Gravitating vortices in an AdS_3 and Minkowski background

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The vortex

- The vortex is a finite energy field configuration in $2+1$ dimensions composed of a complex scalar field ϕ and a gauge field A_μ that have particular boundary conditions. It is stable and non-singular.
- The action contains a potential with a minimum (vacuum) at $|\phi| = v \neq 0$. There is a continuous set of vacuum states determined by the phase. In the vortex, $|\phi| \to v$ as $r \to \infty$.
- The phase of the scalar field can cover $2\,n\,\pi$ in one full rotation around a circle at infinity. Here n is a positive integer called the winding number. So vortices are characterized by their values of v and n.
- Assuming rotational symmetry, the scalar and gauge fields are represented by the functions $f(r)$ and $a(r)$ respectively. Asymptotically, $f \to v$ and $a \to n$ and this yields a finite energy configuration. Without the gauge field, the energy would diverge logarithmically.

Embedding the vortex in Einstein gravity with cosmological constant

The Lagrangian density for the vortex coupled to Einstein gravity with cosmological constant is given by

$$\mathcal{L} = \sqrt{-g}\left[\alpha\,(R-2\Lambda) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(D_\mu\phi)^\dagger(D^\mu\phi) - \frac{\lambda}{4}(|\phi|^2 - v^2)^2\right]$$

where R is the Ricci scalar, Λ is the cosmological constant, $F_{\mu\nu}$ is the electromagnetic field tensor and the covariant derivatives are defined in the usual fashion by

$$D_\mu\phi = \partial_\mu\phi + i\,e\,A_\mu\phi\,.$$

The constant α is equal to $1/(16\pi G)$ where G is Newton's constant and the constant v is the non-zero VEV of the scalar field which spontaneously breaks the local $U(1)$ symmetry.

Rotationally symmetric ansatz

We consider rotationally symmetric static solutions so that the 2+1-dimensional metric has the form

$$ds^2 = -B(r)\,dt^2 + \frac{1}{A(r)}\,dr^2 + r^2 d\theta^2\,.$$

We make the following ansatz for the scalar and gauge fields

$$\phi(\mathbf{x}) = f(r)e^{in\theta} \text{ and } A_j(\mathbf{x}) = \epsilon_{jk}\,\hat{x}^k\,\frac{a(r)}{e\,r}$$

where n is the winding number.

The magnetic field is given by $B = F_{21} = \dfrac{1}{e\,r}\dfrac{da}{dr}$.

Inserting ansatz into Lagrangian

Substituting the ansatz into the Langrangian density we obtain

$$\mathcal{L} = \sqrt{B/A}\, r \left(\alpha(R - 2\Lambda) - (\lambda/4)(f^2 - v^2)^2 - \frac{(f')^2 A}{2} - \frac{(n-a)^2 f^2}{2r^2} - \frac{A(a')^2}{2e^2 r^2} \right)$$

where a prime denotes derivative with respect to r.

In terms of the metric functions A and B the Ricci scalar is given by

$$R = -\frac{A'}{r} + \frac{(B')^2 A}{2B^2} - \frac{B'' A}{B} - \frac{A' B'}{2B} - \frac{B' A}{rB}\,.$$

Equations of motion

There are four functions of r : A, B, f and a. The four equations of motion can be reduced to three by eliminating B from the equations:

$$\begin{aligned}2e^2\left(n-a\right)^2 f^2 - 2e^2r^2v^2\lambda f^2 + e^2r^2\lambda f^4 + e^2r\left(rv^4\lambda + 8r\alpha\Lambda + 4\alpha A'\right)\\ + 2A\left((a')^2 + e^2r^2(f')^2\right) = 0 \qquad (1)\end{aligned}$$

$$\begin{aligned}&-2\left(n-a\right)^2 f + 2r^2v^2\lambda f - 2r^2\lambda f^3 + \frac{rf'}{4e^2\alpha}\Big(-e^2r^2\left(v^4\lambda + 8\alpha\Lambda\right)\\ &-2e^2\left(n-a\right)^2\, f^2 + 2e^2r^2v^2\lambda f^2 - e^2r^2\lambda f^4 + 2A\left((a')^2 + e^2r^2(f')^2\right)\Big)\\ &+ r\left(rA'f' + 2A\left(f' + rf''\right)\right) = 0 \qquad (2)\end{aligned}$$

