The Conformal Frontier

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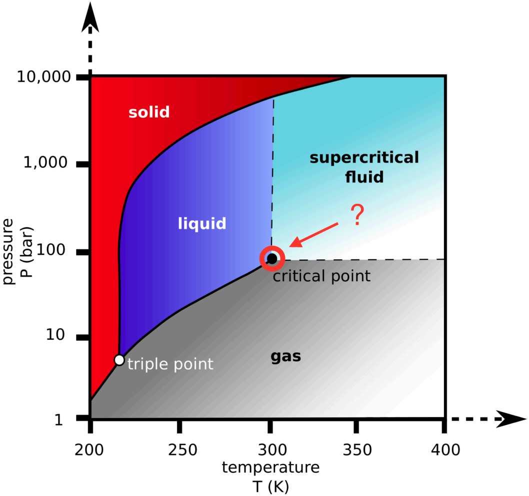
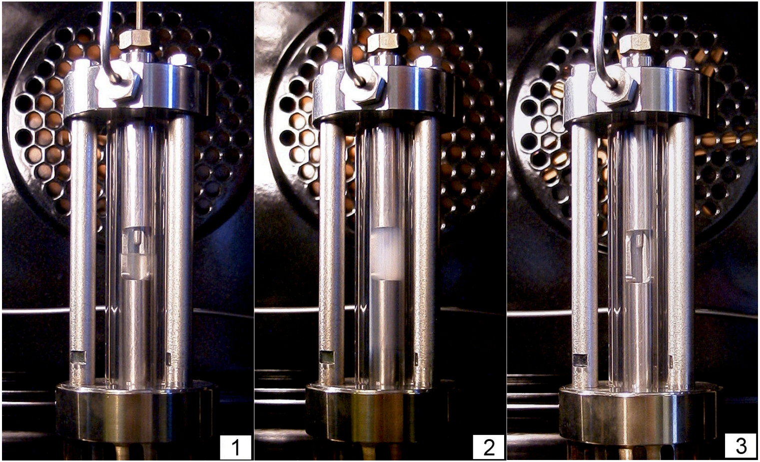
Fermilab Theory Seminar

Phase Diagram for Fluids



What happens at critical points?

Critical Opalescence



- As one heats the fluid (here Ethane) and approaches the critical point, it becomes milky and light cannot pass through.

Correlation Length

- Physically, what happens is that fluctuations of the fluid density occur over longer and longer distances, measured by the *correlation length* ξ:

$$\langle \delta\rho(x)\delta\rho(0)\rangle \sim \begin{cases} e^{-|x|/\xi} & |x| \gg \xi \\ \frac{1}{|x|^{1+\eta}} & |x| \ll \xi \end{cases}$$

- Near the critical point (at fixed $P = P_c$), it diverges as $\xi \sim (T - T_c)_+^{-\nu}$, and the leading behavior is captured by the critical exponents η and ν.

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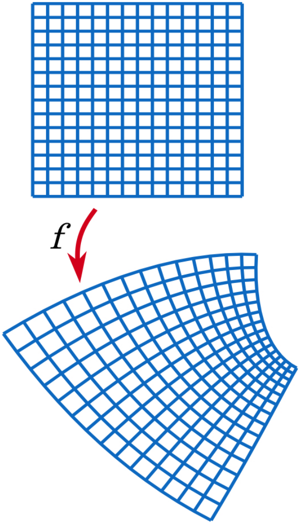
Universality:

In a wide variety of fluids, fluid mixtures, uniaxial magnets, and (2+1)D quantum critical points one can find the same exponents $\nu \sim .63, \eta \sim .04$!!

Scale and Conformal Symmetry

- $\xi \to \infty$ indicates that physical length scales have disappeared, signifying an emergent scale invariance. In fact, it is believed that the symmetry is enhanced to a larger group, the conformal group $\leftrightarrow SO(D+1,1)$.

- This group contains translations, rotations, rescalings, and special conformal transformations (angle-preserving twists).



Conformal Field Theory

- Conformal symmetry allows us to organize fields/local operators according to their behavior under rescalings, rotations, and SCTs

$$\mathcal{O}(\lambda x) = \lambda^{-\Delta}\mathcal{O}(x)$$

- The scaling dimensions Δ are directly related to the critical exponents:

$$\begin{aligned} \Delta_\sigma &= \frac{D-2}{2} + \eta/2 \\ \Delta_\epsilon &= D - 1/\nu \\ &\vdots \end{aligned}$$

and the number of relevant operators ($\Delta < D$) allowed by symmetries controls how many parameters need to be tuned.

3D Ising Model

- The conformal field theory describing liquid-vapor critical points (and uniaxial magnets) is often called the critical 3D Ising model, most simply described using a single scalar charged under a $\mathbb{Z}_2$ symmetry ($\sigma \to -\sigma$):

$$\mathcal{L}_{\text{Ising}} \sim (\partial\sigma)^2 + m^2\sigma^2 + \lambda\sigma^4 + \ldots$$

- At the critical value of m^2/λ^2, the IR fixed point is strongly-coupled and we can't use perturbation theory, so we must try other methods

Conformal Bootstrap

The conformal bootstrap asks if we can use:

1. Conformal Symmetry: $SO(D, 2)$ or $SO(D+1, 1)$
2. Crossing Symmetry
3. Unitarity or Reflection Positivity

to classify and solve conformal field theories.

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- Beautiful success story in 2D $\rightarrow \{\Delta_\sigma, \Delta_\epsilon\} = \{\frac{1}{8}, 1\}$ in 2D Ising
  [Ferrara, Gatto, Grillo '73; Polyakov '74; Belavin, Polyakov, Zamolodchikov '83]
- Exciting progress in $D > 2$ starting in 2008
  [Rattazzi, Rychkov, Tonni, Vichi '08; ...]

