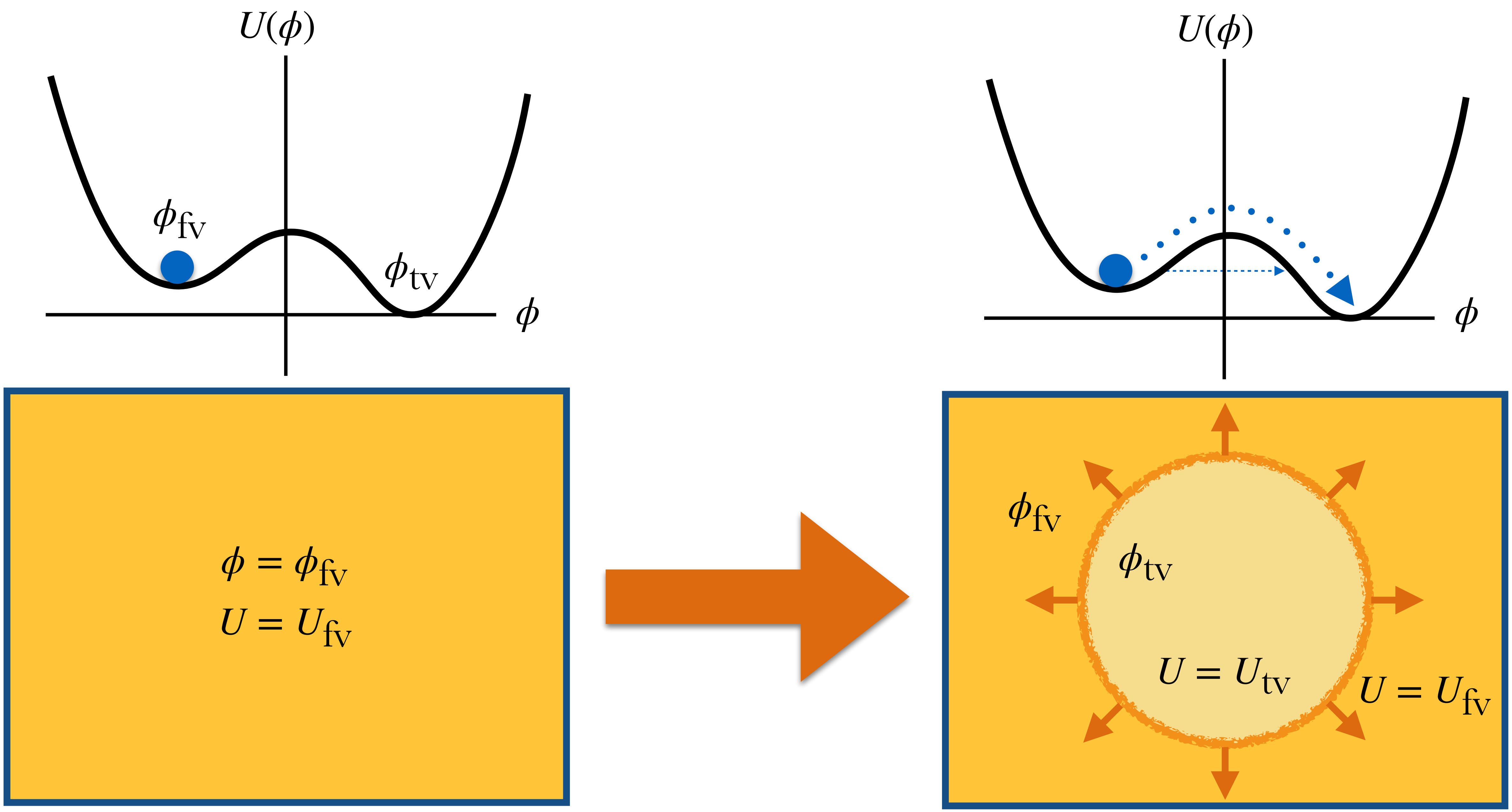




- Tunneling processes include normal (4d) Coleman-De Luccia (CDL); Hawking-Moss (HM); and (for $(4+n)$d theories) the Witten BON

S.R. Coleman and F. De Luccia, PRD 21 (1980) 3305

# Coleman–De Luccia

Tunneling proceeds by nucleating a bubble of the true vacuum, with probability $\Gamma \approx v^4 \exp(-S_E/\hbar)$




After the bubble forms, it expands at an accelerating rate.

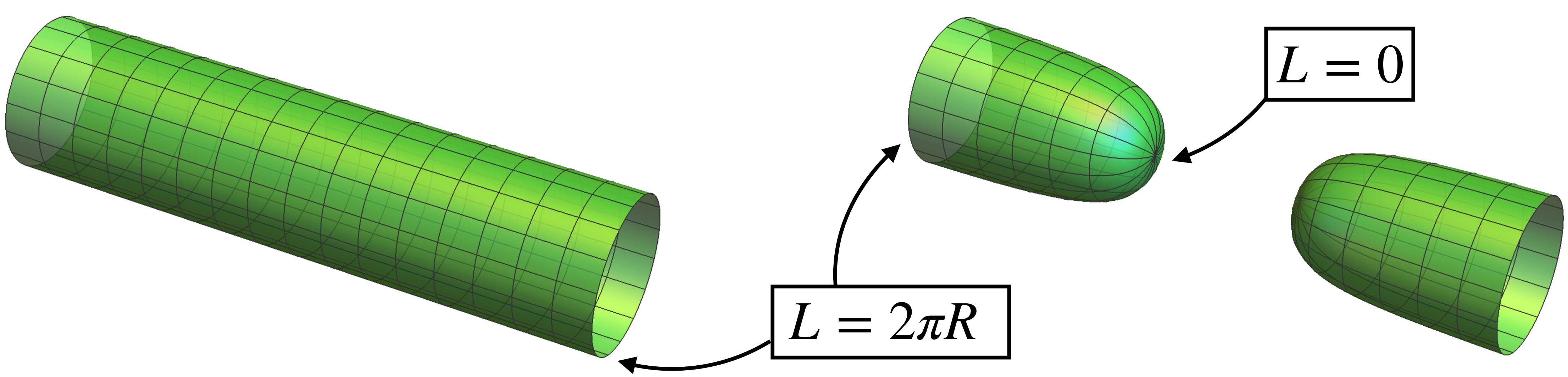
E. Witten, Nucl. Phys. B 195 (1982) 481

# Witten's Bubble of Nothing

- Previous discussion was for generic scalar $\phi$ coupled to gravity, with some $U(\phi)$.

**Exotic possibility**: if $\phi$ is the modulus that sets the size of an extra dimension, then $\phi_{\text{fv}} \to \phi_{\text{tv}}$ implies a change in $L$, from $L = 2\pi R$ to ***something else***:

- $L \to 0$: bubble of nothing (BON) $\longleftarrow$ (this talk)
- $L \to L'$: change in $R$
- $L \to L' \to \infty$: spontaneous decompactification



E. Witten, Nucl. Phys. B 195 (1982) 481

# Witten BON: as 5D gravitational instanton

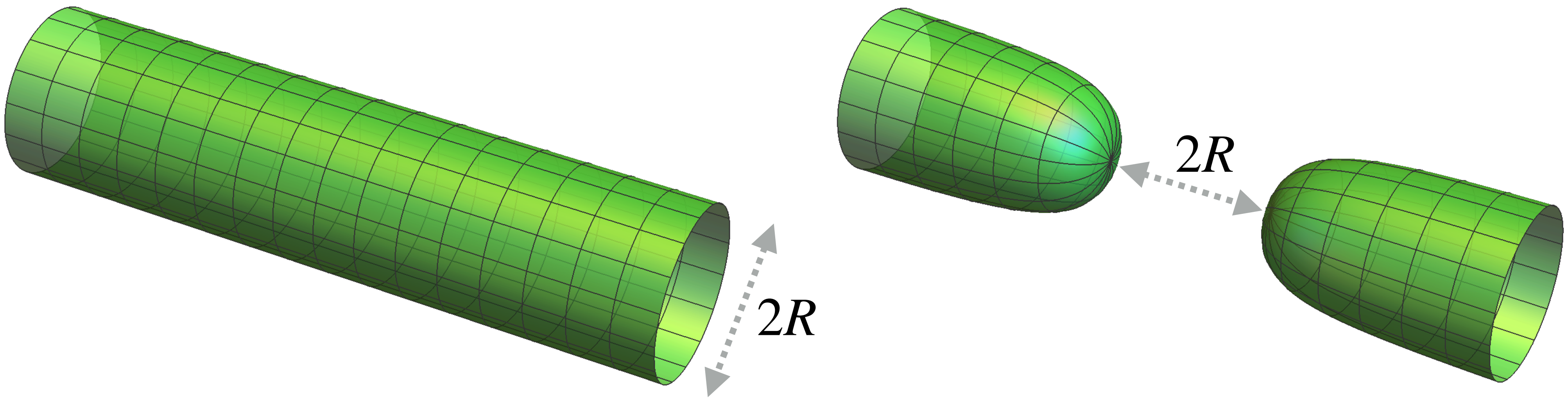
Witten BON solution is 5D Euclidean Schwarzschild:

$$\boxed{ds_5^2 = f dt^2 + f^{-1} dr^2 + r^2 d\Omega_3^2 \qquad f = 1 - (R/r)^2}$$

(Defined on $r \geq R$)

Smooth as $r \to R$ if $t$ is periodic, with $t \sim t + 2\pi R$.

- Identify $t$ with KK circle coordinate $x_5$:




- Euclidean action: $S = -\dfrac{1}{8\pi G_5} \displaystyle\int d^4x \sqrt{h}(K - K_0)$

$= \pi^2 R^2 M_p^2$ (from GHY boundary term)

# Witten BON: as CDL problem

Dimensional reduction: 5D theory $\longrightarrow$ 4D theory with massless scalar $\phi$

$$ds_5^2 = e^{-\sqrt{\frac{2}{3}}\frac{\phi}{M_p}} ds_4^2 + e^{2\sqrt{\frac{2}{3}}\frac{\phi}{M_p}} dx_5^2$$

- For Witten BON:

$$\phi = \frac{M_p}{2}\sqrt{\frac{3}{2}}\log f$$

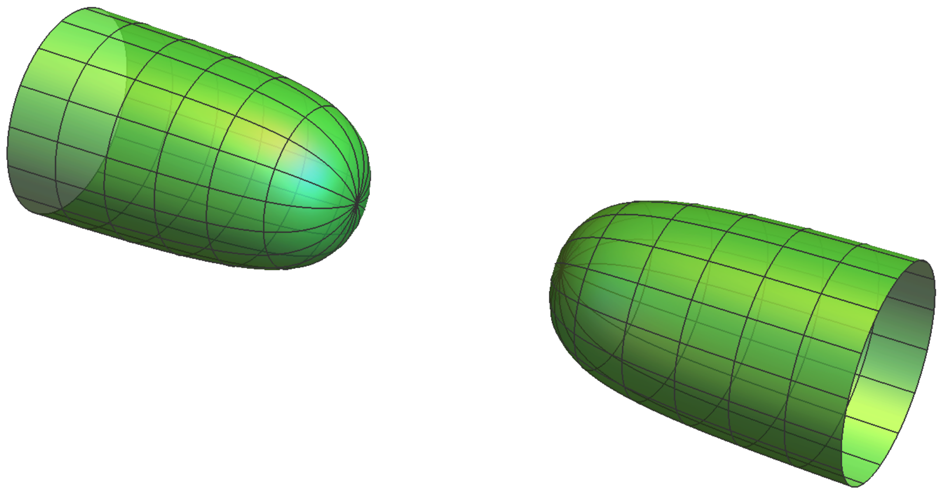
$$ds_4^2 = f^{-1/2}dr^2 + f^{1/2}r^2 d\Omega_3$$

$$f = 1 - \left(\frac{R}{r}\right)^2$$

Looks singular near $r \to R$, but actually caps off smoothly.
At the cap, $R^4 \times S^1 \to S^3 \times R^2$.

$$L(\phi) = 2\pi R \exp\left(\sqrt{\frac{2}{3}}\frac{\phi}{M_p}\right)$$

as $r \to R$: $\phi \to -\infty$, $L \to 0$.



# Witten BON: as CDL problem

- Spherically symmetric $ds_4^2$ matches CDL ansatz of $O(4)$ symmetry:

  $$ds_4^2 = f^{-1/2}dr^2 + f^{1/2}r^2 d\Omega_3^2 = d\xi^2 + \rho(\xi)^2 d\Omega_3^2$$

  for $r \geq R$ or $\xi \geq 0$

- CDL equations of motion for $\phi$ and $\rho$:

$$\boxed{\begin{aligned} &\phi'' + \frac{3\rho'}{\rho}\phi' - \frac{dU}{d\phi} = 0 \\ &\rho'^2 - \left[1 + \frac{1}{3M_p^2}\rho^2\left(\frac{1}{2}\phi'^2 - U(\phi)\right)\right] = 0 \end{aligned}}$$

if $U = \text{const}$,
$\rho^3\phi' = \text{const}$.

- Witten BON has the same EOM, just with $U = 0$.
  The EOM can be integrated, to find exact solutions for $\phi(\xi)$ and $\rho(\xi)$.

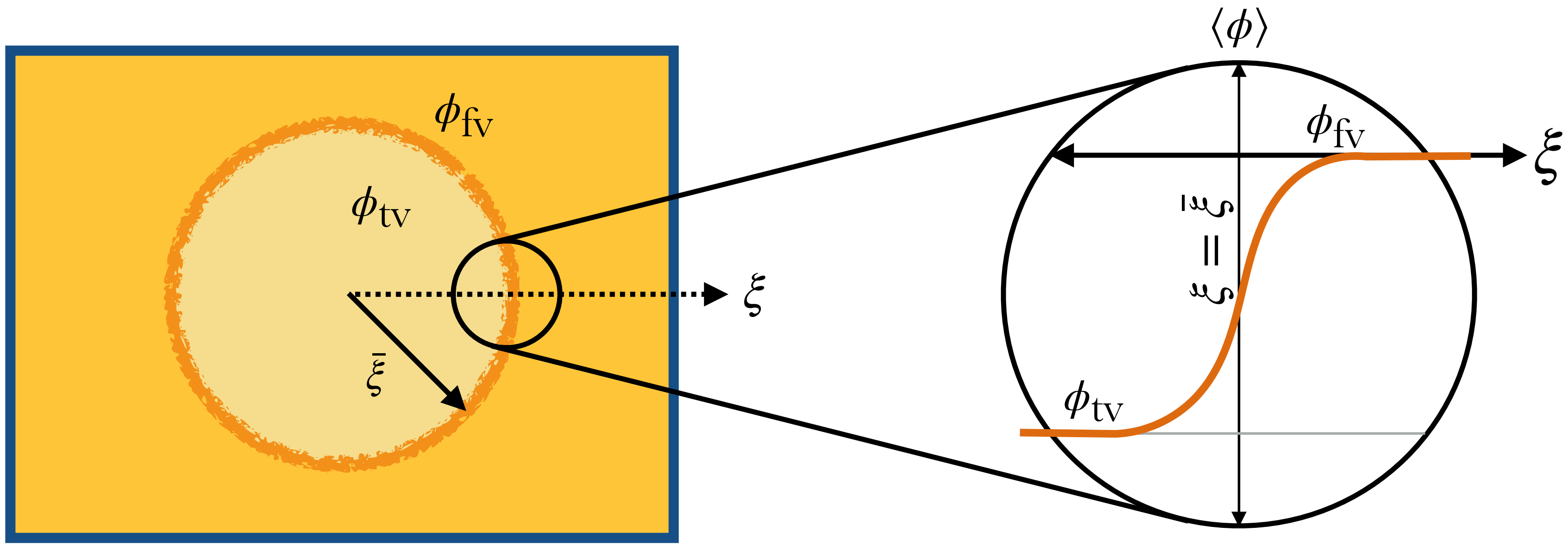
- Main difference between BON and CDL: **initial conditions**