$$\begin{aligned}&2e^2r(n-a)f^2 - 2Aa' + ra'A' + \frac{a'}{4e^2\alpha}\Big(-e^2r^2\left(v^4\lambda + 8\alpha\Lambda\right) - 2e^2\left(n-a\right)^2 f^2\\ &+ 2e^2r^2v^2\lambda f^2 - e^2r^2\lambda f^4 + 2A\left((a')^2 + e^2r^2(f')^2\right)\Big) + 2rAa'' = 0\,. \qquad (3)\end{aligned}$$

Vacuum and asymptotic metric solutions

One can solve analytically for the metric in vacuum (denoted by the subscript '0') by setting $f = v$ and $a = n$ identically in the equations of motion. This yields

$$A_0(r) = B_0(r) = -\Lambda r^2 + C$$

where the integration constant C determines the initial conditions at $r = 0$ (we set $C = 1$).

In the presence of the vortex, the asymptotic form of the metric function $A(r)$ at large radius R is given by

$$A(R) = -\Lambda R^2 + D$$

where the constant D differs from the constant C due to matter.

AdS_3 background

- The spacetime is asymptotically AdS_3, a maximally symmetric spacetime with isometry group $SO(2,2)$.
- It has a timelike Killing vector and hence one can define a conserved energy (the ADM mass).
- Maximally symmetric spacetimes can be viewed as the ground states of General Relativity so that their energy is usually set to be zero. This requires one to subtract out from the Hamiltonian the contribution of the background. This subtraction mechanism is already implemented in the usual formulation of the ADM mass.

Expression for ADM mass in AdS_3

The ADM mass suitably generalized to $2+1$ dimensions is given by

$$M = -2\alpha \lim_{C_t \to R} \oint_{C_t} (k - k_0) \sqrt{\sigma} \, N(R) \, d\theta$$

- $N(R) = [B_0(R)]^{1/2}$ is the lapse; σ_{AB} is the metric on C_t
- k and k_0 are the extrinsic curvature of C_t embedded on the two-dimensional spatial surface of the metric and AdS_3 respectively.

Calculation yields $M = 4\pi\alpha \left(A_0(R) - [A_0(R) A(R)]^{1/2} \right)$.

Since $|(D - C)/A_0(R)| << 1$ this can be simplified to

$$M_{AdS_3} = 2\pi\alpha(C - D) = 2\pi\alpha \left(A_0(R) - A(R) \right)$$

Expression for ADM mass in asymptotically flat space and angular deficit

The ADM mass formula applies to asymptotically flat spacetime where $\Lambda = 0$. In that case we have $A_0(R) = C$ and $A(R) = D$ so that

$$M_{flat} = 4\pi\alpha\left(C - (C\,D)^{1/2}\right) = 4\pi\alpha\left(1 - D^{1/2}\right)$$

where we used $C = 1$.

Asymptotically, the spacetime is conical with angular deficit of

$$\delta = 2\pi(1 - D^{1/2}) = \frac{M_{flat}}{2\,\alpha}\,.$$

However, the spacetime has no conical singularity.

Integral mass formulas

One can solve for the metric $A(r)$ as an integral over matter fields only. For a vortex embedded in an AdS_3 background we obtain

$$M_{AdS_3} = I = n^2\, v^2\, F$$

where F is a dimensionless integral over matter field profiles ($f_1 = f/v$, $a_1 = a/n$),

$$F = \frac{\pi}{2}\int_0^{R_1} \frac{du}{u}\left[u^2\frac{\lambda}{e^2} + f_1^4 u^2 \frac{\lambda}{e^2} + 2f_1^2\left((-1+a_1)^2 - u^2\frac{\lambda}{e^2}\right)\right.$$
$$\left. + \frac{2f_1\left((a_1')^2 + (f_1')^2 u^2\right)\left((1-a_1)f_1 f_1' u^2 + a_1'\left((-1+a_1)^2 + (-1+f_1^2)u^2\frac{\lambda}{e^2}\right)\right)}{u(2a_1' f_1' - a_1'' f_1' u + a_1' f_1'' u)}\right]$$

The point is that I makes no reference to metrics, Newton's constant or the cosmological constant.