Conformal Block Expansion

Can probe spectrum by expanding 4-point functions in conformal blocks:

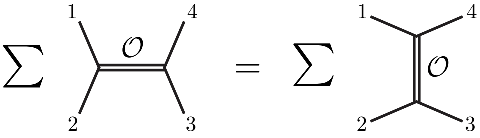
$$\langle \wick{\c1\sigma(x_1)\c1\sigma(x_2)}\,\wick{\c1\sigma(x_3)\c1\sigma(x_4)} \rangle = \sum_{\Delta,\ell} \lambda^2_{\sigma\sigma\mathcal{O}}\, g_{\Delta,\ell}(x_1,x_2,x_3,x_4)$$

- Blocks $g_{\Delta,\ell}(x_1,x_2,x_3,x_4) = \frac{g_{\Delta,\ell}(z,\bar{z})}{x_{12}^{2\Delta_\sigma} x_{34}^{2\Delta_\sigma}}$ known special functions giving the contribution of primary $\mathcal{O} \in \sigma \times \sigma$ with dimension Δ and spin ℓ
- Similar to expansion in spherical harmonics Y_ℓ^m, but for CFTs

Crossing Symmetry

$\langle\sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4)\rangle$ is symmetric under permutations of x_i:

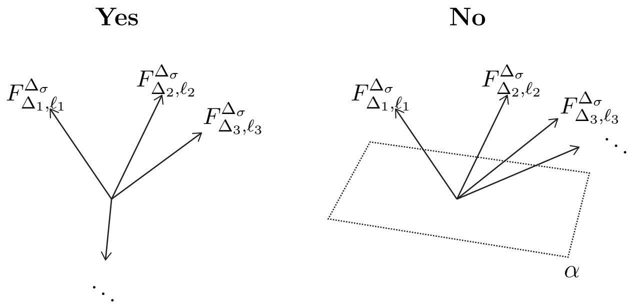
- Switching $x_1 \leftrightarrow x_3$ gives the crossing symmetry condition:



$$\sum_{\Delta,\ell} \lambda^2_{\sigma\sigma\mathcal{O}} \left[g_{\Delta,\ell}(x_1,x_2,x_3,x_4) - g_{\Delta,\ell}(x_3,x_2,x_1,x_4)\right] \;=\; 0$$

- Unknowns are scaling dimensions and coefficients: $\{\Delta, \lambda^2_{\sigma\sigma\mathcal{O}}\}$, with lower bounds $\lambda^2_{\sigma\sigma\mathcal{O}} \geq 0$ and $\Delta \geq \ell + D - 2 - \delta_{\ell,0}\frac{(D-2)}{2}$ from unitarity.

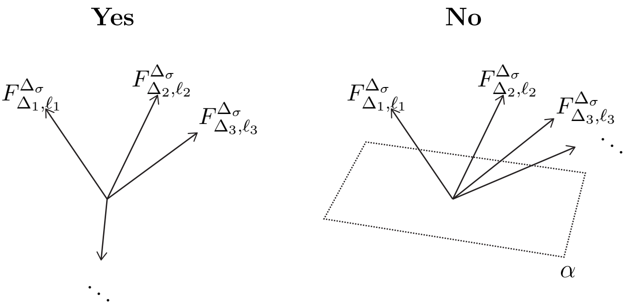
Numerical Approach



- Make some assumption on $\{\Delta, \lambda_{ijk}\}$, search for functional

$$\alpha = \left(\sum_{m+n\leq\Lambda} \alpha_{mn} \partial_z^m \partial_{\bar{z}}^n \Big|_{1/2,1/2} \right) \text{ implying } 0 = \sum(\text{positive})$$

Numerical Approach



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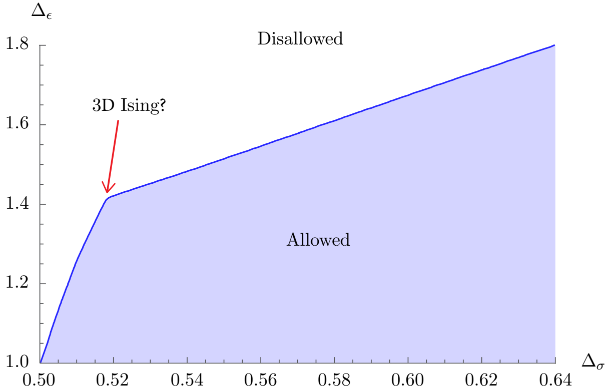
  $\alpha = \left(\sum_{m+n\leq\Lambda} \alpha_{mn} \partial_z^m \partial_{\bar{z}}^n \Big|_{1/2,1/2} \right)$ implying $0 = \sum$(positive)

- Find them by solving semidefinite programs: `SDPB 2.0`
  https://github.com/davidsd/sdpb [Simmons-Duffin '15; Landry, Simmons-Duffin '19]

Bootstrap software repository: http://gitlab.com/bootstrapcollaboration

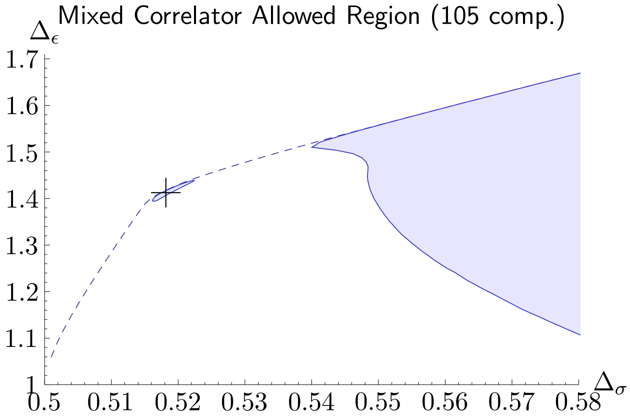
3D Dimension Bounds



[El-Showk, Paulos, DP, Rychkov, Simmons-Duffin, Vichi, '12; '14]

- Upper bound on first $\mathbb{Z}_2$-even scalar in $\sigma \times \sigma \sim \mathbb{1} + \epsilon + \ldots$ from $\langle \sigma\sigma\sigma\sigma \rangle$