M. Dine, P.J. Fox, E. Gorbatov, JHEP 09 (2004) 037 [hep-th/0405190]

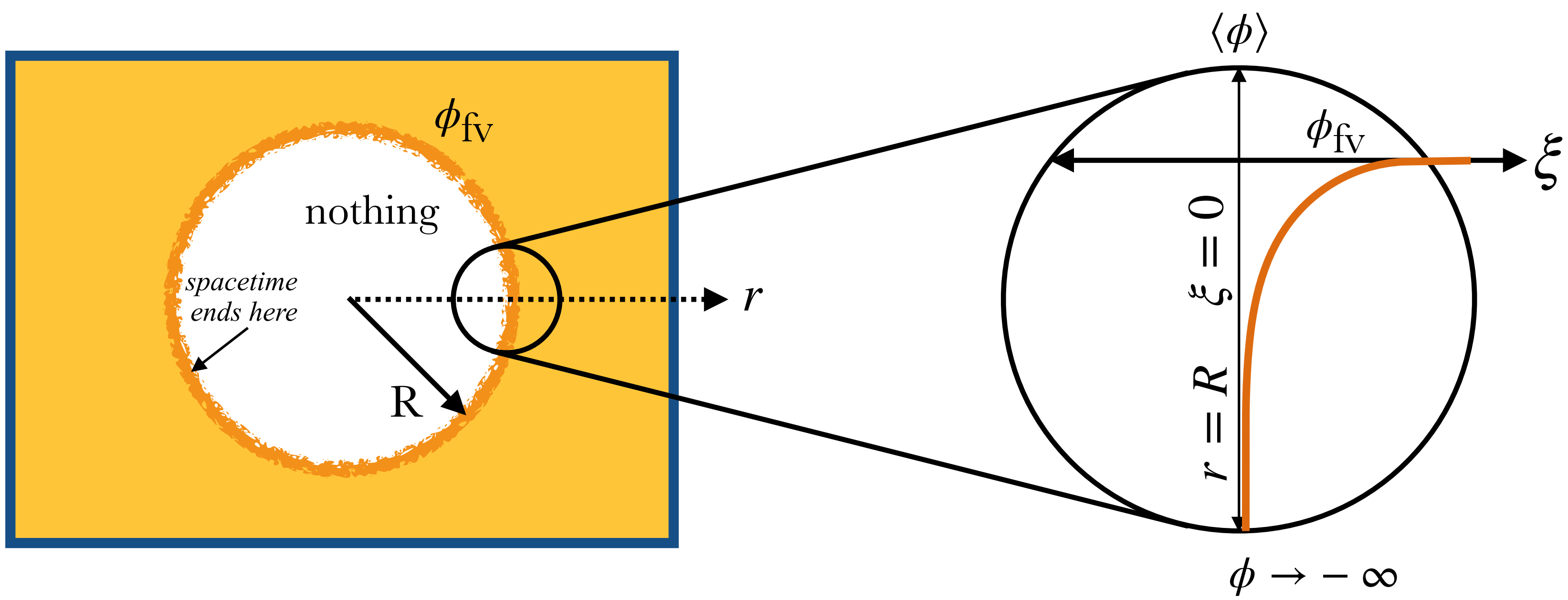
# Witten BON: as CDL problem

- CDL: near $\xi \sim 0$: $\phi = \phi_0$, $\phi' = 0$
- BON: near $\xi \sim 0$ $(r \simeq R)$: $\phi \to -\infty$, $\phi' \to +\infty$

CDL: bubble interior is false vacuum




BON: spacetime ends at $r = R$ No interior



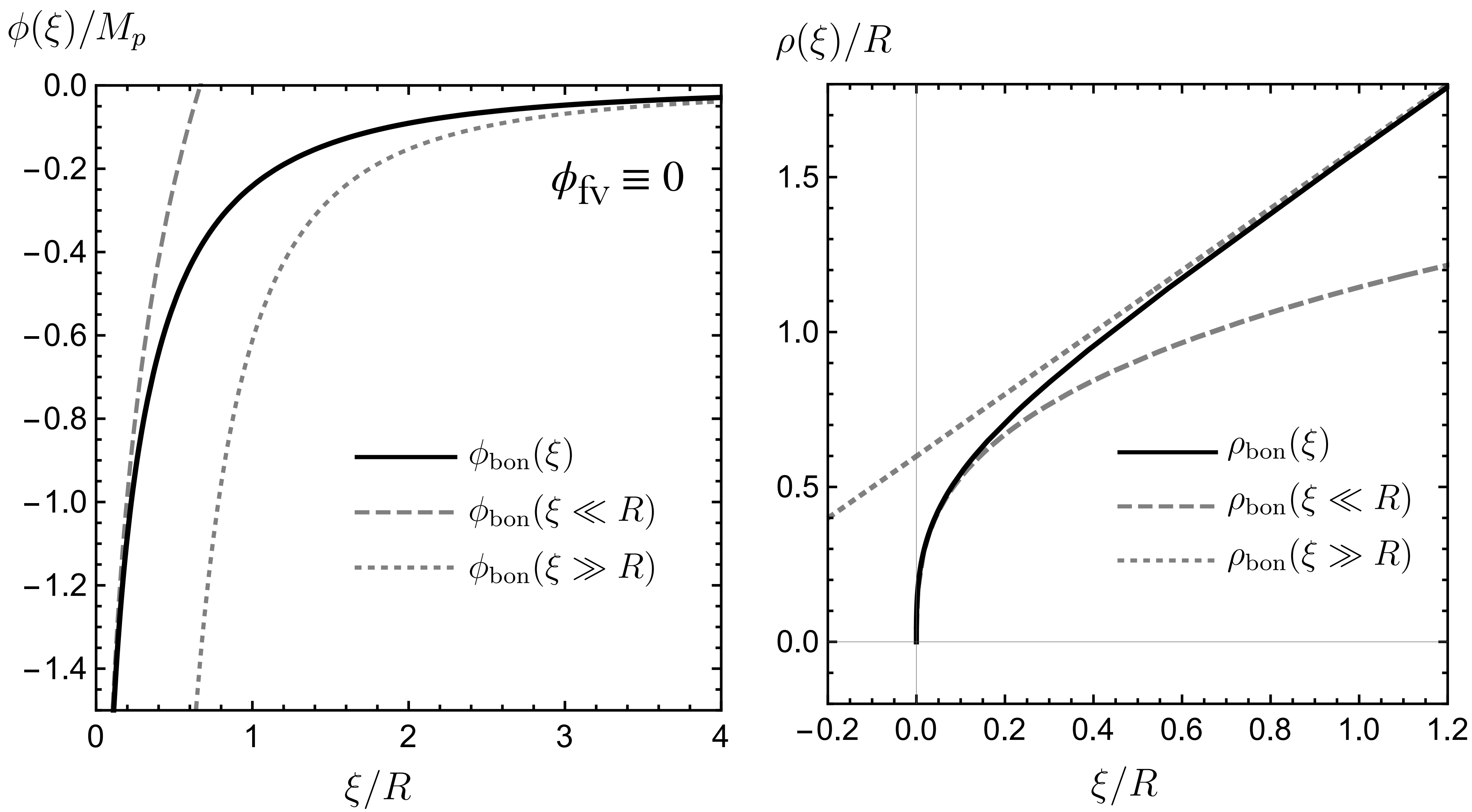
# Witten BON: as CDL problem

- Exactly solvable. Near the horizon,

$$\boxed{\rho^3\phi' = \text{const}}$$

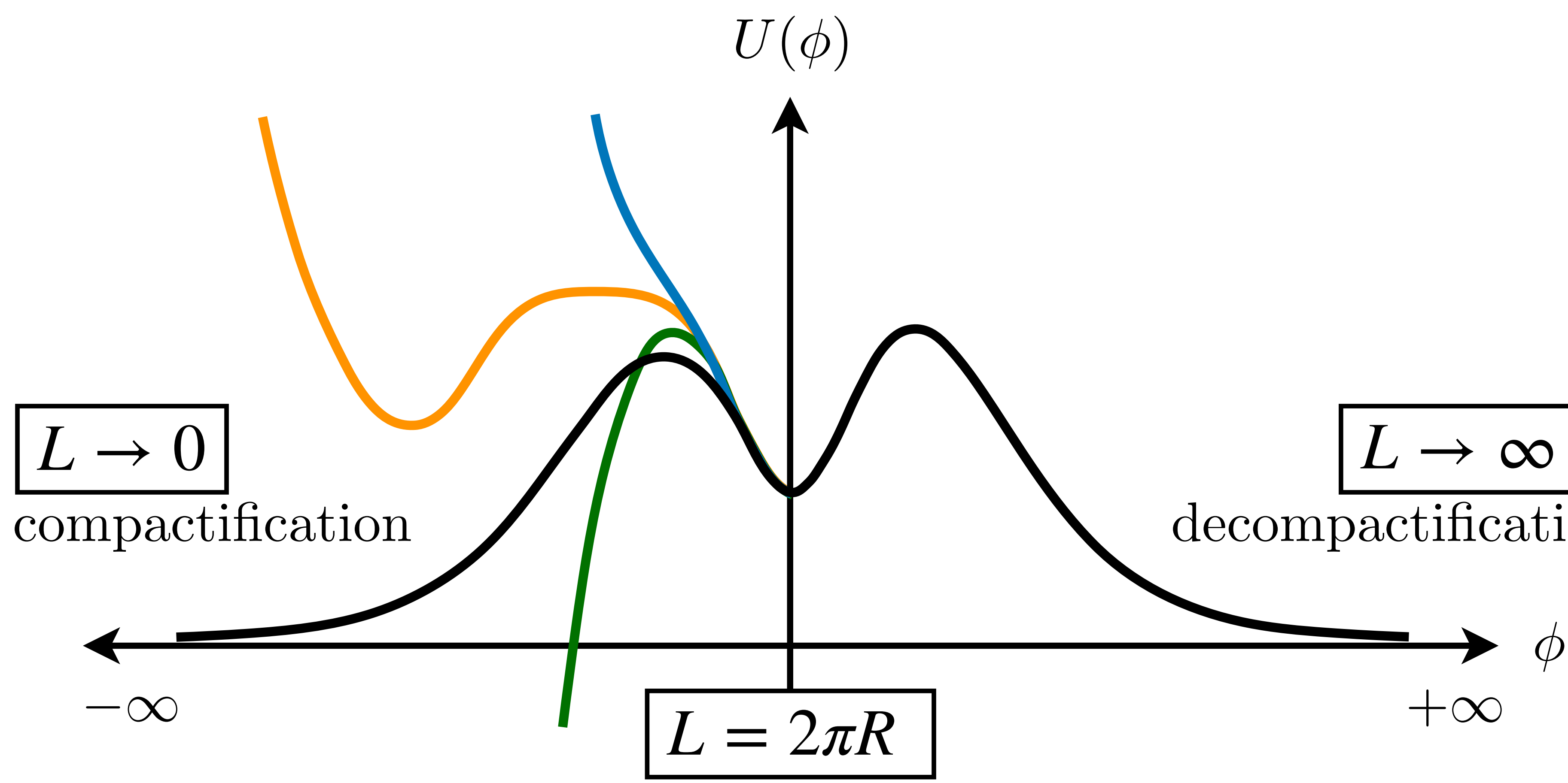
$$\boxed{\phi_{\text{bon}}(\xi \ll R) \simeq M_p\sqrt{\frac{2}{3}}\log\left(\frac{3\xi}{2R}\right), \qquad \rho_{\text{bon}}(\xi \ll R) \simeq R\left(\frac{3\xi}{2R}\right)^{1/3}, \qquad \rho^3\phi' = \sqrt{\frac{3}{2}}M_pR^2}$$

- Far away from the bubble, $\phi \approx \phi_{\text{fv}}$ is constant, and $\rho \approx \xi$ is flat



# BON for the stabilized modulus

- The CDL formalism provides a natural way to include the effects from modulus stabilization and the de Sitter vacuum: $U(\phi)$




- Use CDL equations of motion with BON boundary conditions:

$$\phi(\xi \sim 0) \simeq M_p \sqrt{\frac{2}{3}} \log \frac{3\xi}{2R}$$

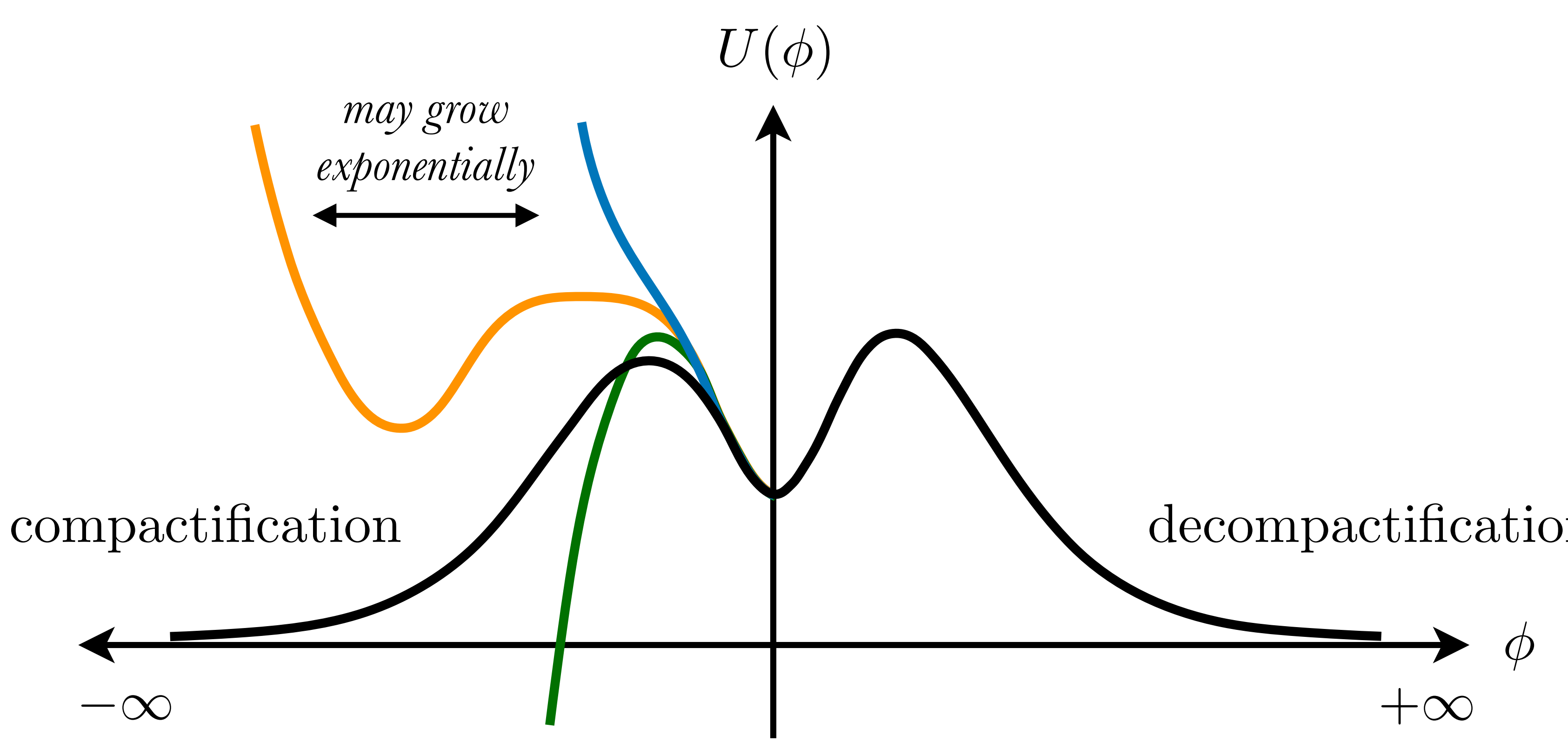
$$\rho(\xi \sim 0) \simeq R \left(\frac{3\xi}{2R}\right)^{1/3}$$

$$\phi'' + \frac{3\rho'}{\rho}\phi' - \frac{dU}{d\phi} = 0$$

$$\rho'^2 - \left[1 + \frac{1}{3M_p^2}\rho^2 \left(\frac{1}{2}\phi'^2 - U(\phi)\right)\right] = 0$$