Integral mass formula for the asymptotically flat case

For a vortex under gravity in an asymptotically flat spacetime the integral representation for the ADM mass is given by

$$M_{flat} = 4\pi\alpha\left(1 - \sqrt{1-\chi}\right)$$

where

$$\chi = \frac{I}{2\pi\alpha}\,.$$

So χ is an integral over matter field profiles just like I.

The no gravity case: interesting result

Though the equations of motion assume gravity the integral I is an integral over matter field profiles only (no reference to any gravitational parameters).

It turns out that the mass of the vortex in the case of no gravity in a *fixed* Minkowski spacetime is given by I. This can be seen by taking the limit of M_{flat} as $\alpha \to \infty$. We therefore have

$$M_{no-gravity} = I\,.$$

Note: even though we evaluate M_{AdS_3} and $M_{no-gravity}$ using the same integral mass formula I, the matter field profiles are different for the two cases leading naturally to different masses.

Two ways to obtain the mass

- We can evaluate M_{AdS_3} and M_{flat} in two ways: using the metric or using the matter.
- In the first method, we extract from the numerical simulation the asymptotic value of the metric. In the second method one substitutes the matter field profiles obtained numerically into the integral mass formulas.
- The two methods must match and this provides a good check on our numerical results. This will be shown in Table I of our numerical results.
- Only one way to calculate mass for the no gravity case as the metric is fixed.

Numerical simulation

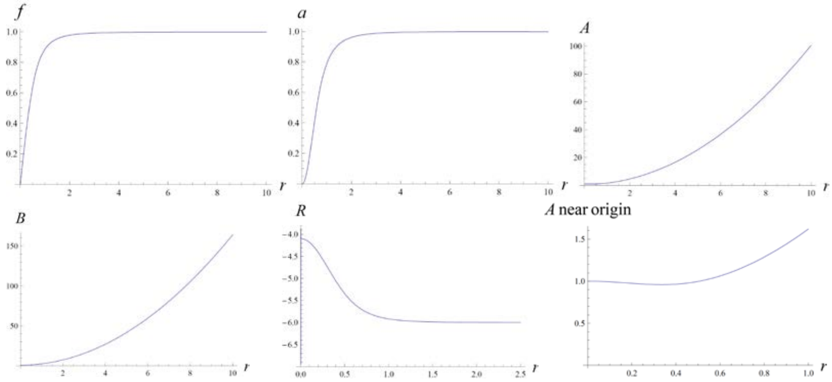
- We solve the three equations of motion numerically for the non-singular profiles of the scalar field $f(r)$, the gauge field $a(r)$ and the metric $A(r)$ using a negative cosmological constant Λ (AdS_3 background).
- We have the following boundary conditions:

$$f(0) = 0 \quad ; \quad a(0) = 0 \quad ; \quad f(R) = v \quad ; \quad a(R) = n \quad ; \quad A(0) = 1$$

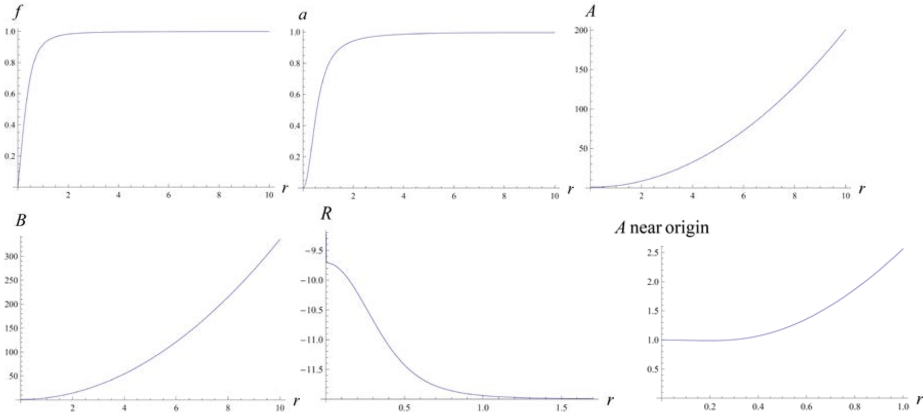
  where R is the computational boundary representing formally infinity. The quantity v is the VEV of the scalar field and n is the winding number of the vortex.
- We obtain the profiles by adjusting $f'(r)$ and $a'(r)$ near the origin to give the final boundary conditions at R where both f and a plateau to their respective values.