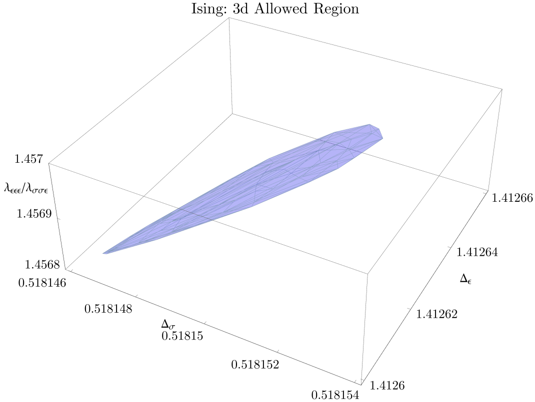
3D Ising Island



[Kos, DP, Simmons-Duffin '14]

- Combine 5 crossing relations from $\langle\sigma\sigma\sigma\sigma\rangle, \langle\sigma\sigma\epsilon\epsilon\rangle, \langle\epsilon\epsilon\epsilon\epsilon\rangle$ and impose that σ and ϵ are the only relevant ($\Delta < 3$) operators

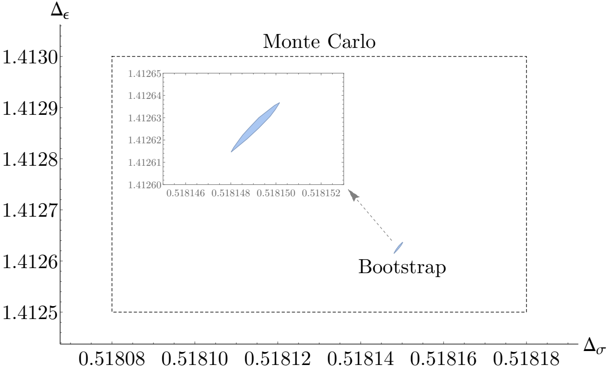
Mixed Correlator Islands



[Kos, DP, Simmons-Duffin, Vichi '16]

- Best bounds: perform "OPE scan" over ratio $r \equiv \lambda_{\epsilon\epsilon\epsilon}/\lambda_{\sigma\sigma\epsilon} \to$ 3d island
- Excludes degenerate exchanged operators at same $\Delta_{\sigma,\epsilon}$ but different λ's

3D Ising Island



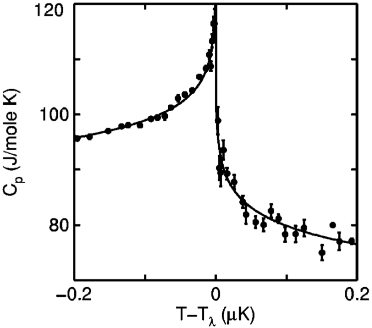
[Kos, DP, Simmons-Duffin, Vichi '16]

- Increase search space to $5 \times 253 = 1265$ components ($\Lambda = 43$)

$$\begin{aligned}\{\Delta_\sigma, \Delta_\epsilon\} &= \{0.518149(1), 1.412625(10)\} \\ \{\lambda_{\sigma\sigma\epsilon}, \lambda_{\epsilon\epsilon\epsilon}\} &= \{1.0518537(41), 1.532435(19)\}\end{aligned}$$

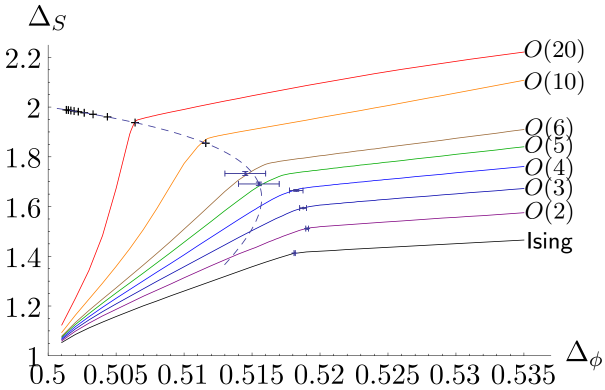
3D O(N) Models

Generalization to N scalars ϕ_i: $\mathcal{L}^{int}_{O(N)} \sim \lambda(\phi_i \phi^i)^2$



- $N = 2$: Superfluid (λ) transition in ^{4}He [Lipa et al, '96; '03]
- $N = 3$: Isotropic ferromagnets (Fe, Co, Ni, ...)
- Large N: Solvable in $1/N$ expansion

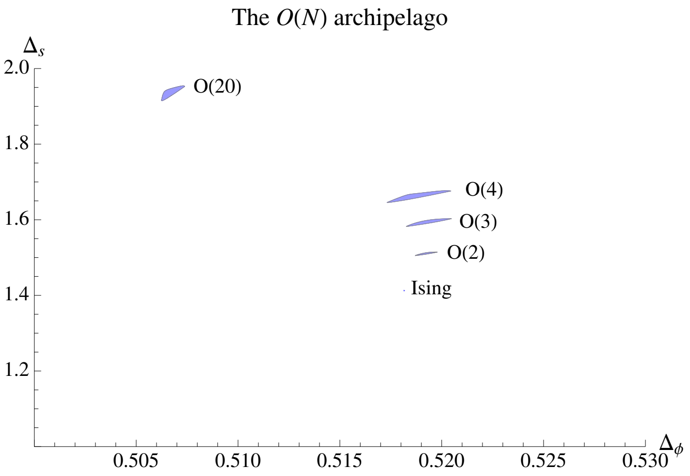
3D $O(N)$ Bounds



[Kos, DP, Simmons-Duffin '13]

- Extension to $\langle \phi_i \phi_j \phi_k \phi_l \rangle$, where ϕ_i is $O(N)$ vector
- Large N: matches $1/N$ expansion, Small N: matches experiment!