# BON for the stabilized modulus

- Unlike CDL, the BON does not require $U(\phi) < U_{\text{fv}}$ at any point. $U(\phi_{\text{fv}})$ can be a global minimum, for example
- In fact, $U(\phi)$ can grow **exponentially large** as $\phi \to -\infty$, as long as it grows more slowly than $U \propto \exp\left(-\sqrt{6}\phi/M_p\right)$.
- For $M_p/\phi \cdot \log U > -\sqrt{6}$, the leading small $\xi$ behavior is unchanged

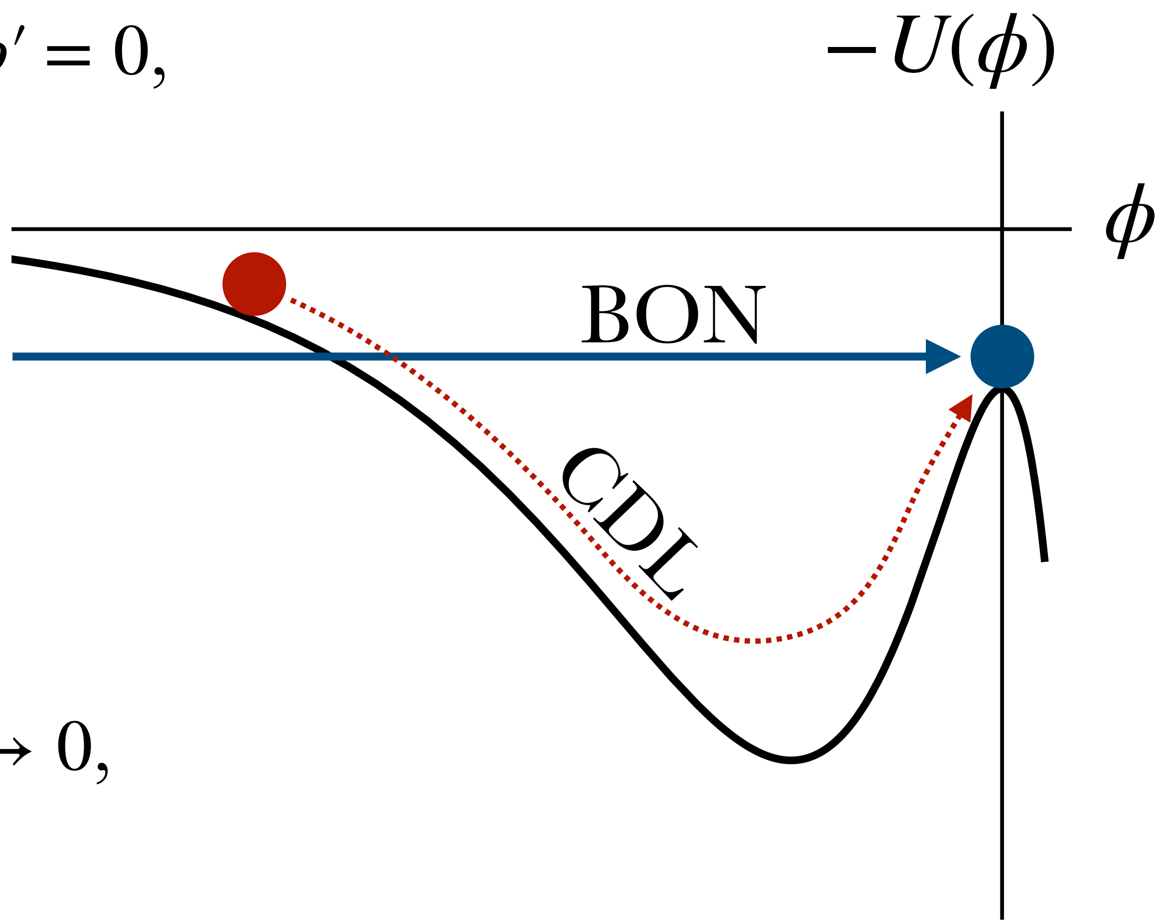


# Bubbles and bounce solutions

Solving the CDL EOM is similar to a kinematic problem in classical mechanics: a particle rolling in the inverted potential, with some friction

- Normal CDL: start at rest, with $\rho^3\phi' = 0$, at some $\phi(0) = \phi_0$

- BON: start with infinite velocity at $\phi(0) \to -\infty$, with $\rho^3\phi' \propto R^2$



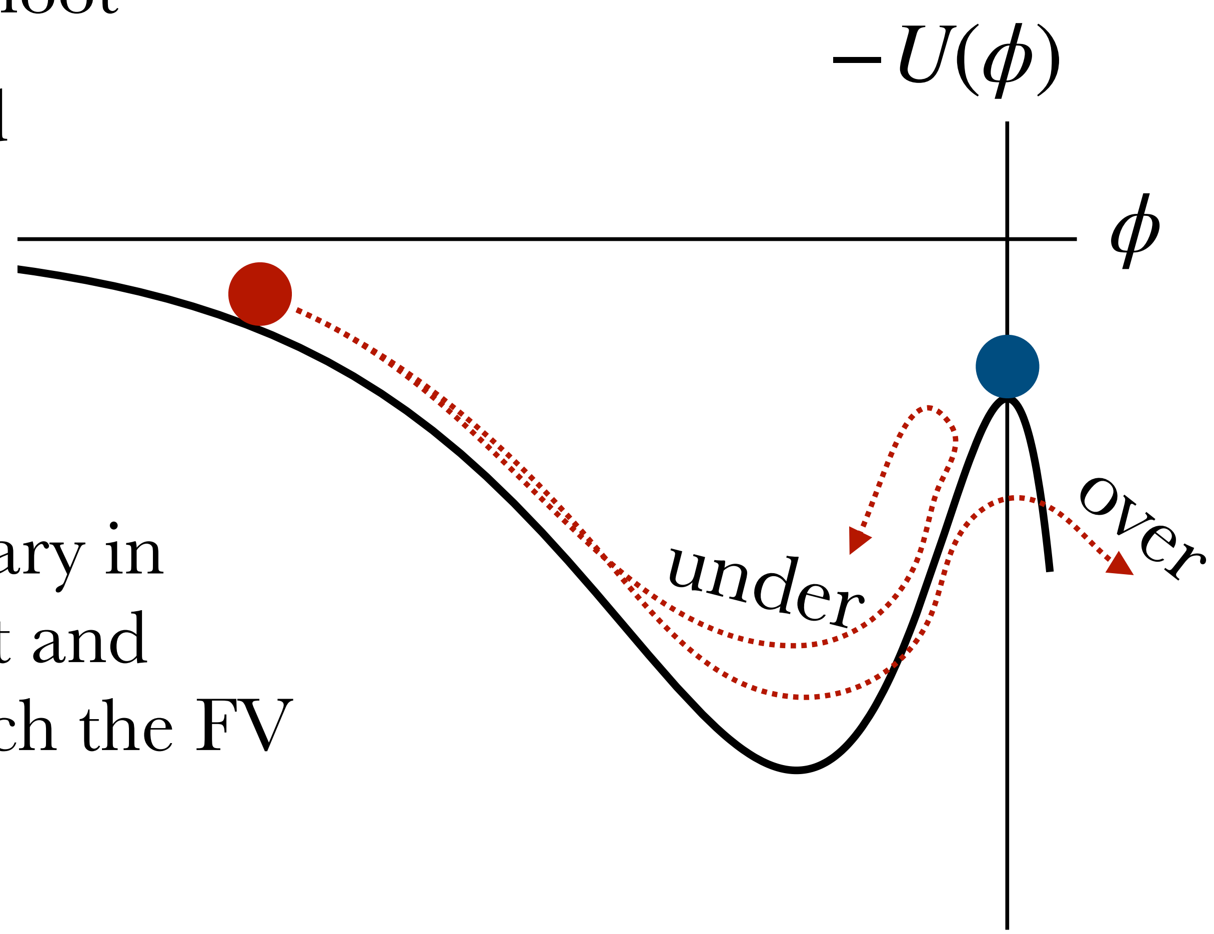

CDL and BON solutions end with $\phi' \to 0$, and with $\phi$ in the vicinity of $\phi \sim \phi_{\rm fv}$.

# Bubbles and bounce solutions

Generic initial conditions either approach $\phi \to \phi_{\mathrm{fv}}$ with the wrong $\phi' \neq 0$, or never approach $\phi \to \phi_{\mathrm{fv}}$ at all.

Simple solution method: "point-and-shoot"

- **undershoot** solutions turn around before reaching the FV
- **overshoot** solutions pass the FV with too much "speed", $\phi' \neq 0$
- **bounce** solutions lie on the boundary in parameter space between overshoot and undershoot solutions. They approach the FV with $\phi' \to 0$



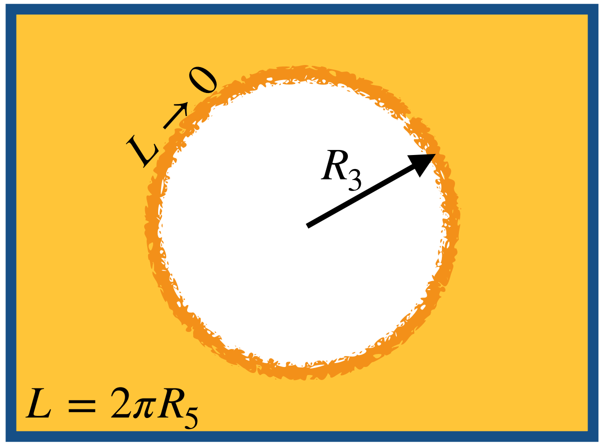

(This is how our numeric calculation works)

# Bubbles and bounce solutions

- For the $U = 0$ Witten solution, the size of the bubble (in 3D space) and the radius of the KK extra dimension are equal: both radii are $R$.
- With $U \neq 0$, these two quantities are no longer necessarily equal:

$R \to R_5$ : proper radius of KK direction

$R \to R_3$ : size of bubble, $R_3 \equiv \eta R_5$




- BON initial conditions:

$$\phi(\xi) \simeq M_p \sqrt{\frac{2}{3}} \log\left(\frac{3\xi}{2R_5}\right), \qquad \text{and} \qquad \rho(\xi) \simeq \eta R_5 \left(\frac{3\xi}{2R_5}\right)^{1/3}$$

- Treating $R_5$ as a fixed property of the UV theory, the bubble size $R_3 = \eta R_5$ defines the initial conditions, and determines whether the eventual solution is of overshoot, undershoot, or bounce type

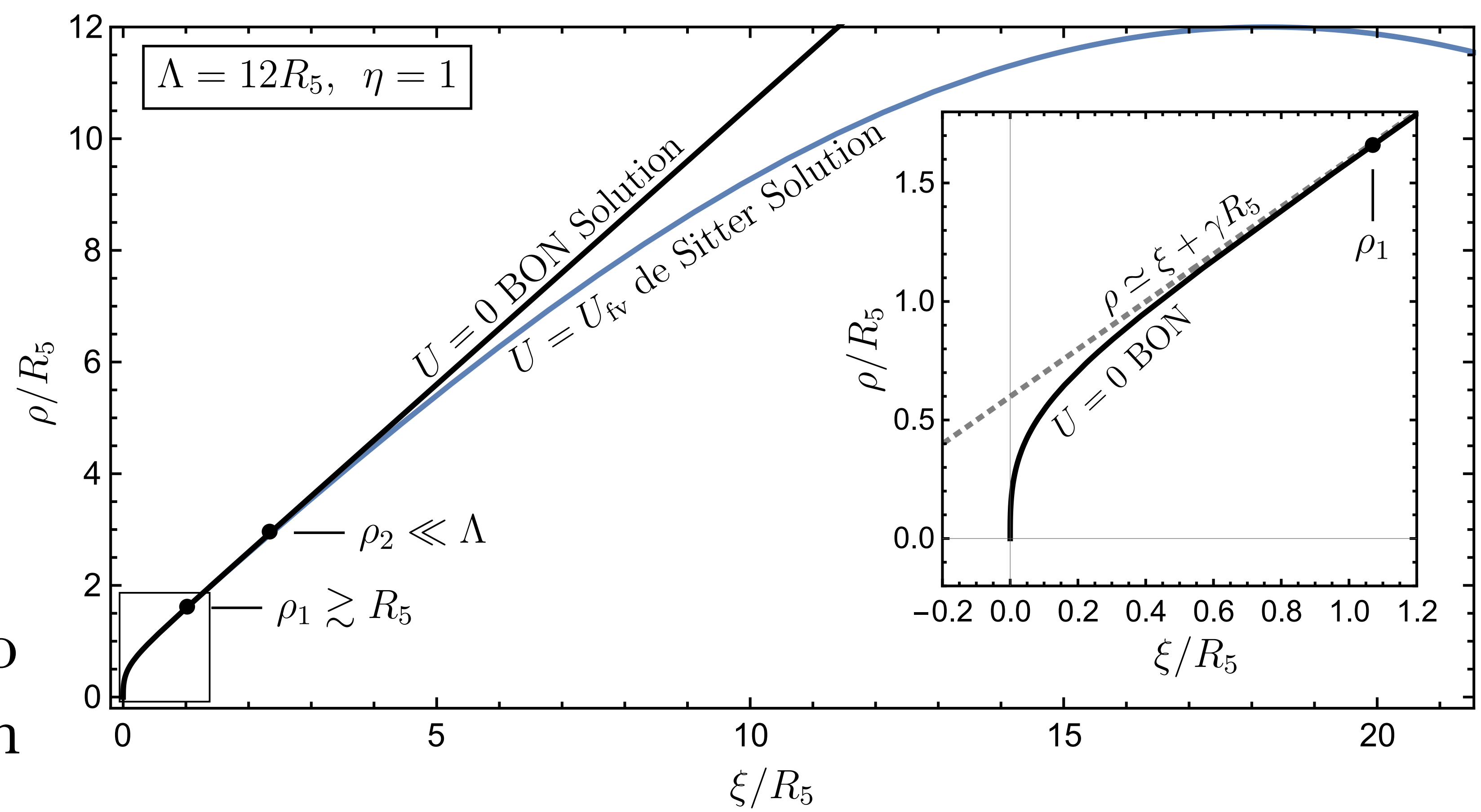
# Solving the EOM: Analytic method

1. as $\xi \to 0$, the U-dependent terms are small compared to $\phi''$ and $\phi'\rho'/\rho$
2. when U is constant, there is an exactly conserved quantity, $(\rho^3\phi')$.
3. when $\rho' \simeq 1$, the EOM may solved exactly, even for nontrivial $U$

$$\boxed{\frac{1}{\rho^3}\frac{d}{d\xi}\left(\rho^3\phi'\right) = \frac{dU}{d\phi} \qquad \rho'^2 = 1 + \frac{3\rho^2}{M_p^2}\left(\frac{1}{2}\phi'^2 - U(\phi)\right)}$$

Split into three regions:

1. inner "core": $U(\phi)$ is unimportant, $\rho^3\phi' \simeq$ const $\propto R^2$
2. bubble "wall": $U(\phi)$ causes $\rho^3\phi'$ to decrease, $\rho^3\phi' \to 0$
3. bubble exterior: $\phi \simeq \phi_{fv}$ asymptotes to de Sitter false vacuum