Plots of numerical results for AdS_3 background

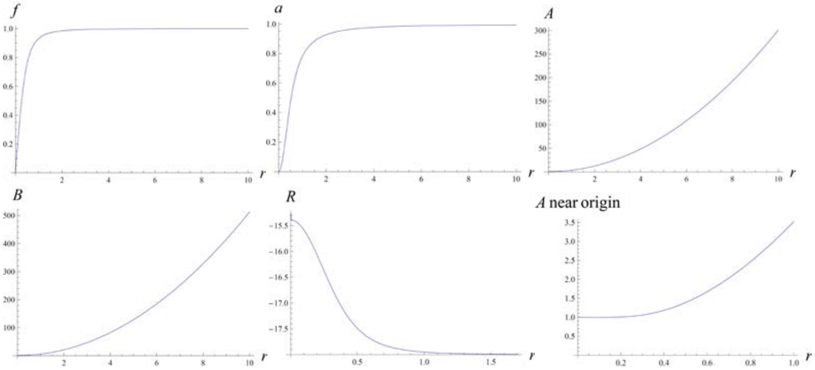
We ran numerical simulations for five different cases determined by the values of the three parameters (n, v, Λ).



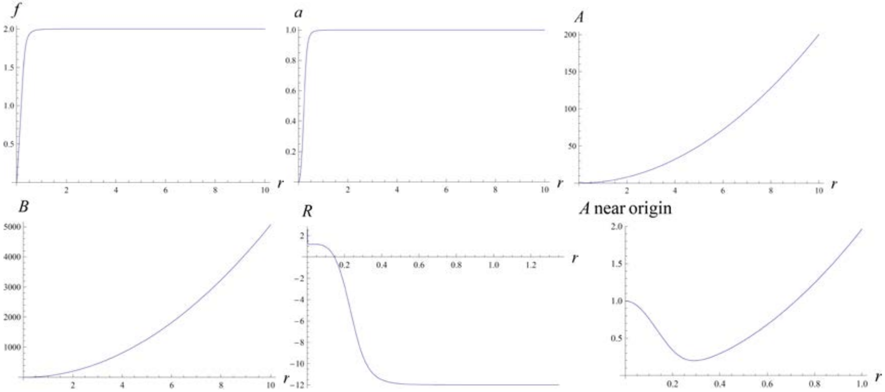
Case I: $n = 1, v = 1$ and $\Lambda = -1$. The matter functions f and a plateau at their respective values of $v = 1$ and $n = 1$. The metric functions A and B are positive throughout and hence there is no event horizon. The Ricci scalar plateaus at -6 which agrees with the expected 6Λ. The mass results of all five cases are summarized in Table 1.



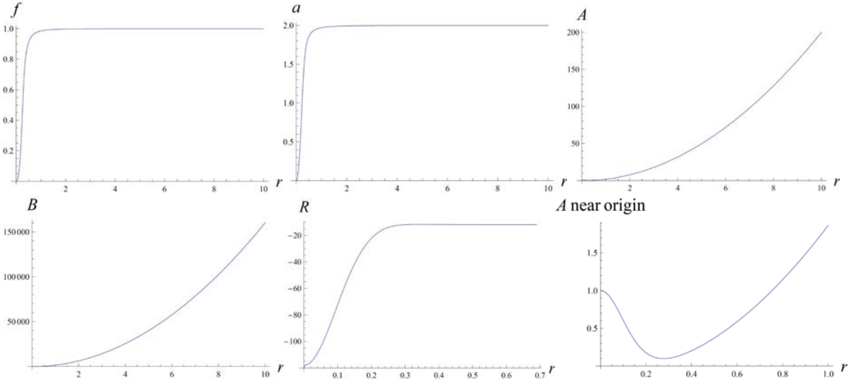
Case II: $n = 1, v = 1$ and $\Lambda = -2$. The Ricci scalar plateaus at -12 which agrees with the expected 6Λ. The core of the matter profile f is smaller (the vortex is more compressed) than in case I and its mass is greater.