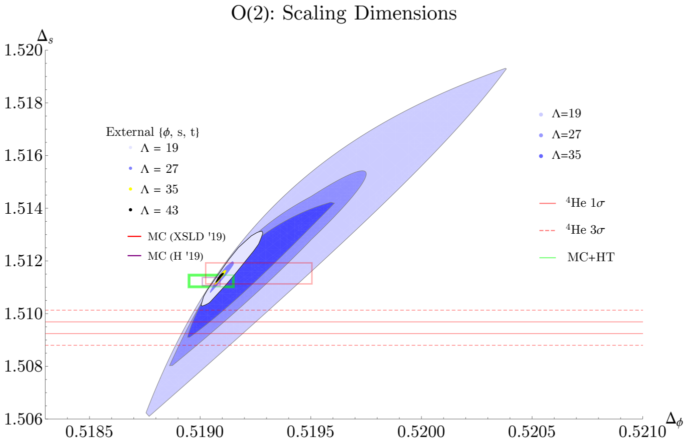
$O(N)$ Archipelago from Mixed Correlators



[Kos, DP, Simmons-Duffin, Vichi '15; '16]

- With $\{\phi_i, s\}$ correlators, again find islands (Input: $\Delta_{\phi'} > 3, \Delta_{s'} > 3$)

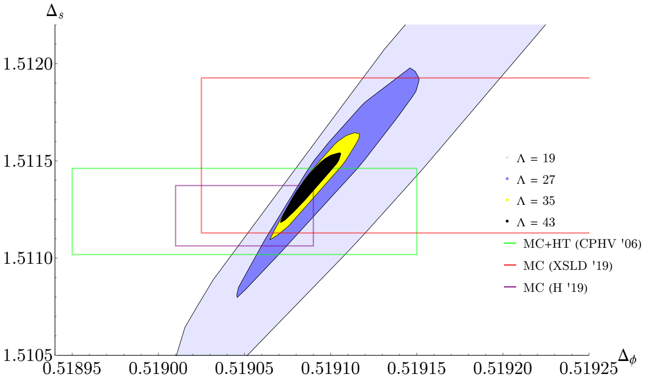
$O(2)$ from $\{\phi_i, s, t_{ij}\}$ System



[Chester, Landry, Liu, DP, Simmons-Duffin, Su, Vichi '19]

- Best result from $\{\phi_i, s, t_{ij}\}$ system (22 crossing equations)
- Resolves 8σ discrepancy between lattice and expt (^{4}He)

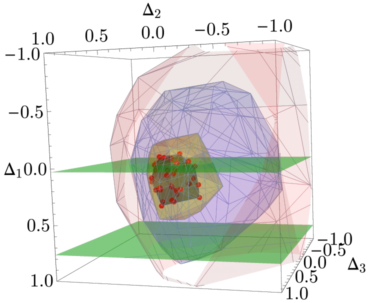
$O(2)$ from $\{\phi_i, s, t_{ij}\}$ System

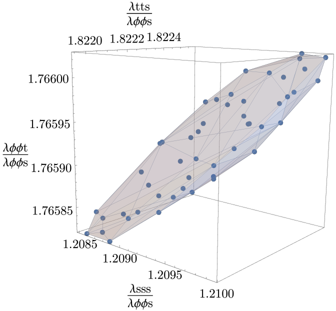


[Chester, Landry, Liu, DP, Simmons-Duffin, Su, Vichi '19]

$$\begin{aligned} \text{Bootstrap: } \alpha &= -0.01527(21) \\ \text{MC: } \alpha &= -0.01507(21) \quad \text{[Hasenbusch '19]} \\ {}^4\text{He: } \alpha &= -0.0127(3) \quad \text{[Lipa et al, '96; '03]} \end{aligned}$$

$O(2)$ from $\{\phi_i, s, t_{ij}\}$ System





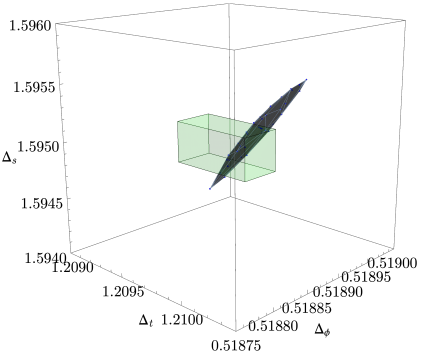
[Chester, Landry, Liu, DP, Simmons-Duffin, Su, Vichi, '19]

- Best results require 6d search over $\{\Delta_\phi, \Delta_s, \Delta_t, \frac{\lambda_{sss}}{\lambda_{\phi\phi s}}, \frac{\lambda_{tts}}{\lambda_{\phi\phi s}}, \frac{\lambda_{\phi\phi t}}{\lambda_{\phi\phi s}}\}$

$$\{\Delta_\phi, \Delta_s, \Delta_t\} = \{0.519088(17), 1.51136(18), 1.23629(9)\}$$

$$\{\frac{\lambda_{sss}}{\lambda_{\phi\phi s}}, \frac{\lambda_{tts}}{\lambda_{\phi\phi s}}, \frac{\lambda_{\phi\phi t}}{\lambda_{\phi\phi s}}\} = \{1.20926(46), 1.82227(19), 1.765918(64)\}$$

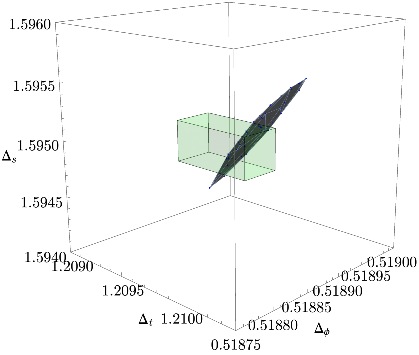
$O(3)$ from $\{\phi_i, s, t_{ij}\}$ System



Black (Bootstrap): [Chester, Landry, Liu, DP, Simmons-Duffin, Su, Vichi '20]
Green (Monte Carlo): [Hasenbusch, Vicari '11; Hasenbusch '20]

- Best $O(3)$ island: $\{\Delta_\phi, \Delta_s, \Delta_t\} = \{0.51894(5), 1.5949(6), 1.2095(2)\}$
- Open question: is $\phi^{\{i}\phi^j\phi^k\phi^{l\}}$ relevant or irrelevant in $O(3)$ model?