# Solving the EOM: Necessary assumptions

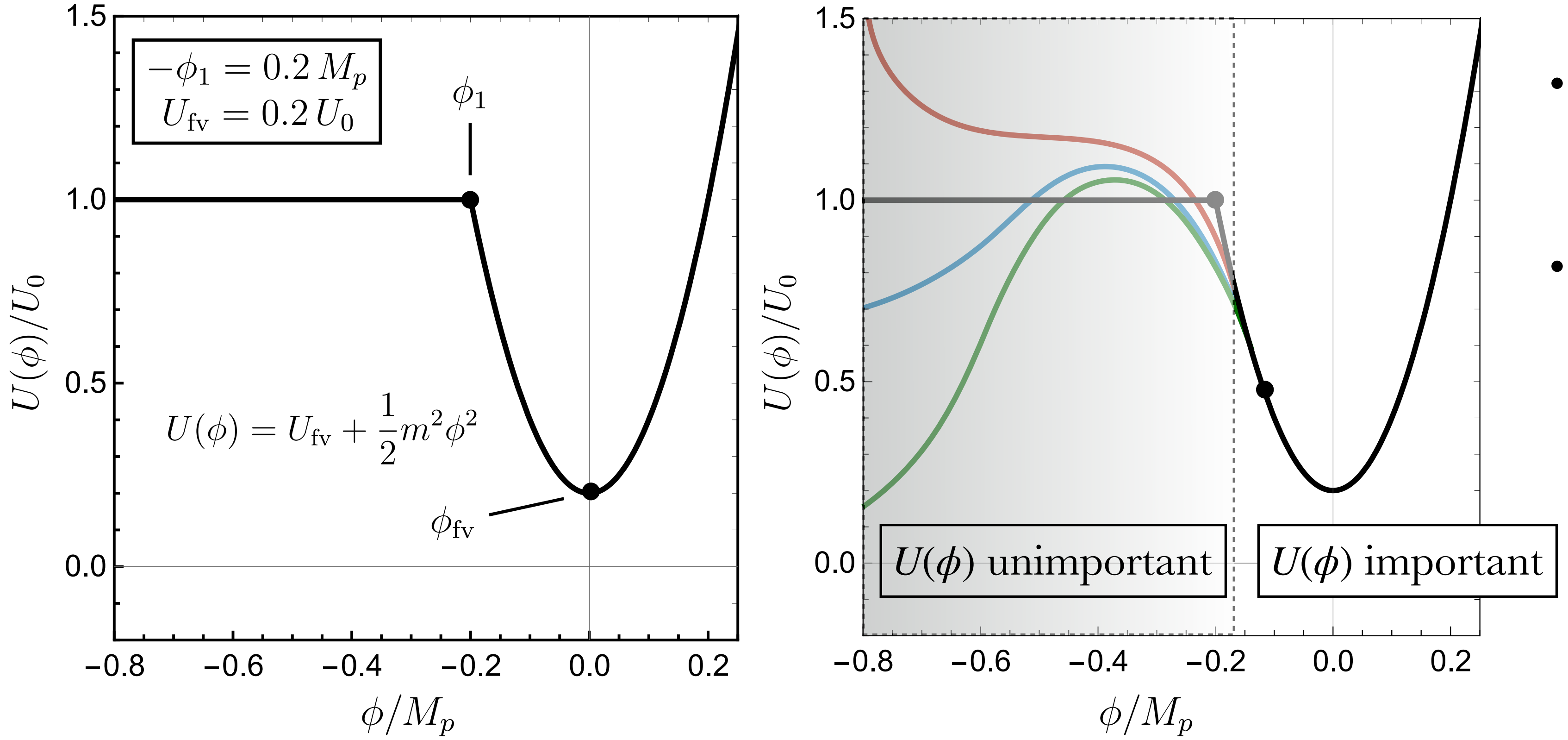
For BON existence:

1. $U(\phi \to -\infty)$ does not grow faster than $U \propto \exp\left(-\sqrt{6}\phi/M_p\right)$

For analytic solubility:

2. $\rho' \simeq 1$ in the transition region (implying $U_0 R_5^2 \lesssim M_p^2$)
3. $U(\phi)$ is approximately quadratic (or linear) in the transition region

Example (2105.10507): piecewise quadratic $U(\phi) = U_{\text{fv}} + \frac{1}{2}m^2\phi^2$

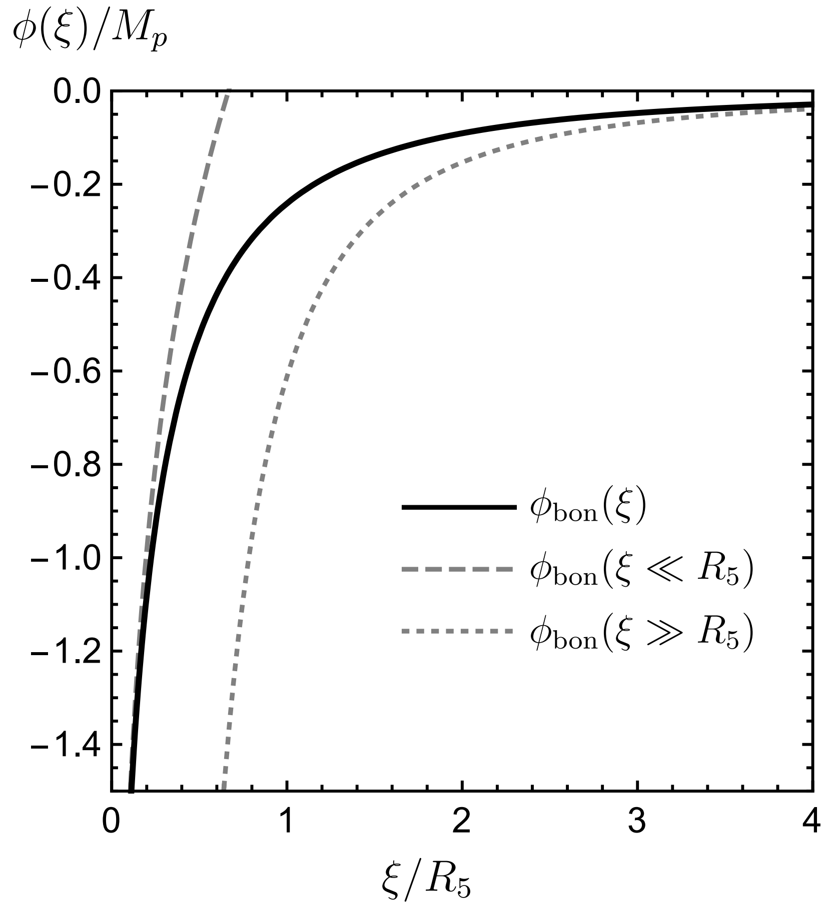



- $U(\phi)$ becomes important around $\phi \sim -m^2 R_5^2 M_p$
- If a $U(\phi)$ is $\sim$ quadratic for $\phi \gtrsim -m^2 R_5^2 M_p$, then it is well approximated by the piecewise model

# Solving the EOM: Analytic results

- Bubble "core": $\phi(\xi) \approx \phi_{\text{bon}}$

$$\phi_{\text{bon}}(\xi \gg R_5) \simeq -\frac{M_p}{2}\sqrt{\frac{3}{2}}\left(\frac{R_5}{\xi}\right)^2$$




- Bounce solution: only $K_1(z)$, no $I_1(z)$
- This selects the "correct" value of $R_3$

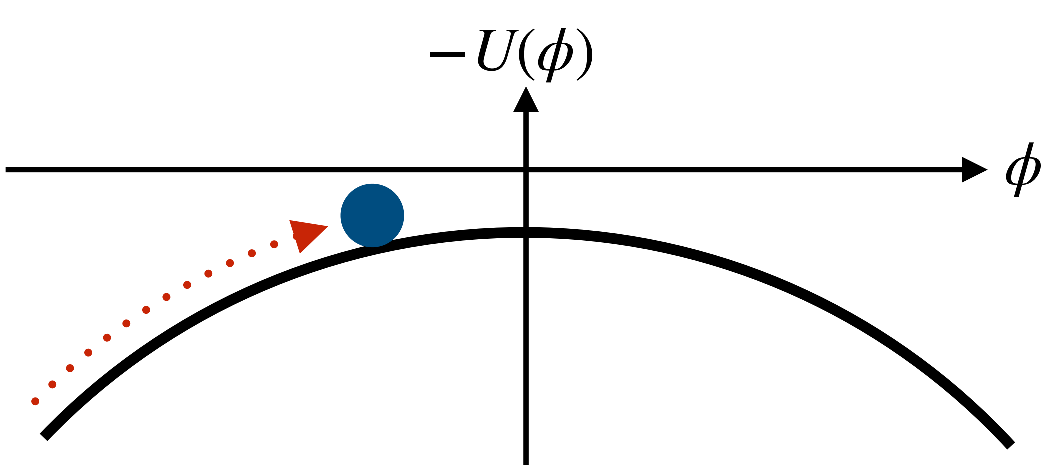
- Bubble "wall": $\phi(\xi) \approx$ Bessel $K_1$

$$\phi(\xi \ll m^{-1}) \approx -M_p\sqrt{\frac{3}{2}\frac{R_5^2}{\xi^2}}$$

$$\phi(\xi \gg m^{-1}) \propto -\frac{e^{-m\xi}}{(m\xi)^{3/2}}$$

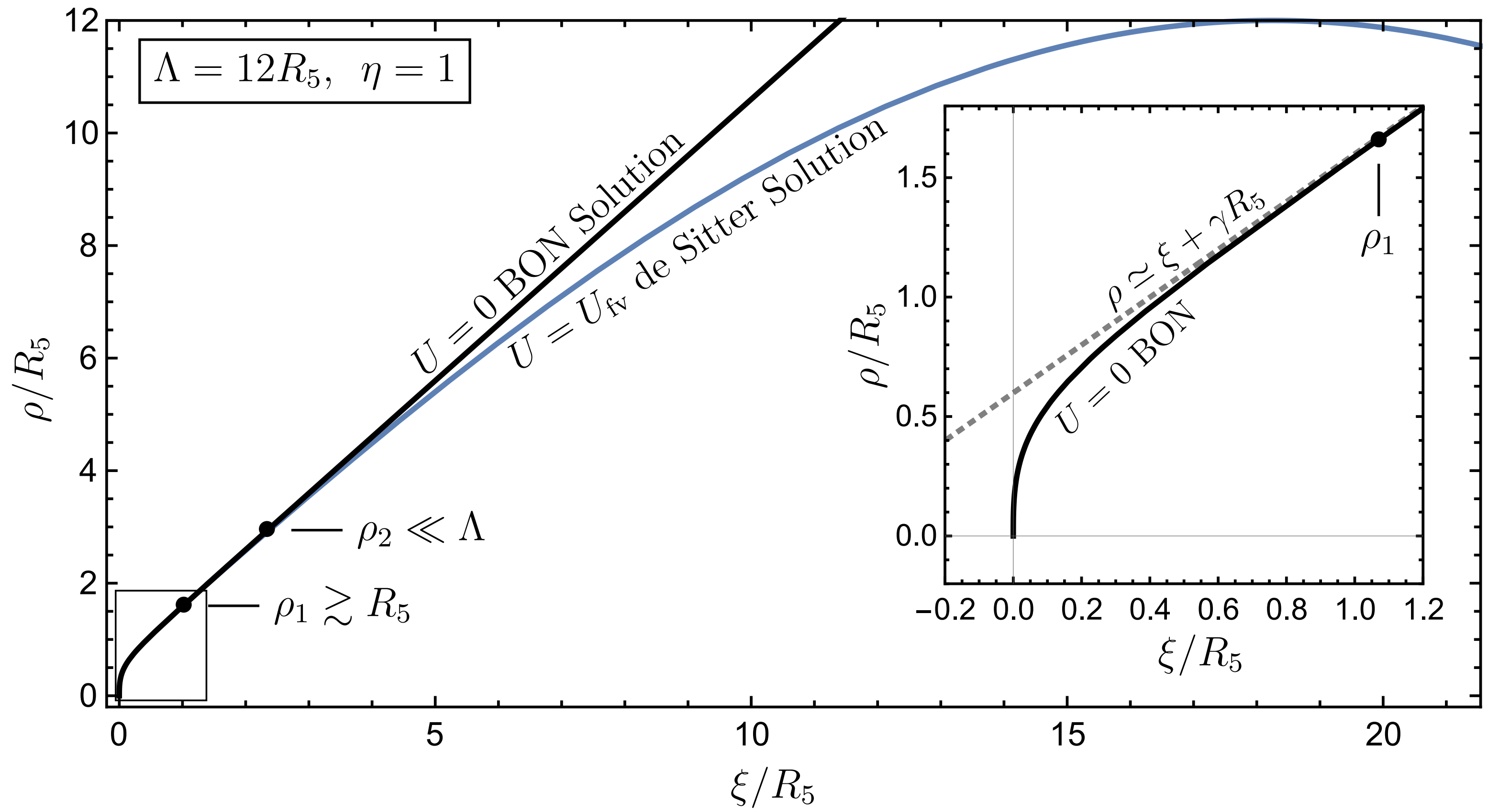
Interpolates between Witten solution ($U = 0$) and an exponentially damped approach to the false vacuum

- Outside the bubble: $\phi(\xi) \approx \phi_{\text{fv}}$ for $\xi \gg \mathcal{O}(\text{few}) \times m^{-1}$



# Solving the EOM: Analytic results

- Solution for $\rho(\xi)$ has the form: BON $\to$ ($\rho' \simeq 1$) $\to$ $\rho_{\mathrm{dS}}$




$$\boxed{\rho_{\mathrm{dS}} = \Lambda \sin\left(\frac{\xi}{\Lambda}\right)}$$

$$\Lambda \equiv \sqrt{\frac{3M_p^2}{U_{\mathrm{fv}}}}$$

$$\Lambda \ggg R_5$$

- Now that we know $\phi(\xi)$ and $\rho(\xi)$, we can calculate the action…

# Solving the EOM: Action

- Euclidean action: $S_E = -\dfrac{1}{16\pi G_{n+4}} \displaystyle\int \sqrt{g}\,(\mathcal{R} - 2\Lambda_{n+4})$

  dimensional reduction: $S_E = \displaystyle\int d^4x\sqrt{g}\left\{-\frac{M_p^2}{2}\mathcal{R} + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{M_p}{\sqrt{6}}\Box\phi + U(\phi)\right\}$

- On-shell, reduces to the CDL action plus an additional contribution:

$$
\begin{aligned}
\Delta S &\equiv S_E^{(\mathrm{bon})} - S_E^{(\mathrm{dS})} \\
&= \pi^2 M_p\sqrt{\frac{2}{3}}\rho^3\phi'\Big|_{\xi=0}^{\xi=\xi_{\max}} - 2\pi^2\int_0^{\xi_{\max}} d\xi\,\rho^3 U + 2\pi^2\int_0^{\pi\Lambda} d\xi\,\rho_{\mathrm{dS}}^3 U_{\mathrm{fv}} \\
&= \underbrace{\pi^2 M_p^2\frac{R_3^3}{R_5}}_{\text{new BON contribution}} \underbrace{- 2\pi^2\int_0^{\xi_{\max}} d\xi\,\rho^3 U + 2\pi^2\int_0^{\pi\Lambda} d\xi\,\rho_{\mathrm{dS}}^3 U_{\mathrm{fv}}}_{\text{regular CDL action}}
\end{aligned}
$$

Witten solution ($U = 0$): $R_3 = R_5 \equiv R$, and $\Delta S = \pi^2 M_p^2 R^2$.