Case III: $n = 1, v = 1$ and $\Lambda = -3$. The Ricci scalar plateaus at -18 in agreement with the value 6Λ. The matter profile f here is more compressed (the core is smaller) than in the two previous cases and it has the greatest mass of the three.



Case IV: $n = 1, v = 2$ and $\Lambda = -2$. This case differs from the previous three because $v = 2$ instead of $v = 1$. The mass of the vortex is significantly greater now compared to the previous three cases which reflects the v^2 dependence of the integral mass formula. There is a significant dip in the metric A near the origin while it remains positive (no horizon).



Case V: $n = 2, v = 1$ and $\Lambda = -2$. The winding number is now $n = 2$ instead of $n = 1$. It is the most massive case. This reflects the n^2 dependence of the integral mass formula. Again, there is a significant dip in the metric A near the origin while it remains positive (no horizon).

(n,v,Λ)	A_0[10]	A[10]	M_{metric}	$M_{integral}$
I=(1,1,-1)	101	100.588	2.589	2.587
II=(1,1,-2)	201	200.534	2.928	2.931
III=(1,1,-3)	301	300.489	3.211	3.210
IV=(1,2,-2)	201	199.961	6.529	6.529
V=(2,1,-2)	201	199.863	7.142	7.142

Table with values of the metric A_0 and A at $r = 10$, M_{metric} evaluated using A_0 and A and $M_{integral}$ evaluated using the integral mass formula. The two masses match (agree to two decimal places and sometimes at three decimal places).

Numerical results for Minkowski background

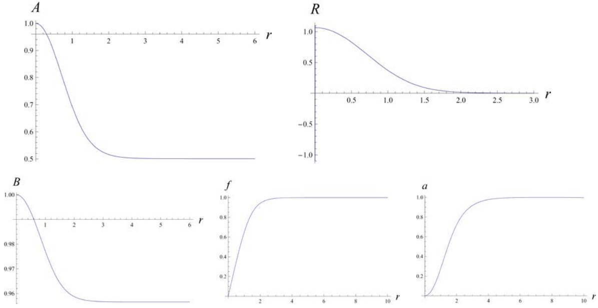


Figure: Case $\alpha = 1$ in asymptotically Minkowski spacetime. This is the case with the strongest gravitational coupling of the three. The Ricci scalar has the highest initial value of the three cases and the metric function A plateaus to the lowest value of D leading to the highest angular deficit. The mass is not too different from the other two cases or from the no gravity case. See Table 2 for values.

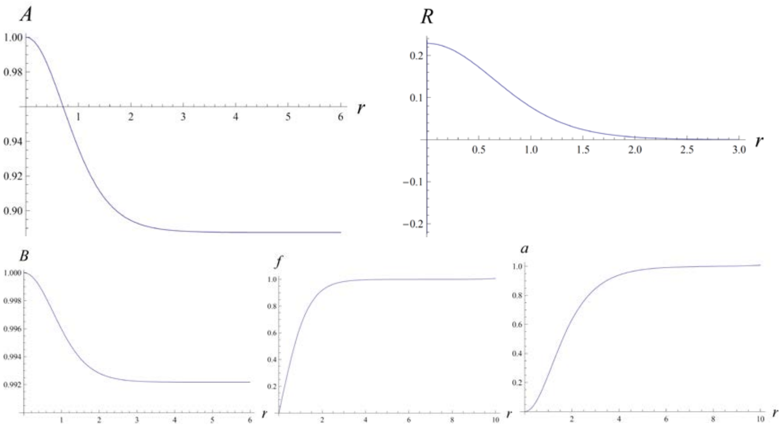


Figure: Case $\alpha = 5$. The Ricci scalar at the origin and the angular deficit is basically five times less than in the $\alpha = 1$ case. The mass is comparable to the other cases.