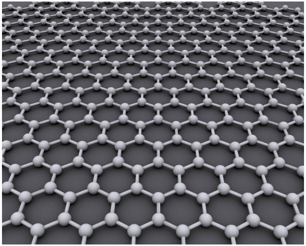
$O(3)$ from $\{\phi_i, s, t_{ij}\}$ System



Black (Bootstrap): [Chester, Landry, Liu, DP, Simmons-Duffin, Su, Vichi, '20]
Green (Monte Carlo): [Hasenbusch, Vicari '11; Hasenbusch '20]

- Using `tiptop` search, find it is relevant: $\Delta_{\phi^{\{i}\phi^j\phi^k\phi^{l\}}} < 2.99056$!
- Proof that critical Heisenberg magnets are unstable to cubic anisotropy, should flow to fixed point with cubic symmetry C_3 rather than $O(3)$

3D Fermion Models (Gross-Neveu-Yukawa)



Interesting CFTs obtained from fixed points involving N fermions:

$\mathcal{L}_{GNY} \sim \frac{g}{2}\sigma\overline{\psi}^i\psi_i + \lambda\sigma^4$ (and variations with multiple scalars)

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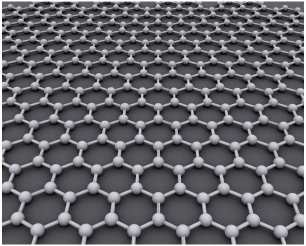

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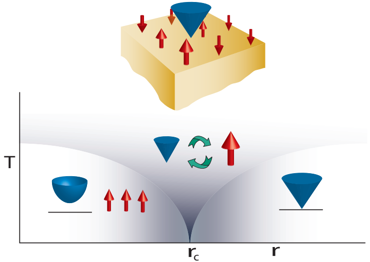
$\mathcal{L}_{GNY} \sim \frac{g}{2}\sigma\overline{\psi}^i\psi_i + \lambda\sigma^4$ (and variations with multiple scalars)

- Large N: Solvable in $1/N$ expansion [Gracey '92; ...]
- $N = 8$: Possible QCPs in D-wave superconductors or graphene [Vojta, Zhang, Sachdev '00; Herbut '06; Classen, Herbut, Scherer '17]
- $N = 4$: Spinless fermions on honeycomb lattice, gapless semiconductors [Raghu, Qi, Honerkamp, Zhang '07; Moon, Xu, Kim, Balents '12; Herbut, Janssen '14]

Minimal 3d SCFT ($N = 1$ Gross-Neveu-Yukawa)



- The $N = 1$ Gross-Neveu-Yukawa model has 3d $\mathcal{N} = 1$ supersymmetry:

$$V = \frac{g}{2}\sigma\overline{\psi}\psi + \frac{g}{8}\sigma^4 \qquad \leftrightarrow \qquad W = \frac{g}{3}\Sigma^3, \quad \Sigma = \sigma + \theta\psi + \theta^2\epsilon$$

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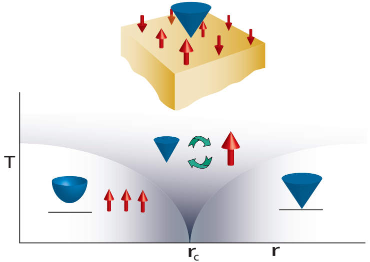

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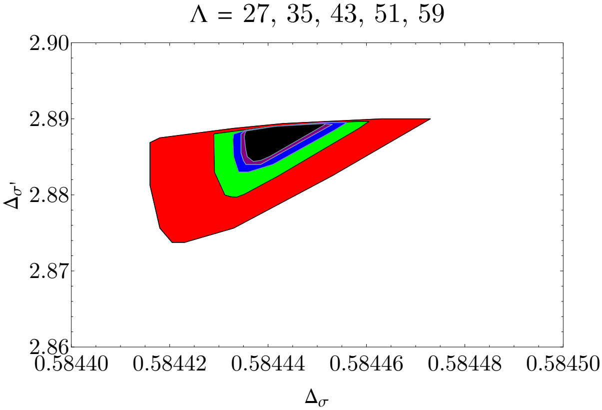
- May be realizable in (3+1)D topological superconductors, with (2+1)D boundary supporting Majorana fermions [Grover, Sheng, Vishwanath '13]

Supersymmetric Island

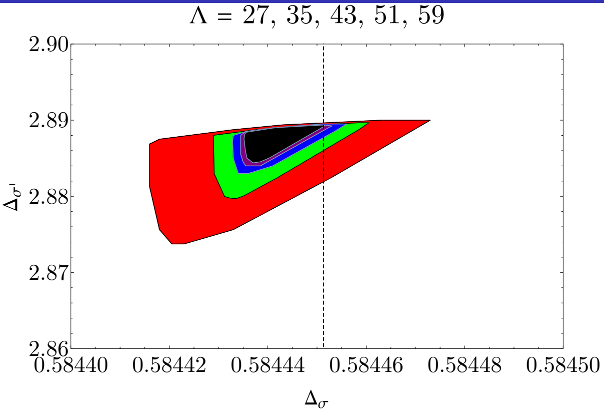


[Rong, Su '18; Atanasov, Hillman, DP '18; Atanasov, Hillman, DP, Rong, Su, in progress]

- $\{\sigma, \epsilon\}$ SUSY system $\rightarrow \Delta_\sigma = .5844435(83)$, $\Delta_{\sigma'} = 2.8869(25)$
  Compare to ϵ-expansion: $\Delta_\sigma = .5837(14)$ [Ihrig, Mihaila, Scherer '18]

(Assumptions: $\mathcal{N} = 1$ SUSY, $\Delta_{\epsilon'} \geq 3, \Delta_{\sigma''} \geq 3$)

Supersymmetric Island



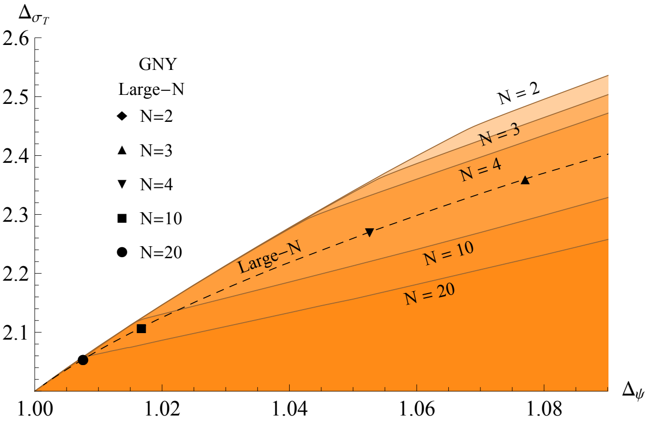
[Rong, Su '18; Atanasov, Hillman, DP '18; Atanasov, Hillman, DP, Rong, Su, in progress]

- Observation: Write $\frac{\lambda_{\epsilon\epsilon\epsilon}}{\lambda_{\sigma\sigma\epsilon}} = \tan(\theta)$ and use SUSY relation $\frac{\lambda_{\epsilon\epsilon\epsilon}}{\lambda_{\sigma\sigma\epsilon}} = f(\Delta_\sigma)$

$$\rightarrow \theta = 1.00010(11)$$

Is there any meaning to $\theta \simeq 1$??