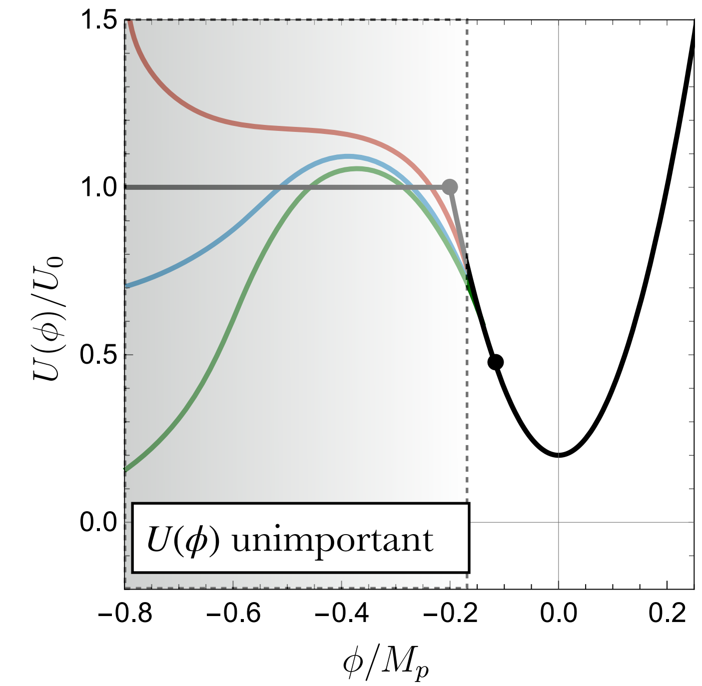
# Bubble Size and Euclidean Action

- Results for $R_3$ and $\Delta S$: as series expansions in small $(mR_5)$:

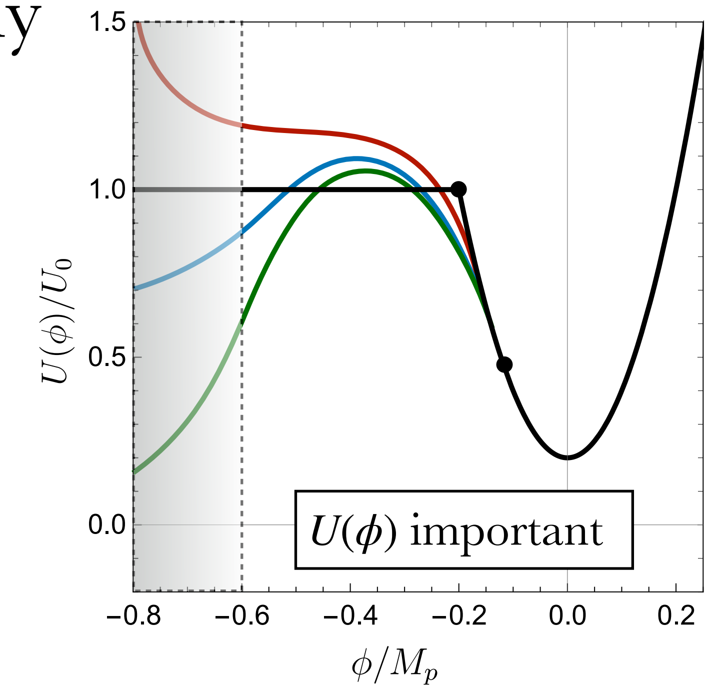
$$R_3 \simeq R_5 \left(1 - \frac{1}{4} m^2 R_5^2 \log(mR_5) + \mathcal{O}(m^2 R_5^2)\right)$$

$$\Delta S \simeq \pi^2 M_p^2 R_5^2 \left(1 - \frac{3}{8} m^2 R_5^2 \log(mR_5) + \mathcal{O}(m^2 R_5^2)\right)$$

where we have assumed $U_0 R_5^2 \ll M_p^2$



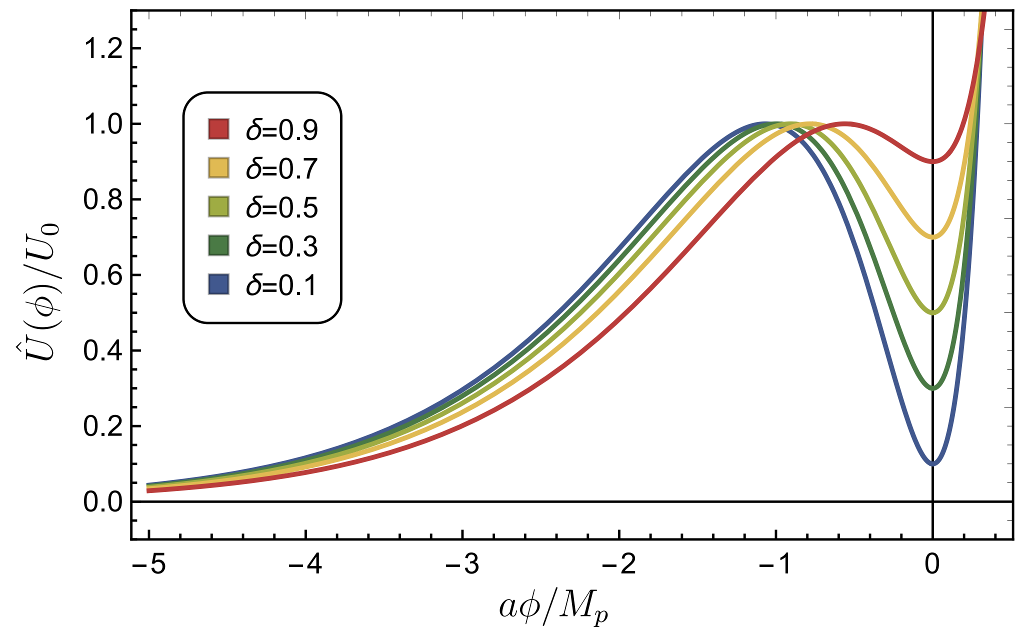

- **Note**: $\Delta S$ not sensitive to $U_{\text{fv}}$ (unlike CDL)
- In this limit ($mR_5 \ll 1$), results from the piecewise model also apply to other $U(\phi)$, if they are approximately quadratic for $-M_p m^2 R_5^2 \lesssim \phi \leq 0$.
    - If $U_0 R_5^2 \gtrsim M_p^2$, need to use numerics:

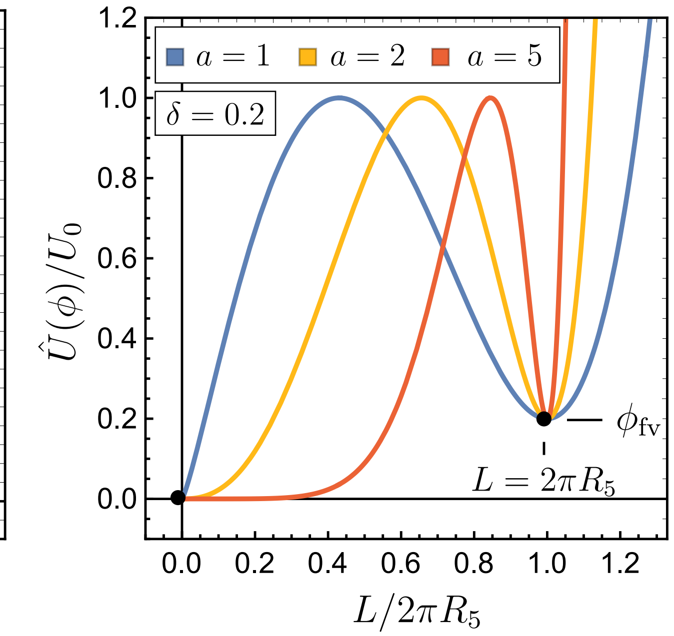


# Numerics For "Realistic" Potential

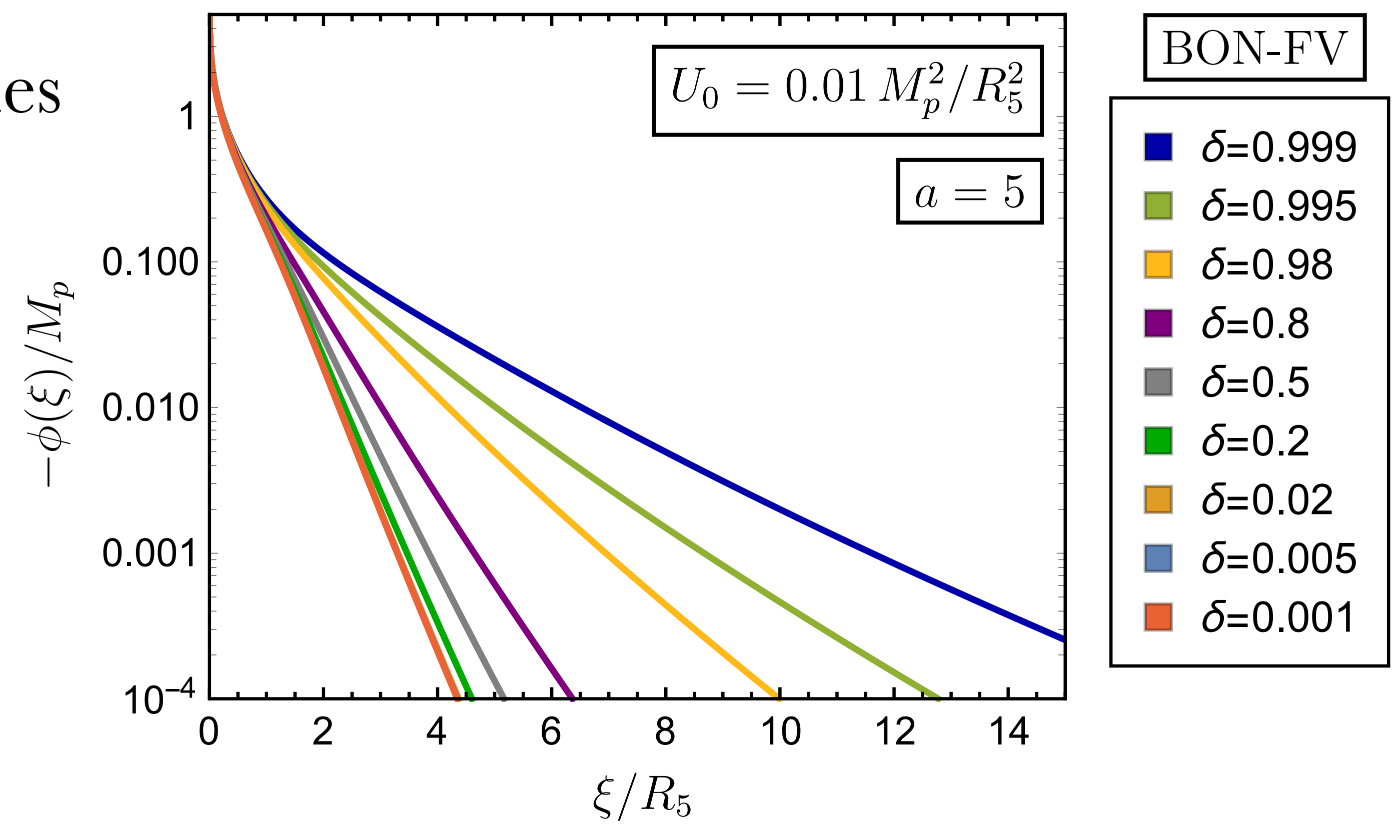
For concreteness, we study a two-parameter potential:

- $\delta$ controls $U_{\text{fv}}$
- $a$ controls $m^2$
  and $L \to 0$ scaling
- $U(\phi) \to 0$ as $L \to 0$
  $(U \to U_0 e^{a\phi/M_p})$




- As expected, $\phi(\xi \lesssim R_5)$ matches the Witten solution
- $\phi(\xi \gg R_5)$ approaches FV exponentially fast (unlike Witten BON), as $\phi \propto \exp(-m\xi)$

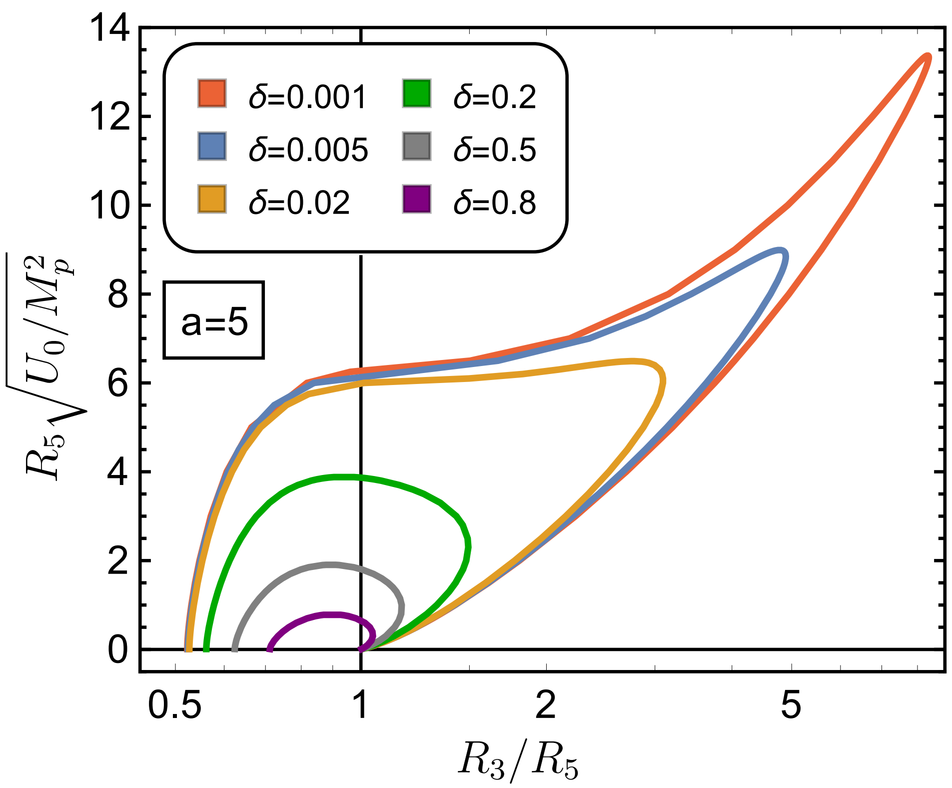


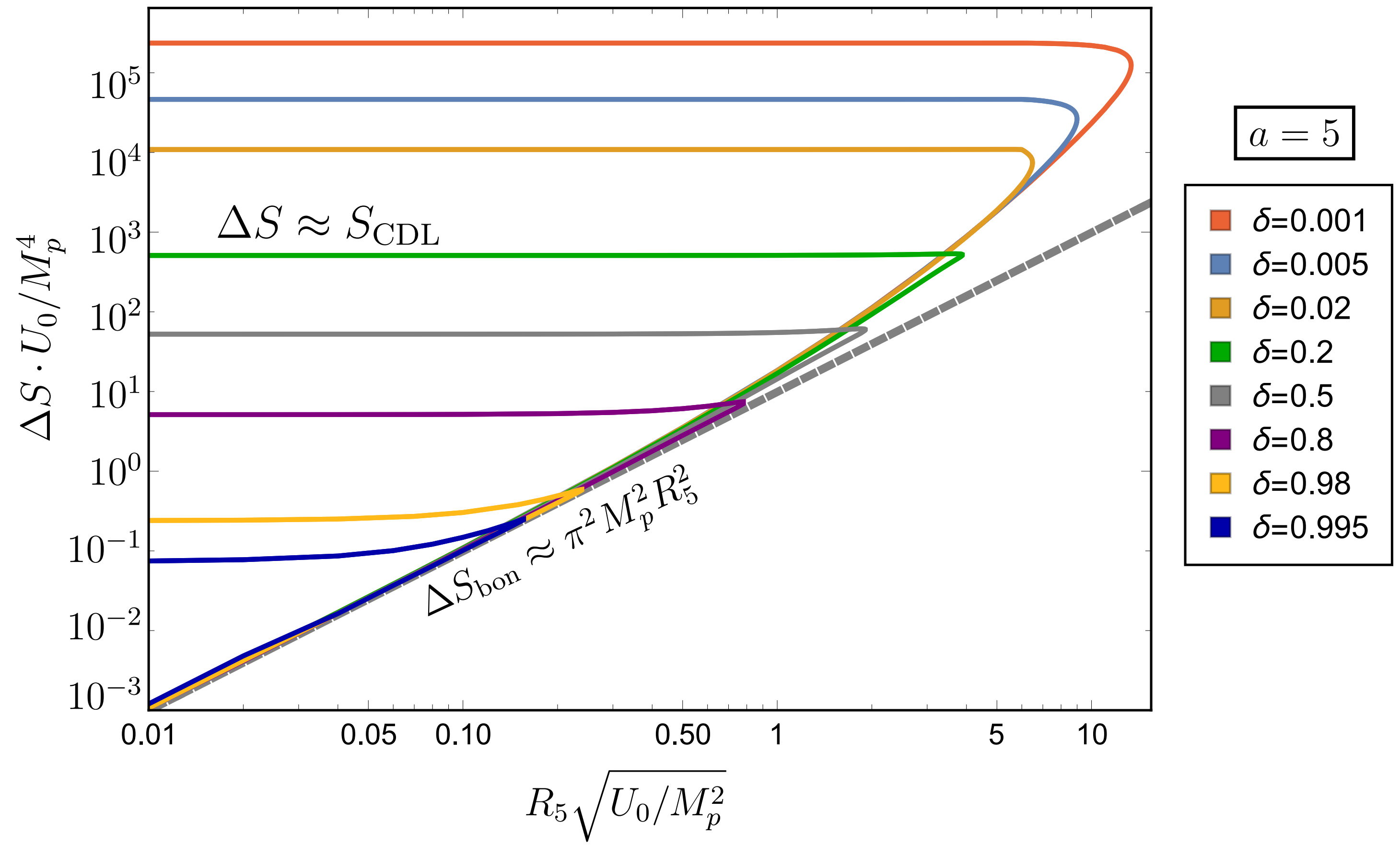
# Numerics For "Realistic" Potential

**Results:** At small $U_0 R_5^2 \ll M_p^2$: $\Delta S \approx \pi^2 M_p^2 R_5^2$, and $R_3 \approx R_5$, as in the Witten BON