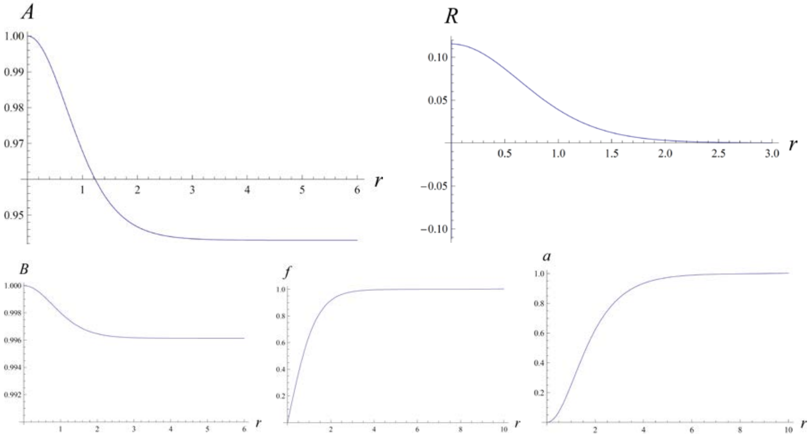


Figure: Case $\alpha = 10$. This is the case with the weakest gravitational coupling where the initial value of the Ricci scalar and the angular deficit are the smallest (basically 10 times smaller than in the $\alpha = 1$ case). The mass is again comparable to the others.

	D	δ	M_{metric}	$M_{integral}$
$\alpha = 1$	0.501	105.2 deg	3.672	3.674
$\alpha = 5$	0.887	20.9 deg	3.643	3.643
$\alpha = 10$	0.943	10.4 deg	3.638	3.638

Table: The quantity D is where the metric function $A(r)$ plateaus to asymptotically and δ is the angular deficit quoted in degrees. The mass M_{metric} is obtained using the metric and $M_{integral}$ is evaluated as an integral over the matter profiles. The two masses should match and they do. The angular deficit has a strong dependence on α: it basically increases tenfold from $\alpha = 10$ to $\alpha = 1$. In contrast, the mass of the vortex hardly changes with α and is not very different from the mass in the no gravity case ($M = 3.634$)

Gravity sheds new light on the logarithmic divergence in the absence of gauge fields

Without gauge fields, it is well known that the vortex has a logarithmic divergence in its energy. This problem persists in the presence of gravity. However, gravity sheds a new light on this problem.

Consider the case where $n \neq 0$, $a = 0$ identically (no gauge field) and $f \to v$ asymptotically. This yields asymptotically the solution

$$\begin{aligned} A(R) &= -\Lambda R^2 - \frac{n^2 v^2}{2\alpha} \ln(R) + D \\ &= -\Lambda R^2 - G\,M \ln(R) + D \quad \text{where } M = 8\pi\, n^2\, v^2 . \end{aligned}$$

The logarithmic term is nothing other than the Newtonian gravitational potential in $2+1$ dimensions with mass parameter M proportional to $n^2\, v^2$, the same mass dependence we encountered previously.

This term implies that the energy (the ADM mass) diverges logarithmically.

Conclusion

- Non-singular vortices (non-black holes) can be constructed in $2+1$ Einstein gravity in both an AdS_3 and Minkowski background.
- Integral mass formulas over the matter field profiles were obtained which complement the mass obtained using the metric. The integral mass formula applies to the no gravity case even though it was originally derived with gravity in mind.
- In the absence of gauge fields, the well-known logarithmic divergence in the energy of the vortex appears as a $2+1$ Newtonian logarithmic gravitational potential in the metric.
- Can one generate a finite energy vortex with logarithmic Newtonian potential when General Relativity is supplemented with a scalar field? One thing to try is switching the boundary conditions: have $\phi \to 0$ asymptotically and $\phi = v$ at the origin. This could work if the Breitonlohner-Freedman bound $V''(0)\,L^2 \geq -(d-1)^2/4$ is satisfied even though $\phi = 0$ is not the minimum of the potential $V(\phi)$.

Thank You