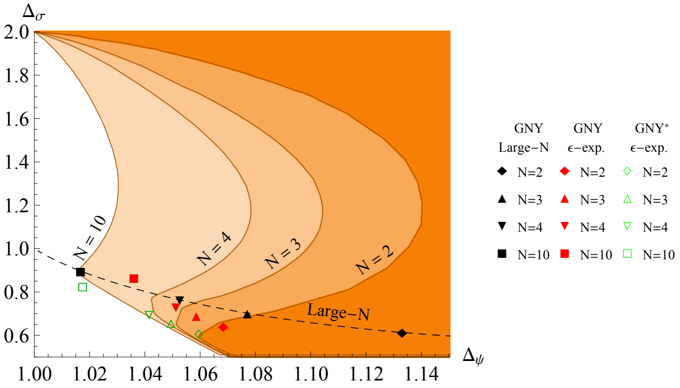
3D O(N) Fermion Bootstrap



[Iliesiu, Kos, DP, Pufu, Simmons-Duffin '17]

- Bootstrap for fermion 4-point functions $\langle \psi_i \psi_j \psi_k \psi_l \rangle$
- Kinks in symmetric tensor bounds match GNY models at large N

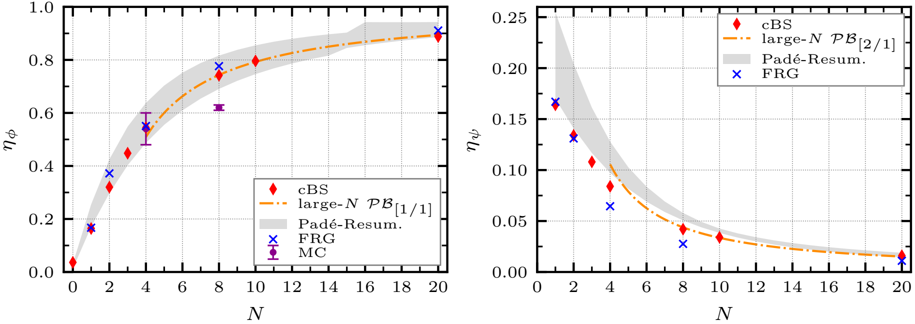
3D O(N) Fermion Bootstrap



[Iliesiu, Kos, DP, Pufu, Simmons-Duffin '17]

- Intricate structure in $\{\Delta_\psi, \Delta_\sigma\}$ plane assuming σ' irrelevant
- Upper kinks plausibly related to GNY models

3D Fermion Bootstrap



[Ihrig, Mihaila, Scherer '18]

- Kink locations showed good agreement with other methods

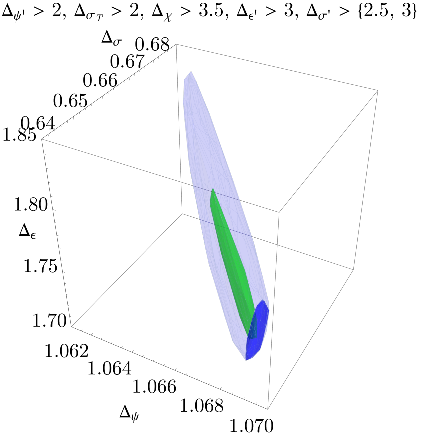
State-of-the-art Fermion Bootstrap

- Our best bootstrap computations for the O(N) GNY models now involve all 4-point functions containing $\{\psi_i, \sigma, \epsilon\}$, giving 38 crossing relations after allowing for all 4-point tensor structures:

$$\sum_{c,\Delta} \vec{\lambda}^T_{c,\Delta} V^i_{c,\Delta}(z,\overline{z}) \vec{\lambda}_{c,\Delta} = 0, \qquad (i = 1, \ldots, 38)$$

- We impose some gap assumptions motivated by large-N expectations, and search for allowed points in the 6d space: $\{\Delta_\psi, \Delta_\sigma, \Delta_\epsilon\}$ and $\{\frac{\lambda_{\psi\psi\sigma}}{\lambda_{\epsilon\epsilon\epsilon}}, \frac{\lambda_{\psi\psi\epsilon}}{\lambda_{\epsilon\epsilon\epsilon}}, \frac{\lambda_{\sigma\sigma\epsilon}}{\lambda_{\epsilon\epsilon\epsilon}}\}$

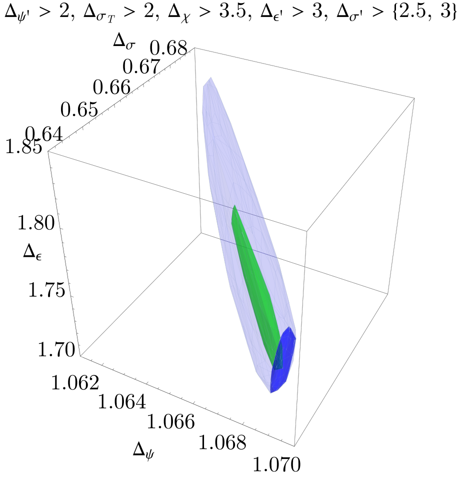
Preliminary Islands for $N = 2$ Gross-Neveu-Yukawa Model



[Erramilli, Iliesiu, Kravchuk, A. Liu, DP, Simmons-Duffin, in progress]

- Preliminary islands from $\{\sigma, \psi_i, \epsilon\}$ system at $\Lambda = 15, 23$
- Gaps motivated by large N estimates and E.O.M. $\partial\psi \sim \sigma\psi$