Interesting new feature: a second branch of solutions at small $U_0 R_5^2$

- On this branch, $\Delta S$ approaches the CDL action, $S_{\text{cdl}}$
- Also, $R_3 < R_5$ by $\mathcal{O}(1)$ fraction

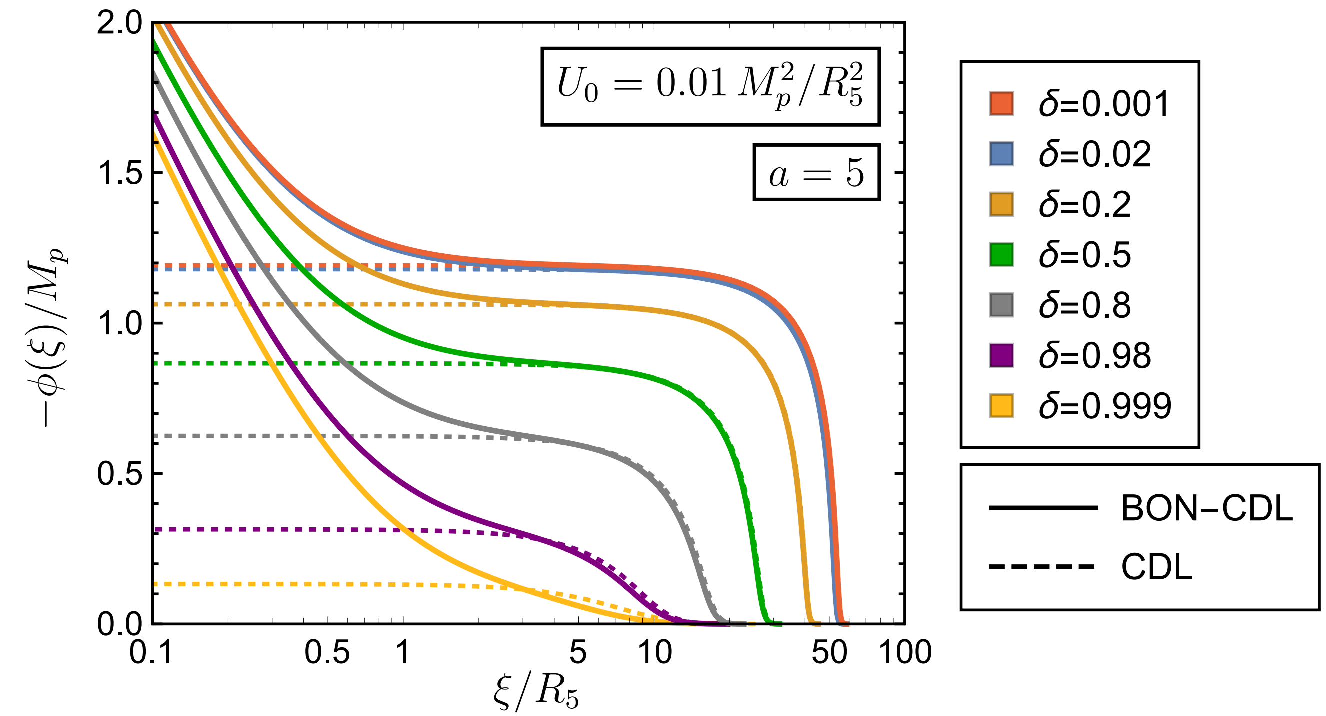


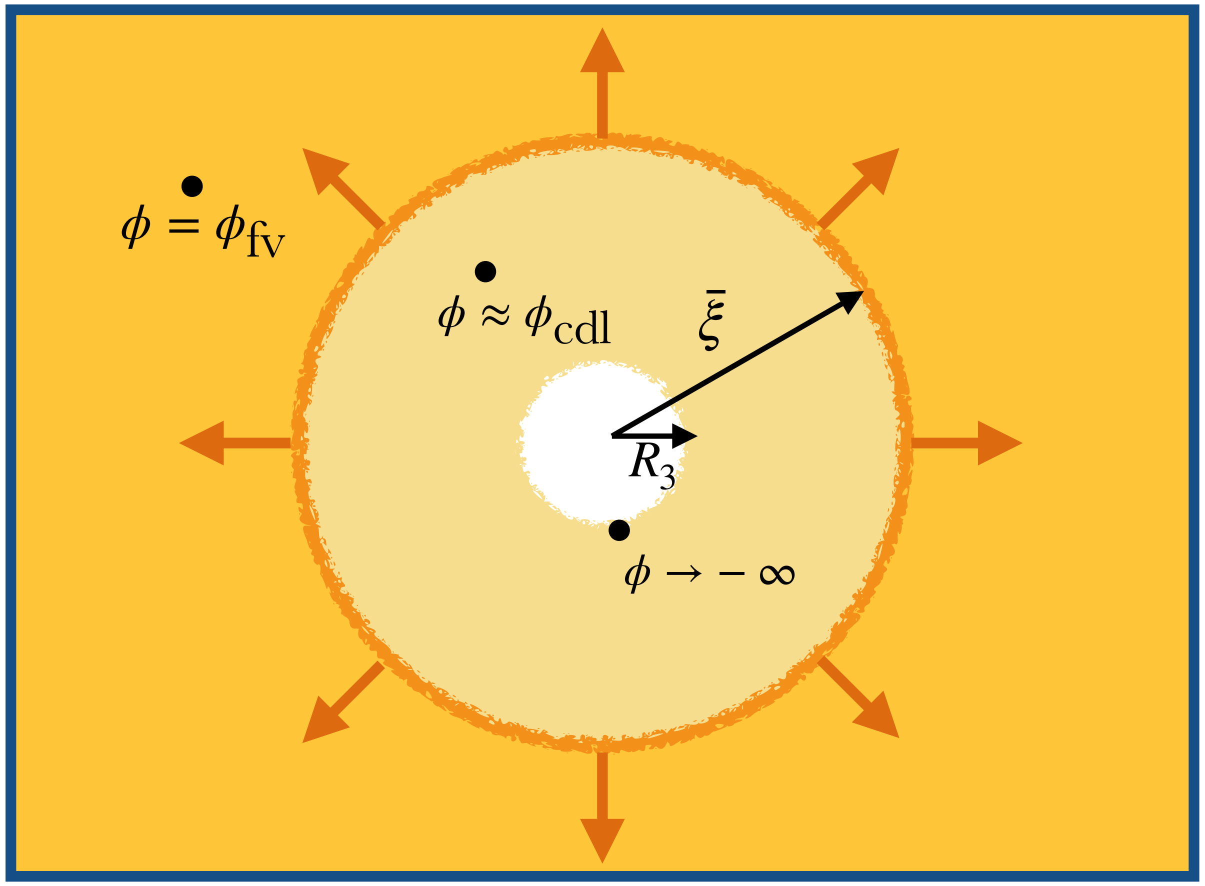




- Interpretation: hybrid bounce solution, combining a BON core and a CDL bubble

# Aside: Hybrid BON-CDL bounce solutions

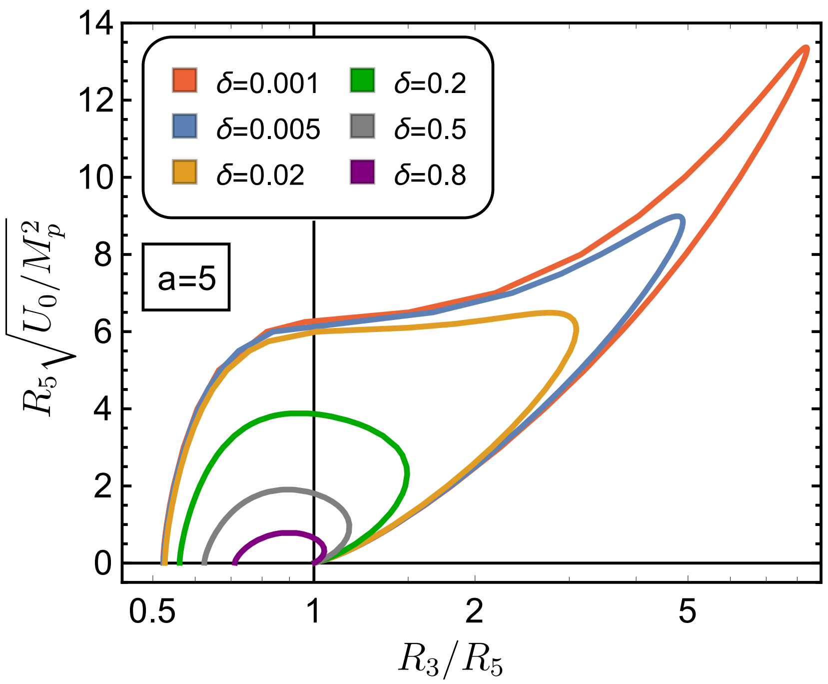
- For $\xi \lesssim R_5$, the solution for $\phi(\xi)$ is BON-like.
- But, for $\xi \gg R_5$, the bounce solution closely matches CDL



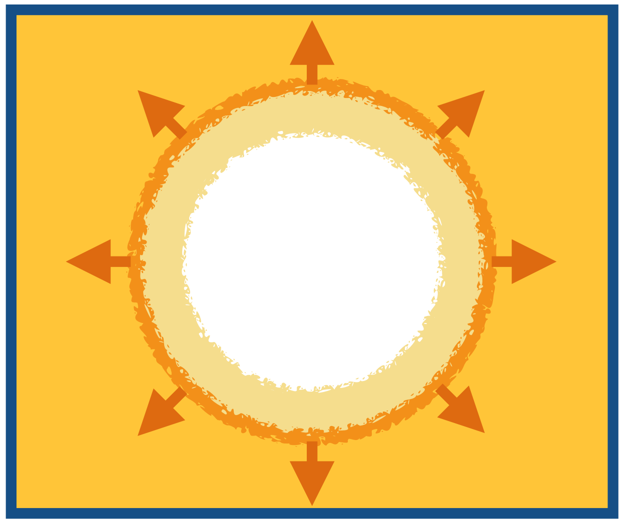



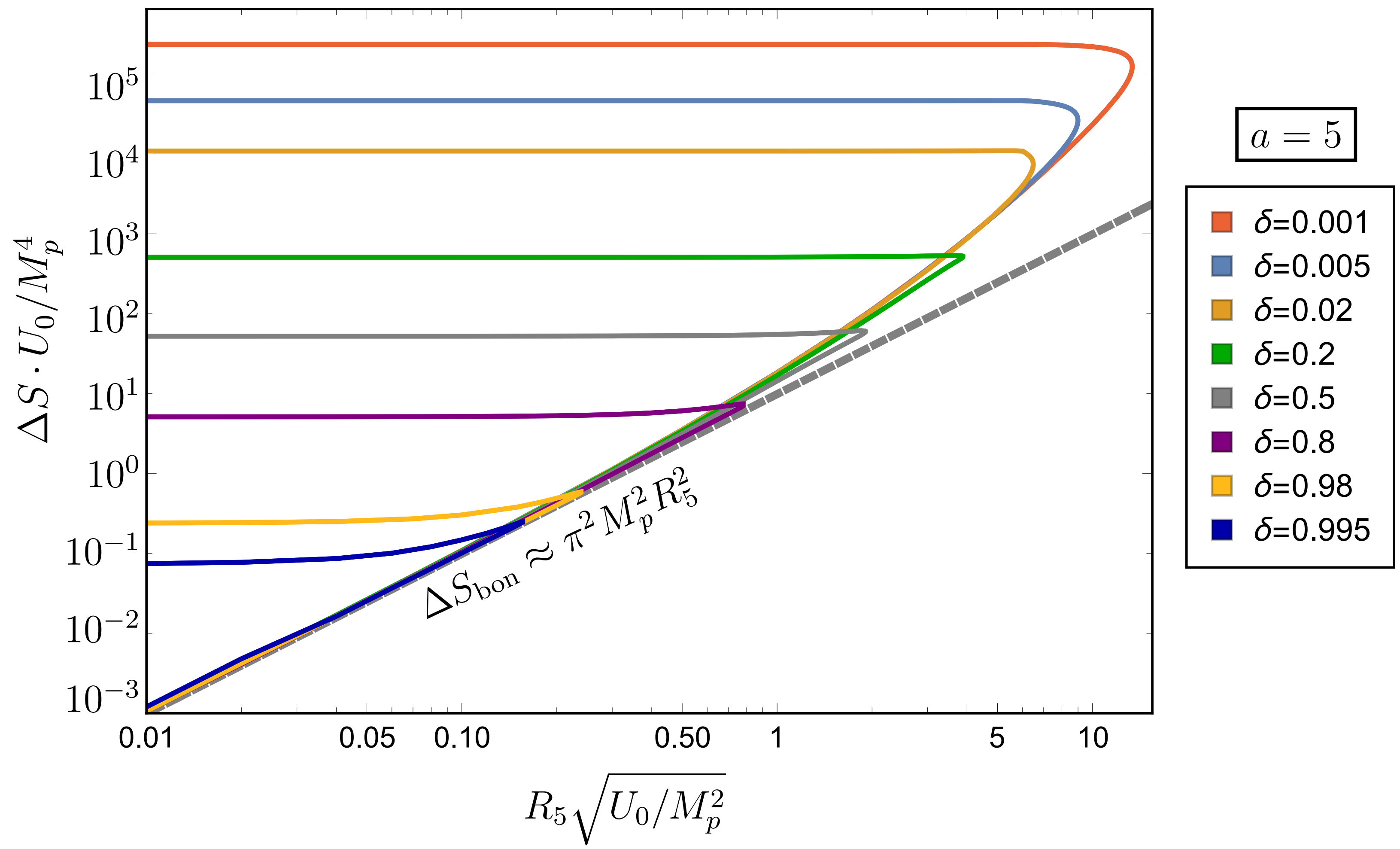

- Of course, this is an $O(4)$ symmetric solution compatible with CDL ansatz and BON boundary condition
- We can expect these for any $U(\phi)$ that has a CDL solution
- Standard BON has smaller $\Delta S$

# Aside: Hybrid BON-CDL bounce solutions



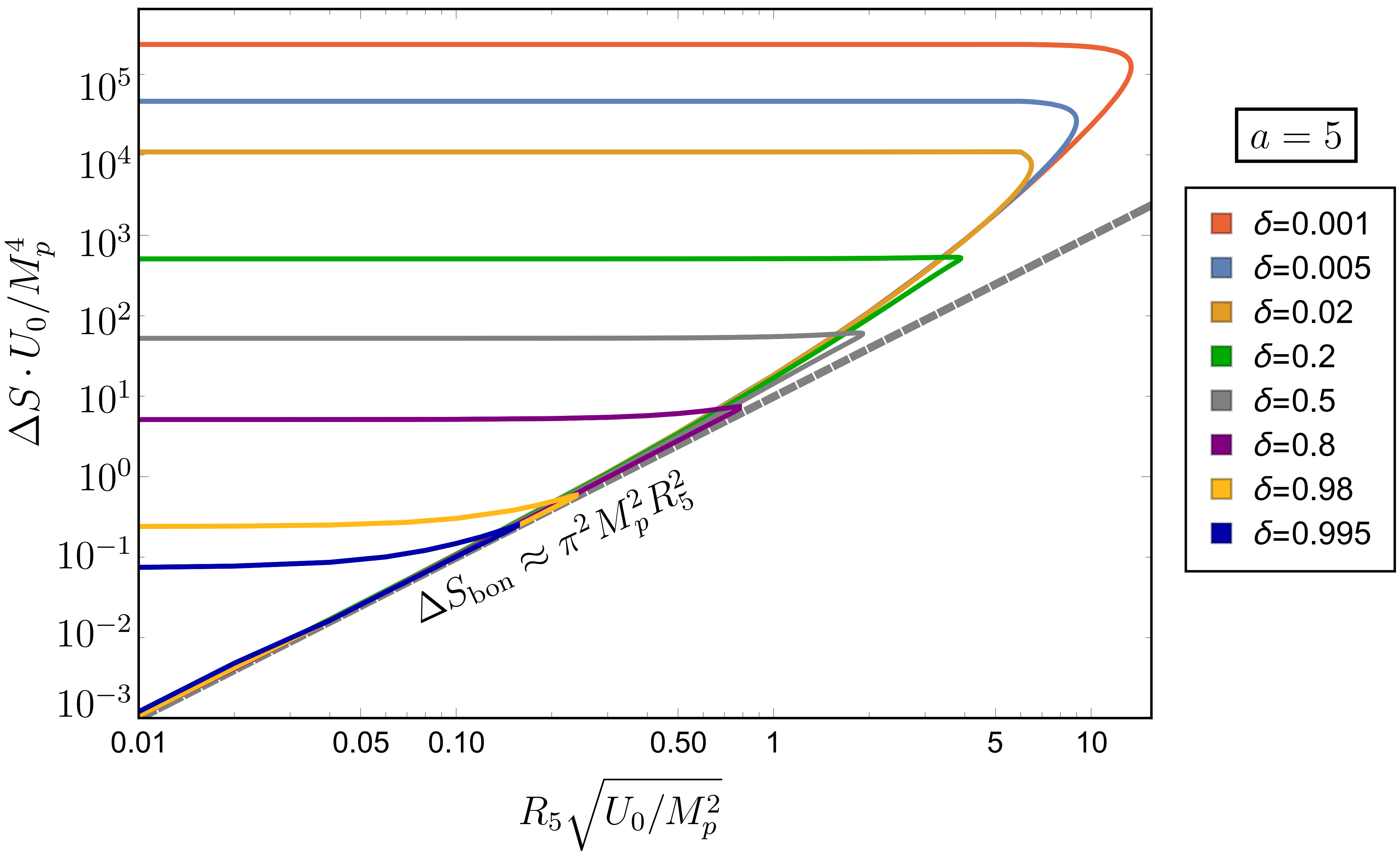

- Numerics show the BON-CDL hybrid bounce merges with the BON solution at large $U_0R_5^2$
- Implies a maximum value of $U_0R_5^2$, above which there is no BON instability





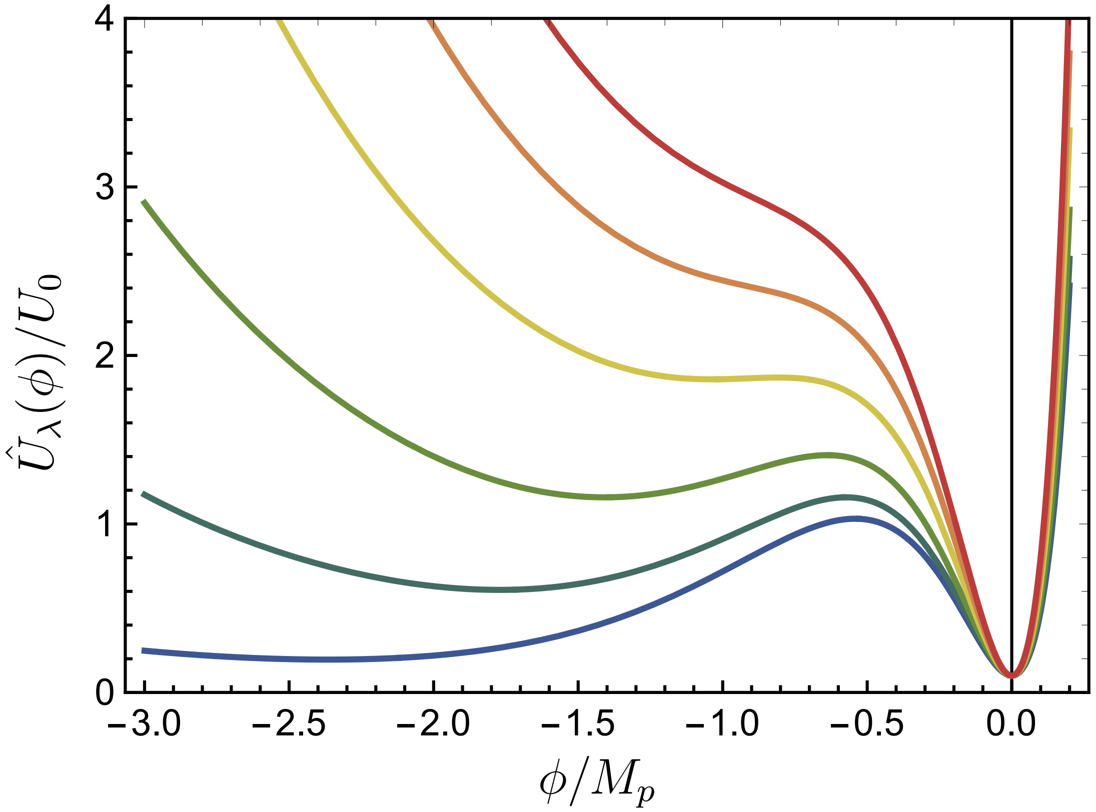
# BON vs CDL Rates

- In the $U_{\text{fv}} \to 0$ limit ($\delta \to 0$), the CDL action diverges as $S_{\text{cdl}} \propto 1/U_{\text{fv}}$
- BON action remains finite, $\Delta S \propto R^2$, and largely independent of $\delta$



# Exponentially Growing Potentials

- BON existence conditions allow for $U \propto \exp(a\phi/M_p)$ in the compactification limit, as long as $a > -\sqrt{6}$
- In the $mR_5 \ll 1$ limit, these $U(\phi)$ should still be approximated by the simple piecewise model




- For example, a 5d cosmological constant, after dimensional reduction, has $a = -\sqrt{2/3}$:

$$U \to U + U_0 \lambda \exp\left(-\sqrt{\frac{2}{3}}\frac{\phi}{M_p}\right)$$

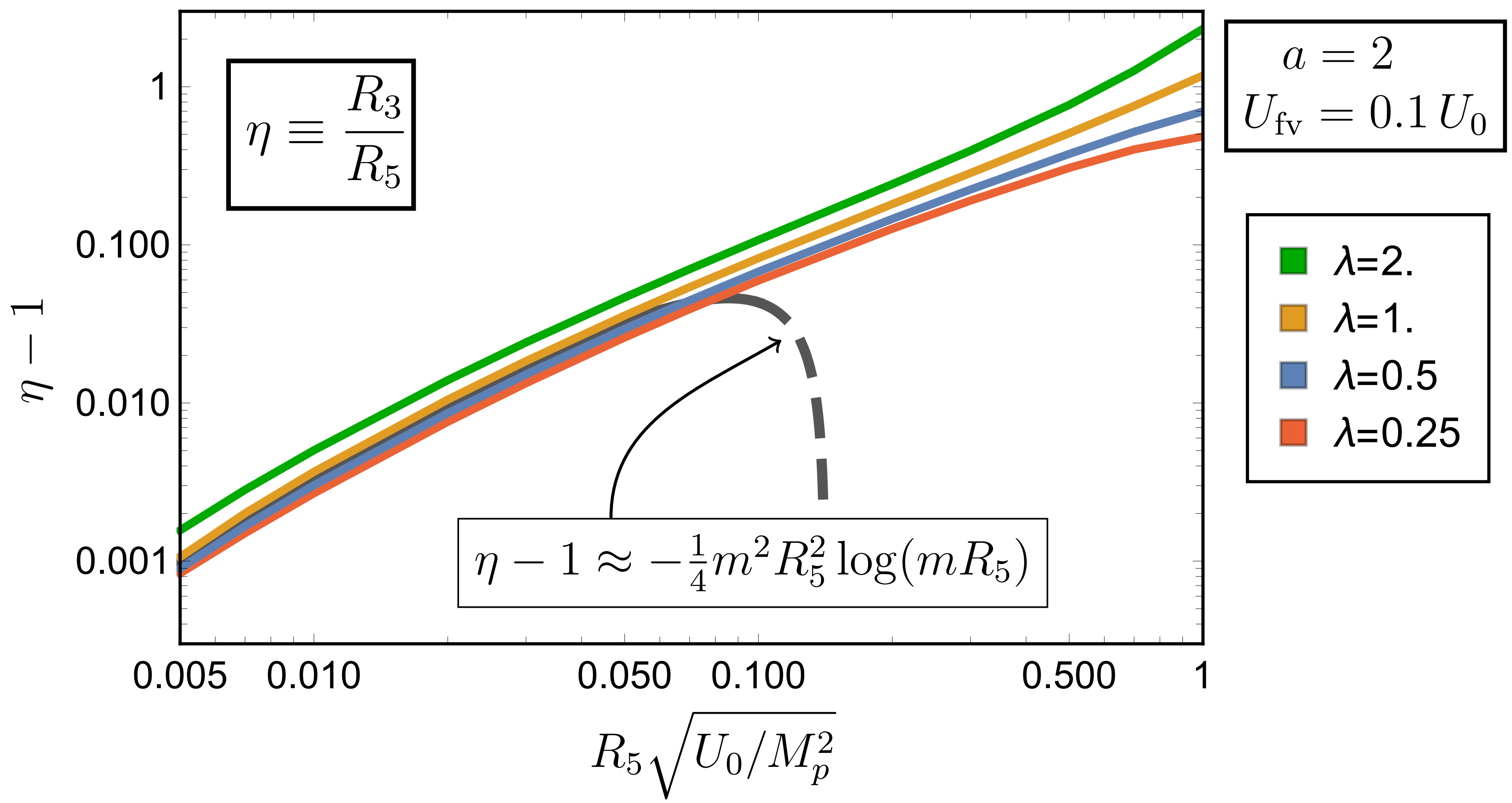
$$\lambda \equiv \frac{\Lambda_5^{\rm CC} M_p^2}{U_0}$$

# Exponentially Growing Potentials

How well do our analytic predictions match the numeric results?

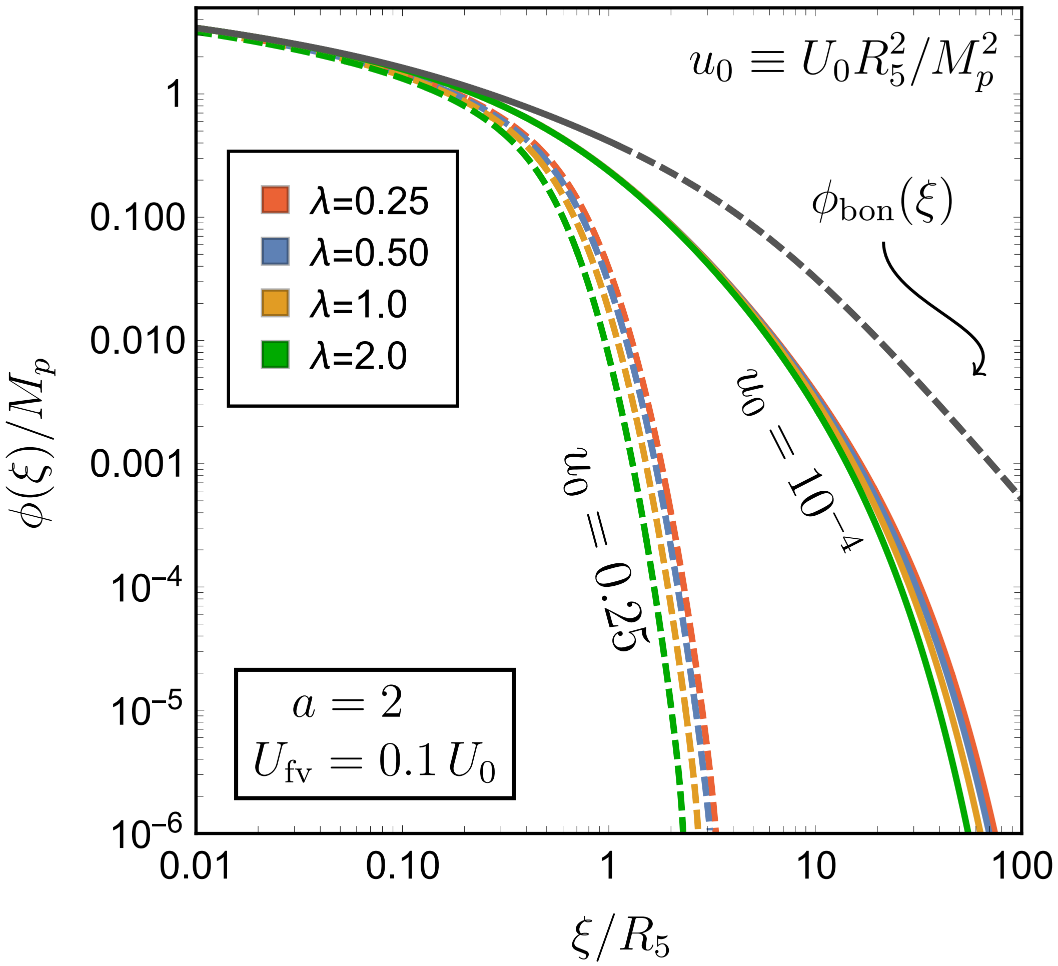
- Actually quite well, for small $U_0 R_5^2 \ll M_p^2$.