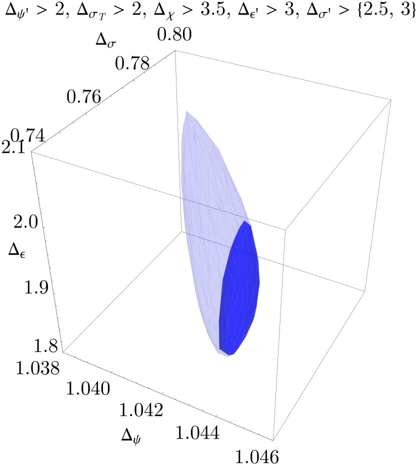
Preliminary Islands for $N=2$ Gross-Neveu-Yukawa Model



[Erramilli, Iliesiu, Kravchuk, A. Liu, DP, Simmons-Duffin, in progress]

$$\{\Delta_\psi, \Delta_\sigma, \Delta_\epsilon\}_{\mathsf{CB}} = \{1.0686(3), 0.651(3), 1.73(2)\} \qquad (\Lambda = 15, \Delta_{\sigma'} > 3)$$
$$\{\eta_\psi, \eta_\sigma, 1/\nu\}_{\mathsf{CB}} = \{0.1371(6), 0.302(6), 1.27(2)\}$$

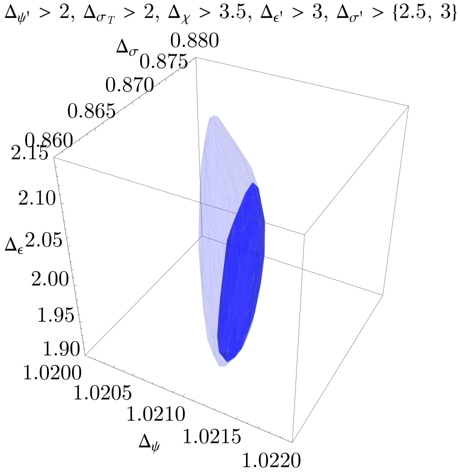
Preliminary Islands for $N = 4$ Gross-Neveu-Yukawa Model



[Erramilli, Iliesiu, Kravchuk, A. Liu, DP, Simmons-Duffin, in progress]

$$\{\Delta_\psi, \Delta_\sigma, \Delta_\epsilon\}_{\text{CB}} = \{1.0434(7), 0.76(1), 1.91(6)\} \qquad (\Lambda = 11, \Delta_{\sigma'} > 3)$$
$$\{\eta_\psi, \eta_\sigma, 1/\nu\}_{\text{CB}} = \{0.0869(14), 0.52(2), 1.09(6)\}$$

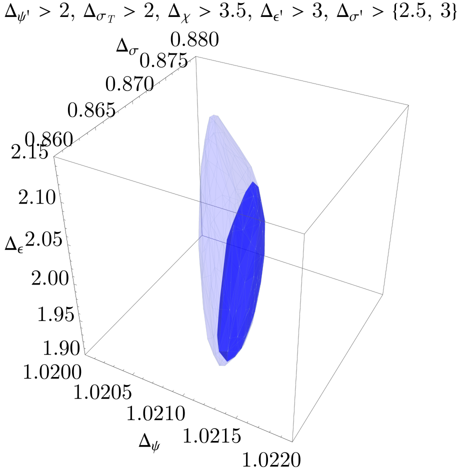
Preliminary Islands for $N = 8$ Gross-Neveu-Yukawa Model



[Erramilli, Iliesiu, Kravchuk, A. Liu, DP, Simmons-Duffin, in progress]

$$\{\Delta_\psi, \Delta_\sigma, \Delta_\epsilon\}_{\mathsf{CB}} = \{1.02115(25), 0.867(6), 2.01(6)\} \qquad (\Lambda = 11, \Delta_{\sigma'} > 3)$$
$$\{\eta_\psi, \eta_\sigma, 1/\nu\}_{\mathsf{CB}} = \{0.0423(5), 0.735(12), 0.99(6)\}$$

Preliminary Islands for $N = 8$ Gross-Neveu-Yukawa Model



[Erramilli, Iliesiu, Kravchuk, A. Liu, DP, Simmons-Duffin, in progress]

$$\{\eta_\psi, \eta_\sigma, 1/\nu\}_{\text{CB}} = \{0.0423(5), 0.735(12), 0.99(6)\}$$

$$\{\eta_\psi, \eta_\sigma, 1/\nu\}_{\text{RG}} = \{0.043, \quad 0.731, \quad 0.982\}$$ [Ihrig, Mihaila, Scherer '18]

$$\{\eta_\psi, \eta_\sigma, 1/\nu\}_{\text{QMC}} = \{0.05(2), \quad 0.60(2), \quad 1.1(1)\}$$ [Liu, Wang, Sun, Meng '20]

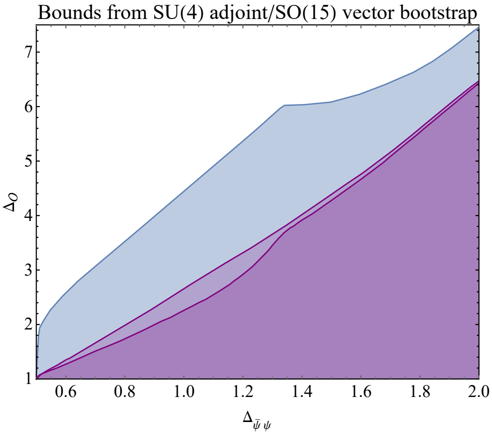
Bootstrapping Gauge Theories

- Many motivations to bootstrap gauge theories: 3d spin liquids and deconfined critical points, 4d physics beyond the SM, dualities, conformal windows ...
- Some concrete targets are $U(1)$ QED$_3$ with $N_f^* < N_f < \infty$ fermions, $SU(N)$ QCD$_4$ with $N_f^* < N_f < \frac{11}{2}N$ fermions, + many more