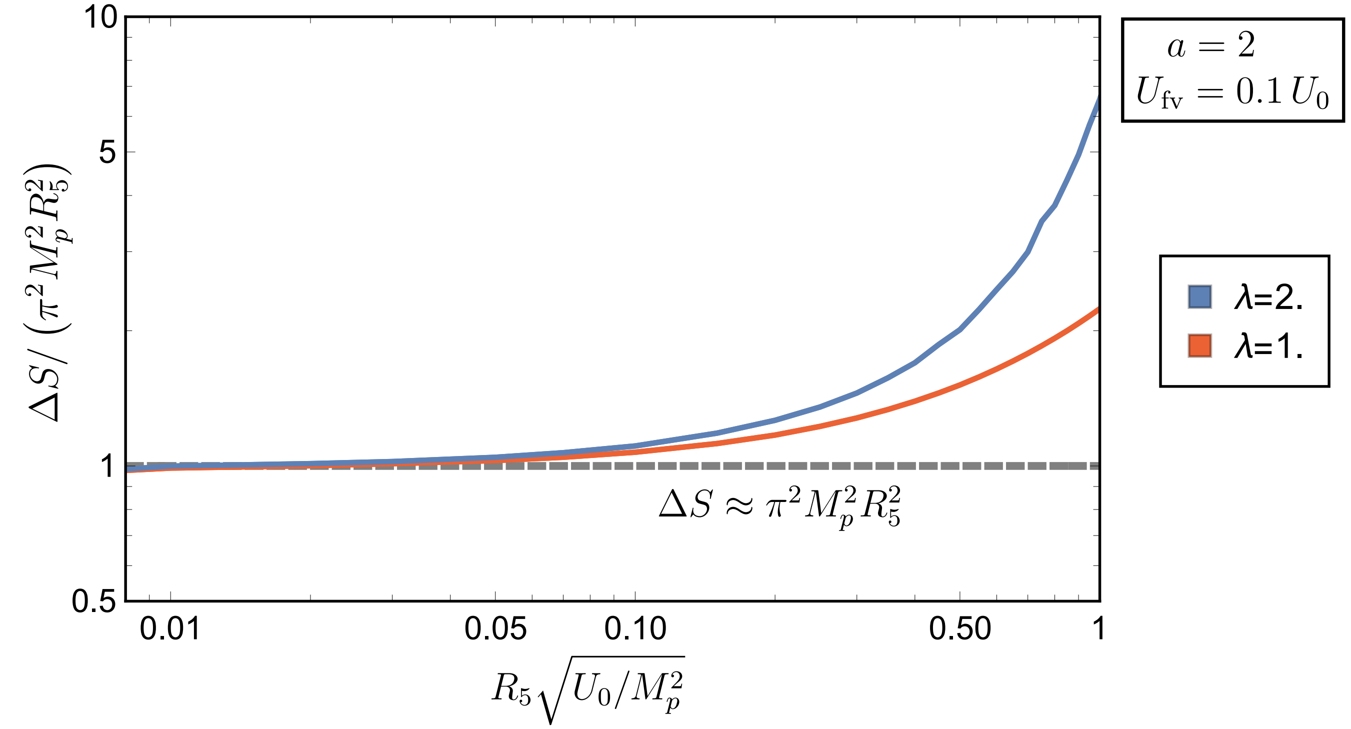
  From analytic model: $\eta \simeq 1 - \frac{1}{4} m^2 R_5^2 \log(m R_5) + \mathcal{O}(m^2 R_5^2)$



# Exponentially Growing Potentials

- BON action is still well approximated by $\Delta S \approx \pi^2 M_p^2 R_5^2$:




Familiar behavior:

- Solution for $\phi(\xi)$ still approximately Witten-like at small $\xi$
- Approaches the false vacuum exponentially fast at $\xi \gg R_5$

# Conclusions

- If $L = 2\pi R_5$ is particularly small, spacetime can undergo a catastrophic decay, with a rate (per Hubble volume)

$$\frac{\Gamma}{H_0^4} \sim \frac{v^4 e^{-\Delta S}}{H_0^4}$$

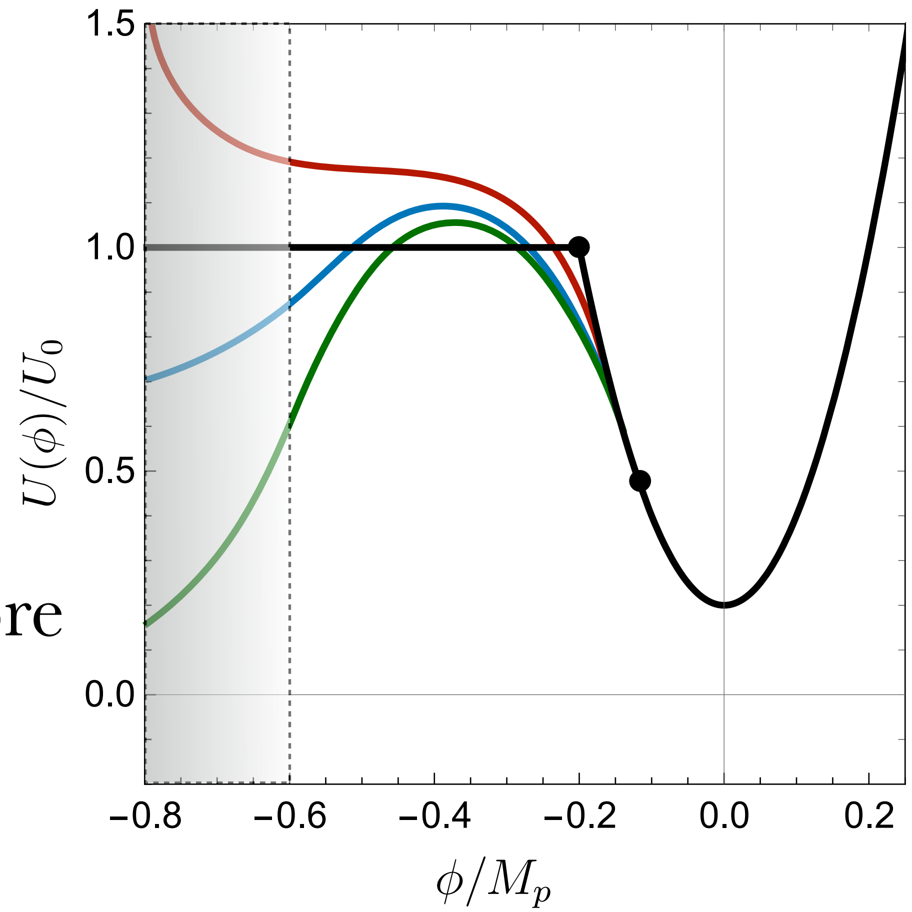
- The fact that the apocalypse has not happened (yet) to our corner of the universe implies

$$\Delta S \gtrsim 560 - \log\left(M_p^4/v^4\right) \qquad \boxed{L \gtrsim 50\, M_p^{-1}}$$

unless the extra dimension is stabilized by a potential that grows more quickly than $U \propto \exp\left(-\sqrt{6}\phi/M_p\right)$, or by a potential with $U_0 \gg M_p^2/R_5^2$ large enough to remove the BON branch of solutions

- Conclusion applies to wide range of stabilizing potentials, even some that grow exponentially in the compactification limit

# Ongoing/Future Research




- Given a particular model of modulus stabilization, what is the BON rate?

  Can extend the $\rho' \simeq 1$ approximation to more complicated potentials, modeling a specific $U(\phi)$ by concatenating several quadratic or linear functions

- Can analyze higher-dimensional compact spaces, e.g. $(4 + n)$d with shrinking $n$-sphere

- Potentials with too-fast exponential growth, e.g. $U_{\text{flux}}(\phi) \propto e^{-3\sqrt{\frac{2n}{n+2}}\frac{\phi}{M_p}}$ may still have BON instabilities: depends on other degrees of freedom, which might screen the flux

# Aside: Flux Compactification

- In $n = 1$, adding flux on the $S^1$ circle generates potential that grows as

$$U_{\text{flux}}(\phi) \propto e^{-3\sqrt{\frac{2n}{n+2}}\frac{\phi}{M_p}} \longrightarrow e^{-\sqrt{6}\frac{\phi}{M_p}}$$

- For this potential, BON initial conditions incompatible with EOM
- For $n$-form flux on $S^n$, same conclusion: $U_{\text{flux}}(\phi)$ prevents BON

This isn't necessarily the final word on the matter: if the flux is screened by pointlike charges, for example, $U(\phi)$ exponent may be less large, and BON could still form. See refs. (Addressing this in followup work)

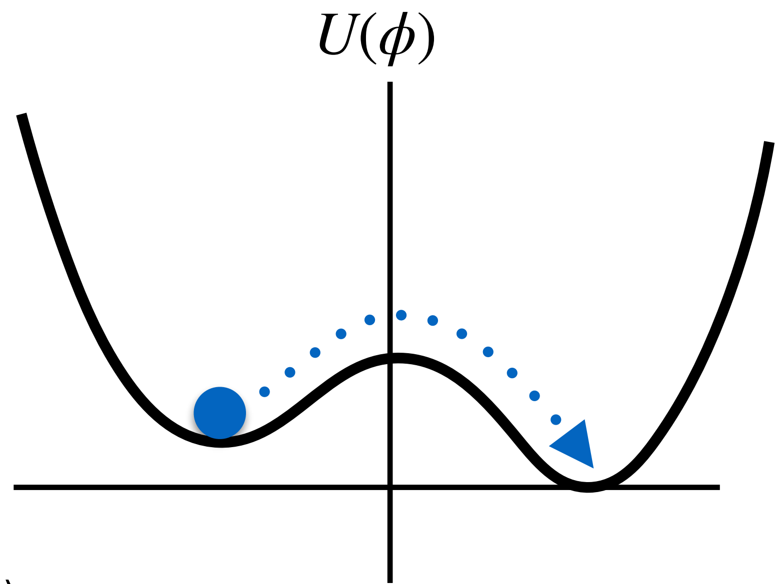
# Aside II: SUSY

- For $n = 1$, SUSY boundary conditions provide topological obstruction to shrinking $S^1$. Not a problem for $n > 1$

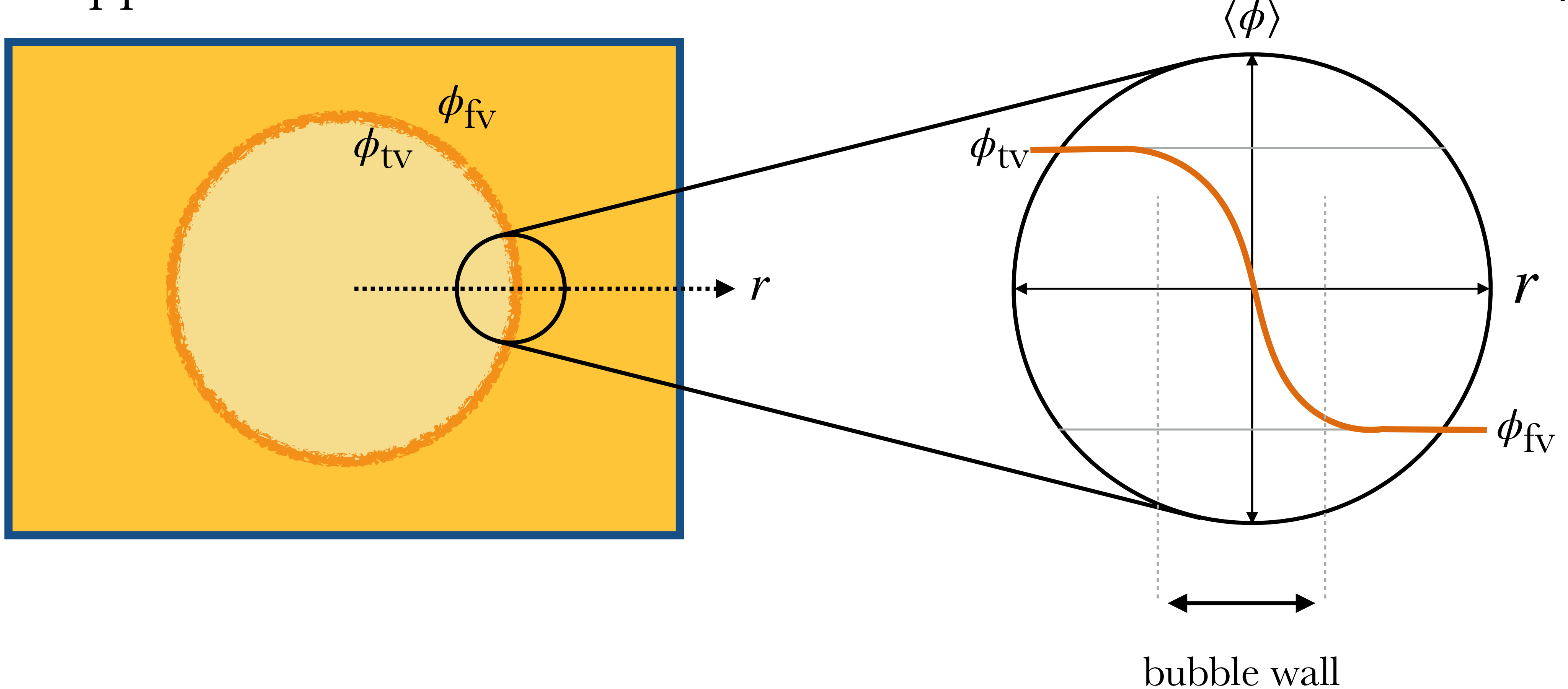
S.R. Coleman and F. De Luccia, PRD 21 (1980) 3305

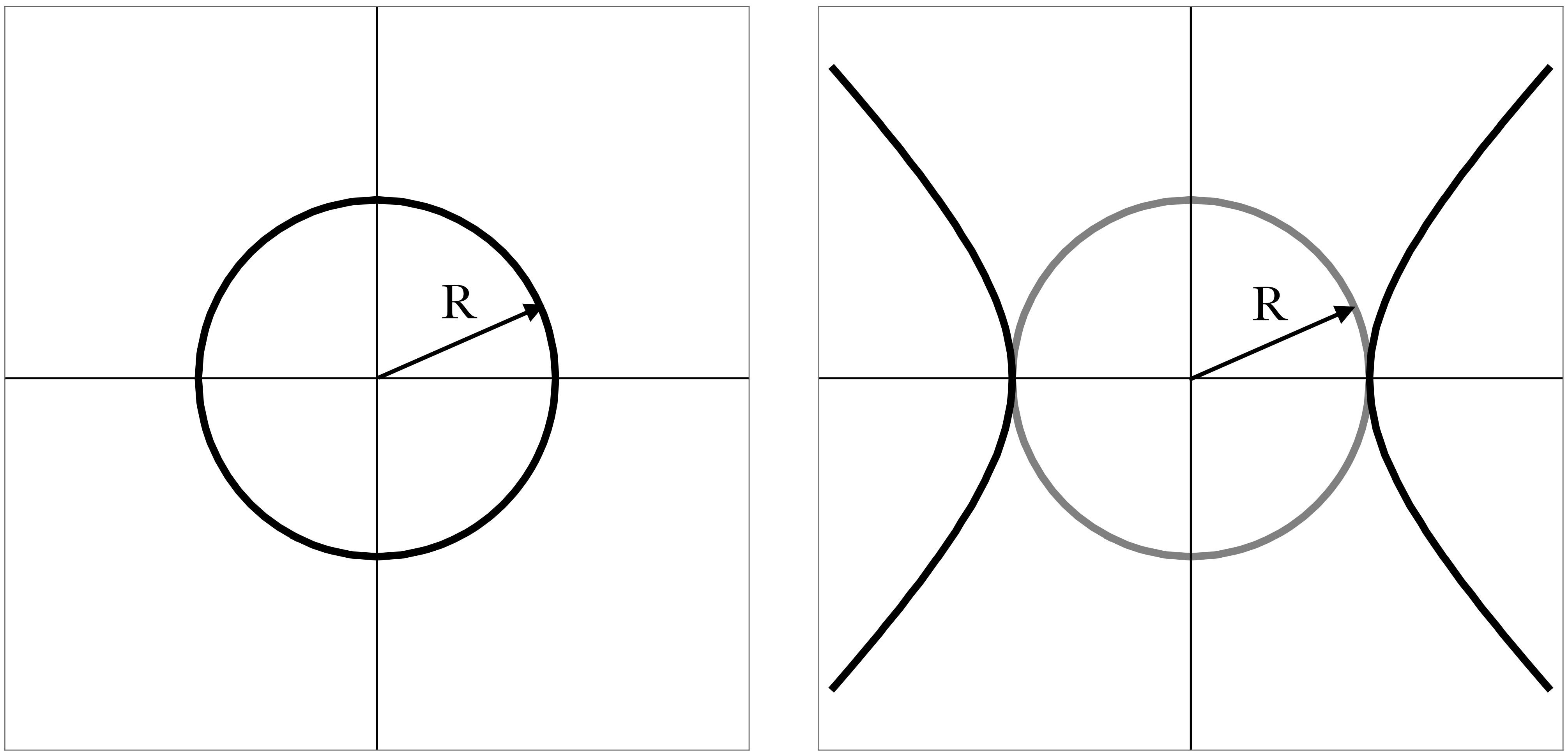
# Coleman–De Luccia (TWA)

- Nucleation rate $\Gamma \approx v^4 \exp(-S_E/\hbar)$, with Euclidean action $S_E$




- Ansatz: spherical bubble, $ds_4^2 = d\xi^2 + \rho(\xi)^2 d\Omega_3^2$
- In the thin-wall limit, $S_E$ has closed-form approximate solution

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