Bootstrapping Gauge Theories

- On the other hand, bootstrapping theories like QED_3 or QCD_4 is hard because we can only use gauge-invariant operators: $\overline{\psi}\psi$, $(\overline{\psi}\psi)^2$, etc
- E.g., hard to use correlation functions of $\overline{\psi}\psi$ to distinguish $U(1)$ QED_3 from $SU(N)$ QCD_3, or QCD_4 theories with different gauge groups

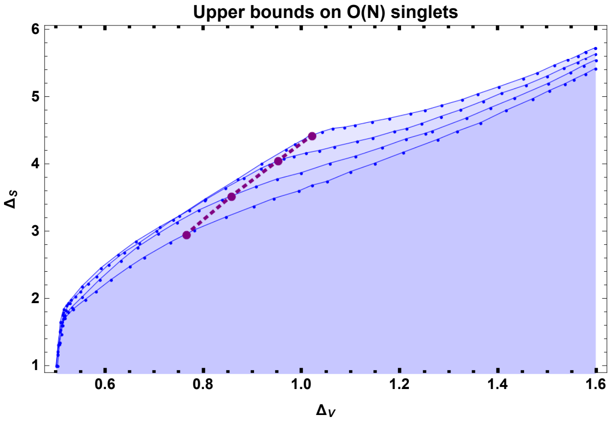
Bootstrapping $SU(N_f = 4)$ QED$_3$ from $\overline{\psi}\psi$ Correlators



[Z. Li '18; DP, Z. Li '20]

- Singlet (upper), $S\overline{S}$ (middle), and $\{S\overline{S}, A\overline{A}\}$ (lower) reps of $SU(4)$
- Funny transition around $\Delta_{\overline{\psi}\psi} \sim 1.35$ (close to $2 - \frac{64}{\pi^2 N_f} \sim 1.46$), but singlet bound is far from expected value $\sim 3 - 4$

Bootstrapping $SU(N_f)$ QED$_3$ from $\overline{\psi}\psi$ Correlators



[Z. Li '18]

- Coincidence with $SO(N = N_f^2 - 1)$ singlet bounds (here $N = 5, 6, 7, 8$)
- Tracking kink to smaller N_f, forces singlet to become relevant around $N_f \sim 2.5$, maybe a clue that $N_f = 2$ is outside conformal window?

Bootstrapping QED$_3$ with N_f fermion flavors

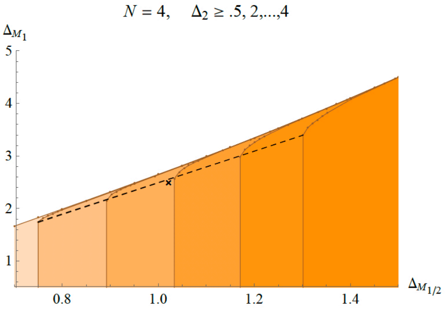
- Complementary progress was made in [Chester, Pufu '16] by bootstrapping correlators of *monopole operators*, charged under $J_T^\mu = \epsilon^{\mu\nu\rho} F_{\nu\rho}$
- We know various properties of them at large N_f, e.g. the lightest monopole M_q with $q = 1/2$ has dimension $\Delta_{M_{1/2}} = 0.265 N_f - 0.0383$

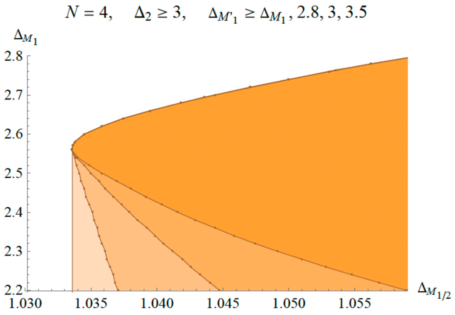
Large N_f Estimates for $N_f = 4$ QED$_3$

$\mathrm{SO}(2)_T$	$\mathrm{SU}(4)$	j	Δ_1	Δ_2	OPE
S	(000) (singlet)	0	$4 + \frac{64(2\pm\sqrt{7})}{3\pi^2 N_f} = \begin{matrix}6.510\\3.651\end{matrix}$	5.00^*	λ_{rrO}, λ_{MMO}
S	(211) (Adj)	0	$4 + \frac{8(25\pm\sqrt{2317})}{3\pi^2 N_f} = \begin{matrix}8.940\\2.437\end{matrix}$	5.00^*	λ_{rrO}
S	(211) (Adj)	1	2.00 $[J_f]$	4.00	λ_{rrO}, λ_{MMO}
S	(220) (A$\bar{\text{A}}$)	0	$4 - \frac{64}{\pi^2 N_f} = 2.379$	6.00	λ_{rrO}, λ_{MMO}
A	(000) (singlet)	1	2.00 $[J_T]$	3.00	λ_{MMO}
A	(211) (Adj)	0	$2 - \frac{64}{3\pi^2 N_f} = 1.460$ $[r = \bar{\psi}\psi]$	4.00	λ_{MMO}
A	(220) (A$\bar{\text{A}}$)	1	4.00	6.00	λ_{MMO}
T	(000) (singlet)	0	4.424	6.156	λ_{MMO}
T	(211) (Adj)	1	2.692	4.424	λ_{MMO}
T	(220) (A$\bar{\text{A}}$)	0	$0.673 N_f - 0.194 = 2.498$ $[M_1]$	6.156	λ_{MMO}
V	(110) (M)	0	$0.265 N_f - 0.0383 = 1.022$ $[M_{1/2}]$	3.888	λ_{rMO}
V	(110) (M)	1	2.474	3.060^*	λ_{rMO}
V	(200) (S)	0	3.888	4.474^*	λ_{rMO}
V	(200) (S)	1	2.474	3.888	λ_{rMO}
V	(321) (AAdj)	0	3.888	5.303	λ_{rMO}
V	(321) (AAdj)	1	3.888	4.924	λ_{rMO}
S	(310) (S$\bar{\text{A}}$)	1	5.00	6.00	λ_{rrO}
S	(422) (S$\bar{\text{S}}$)	0	$4 + \frac{64}{3\pi^2 N_f} = 4.540$	6.00	λ_{rrO}

[Chester, Pufu '16; Chester, Iliesiu, Mezei, Pufu '17; Albayrak, Erramilli, Z. Li, DP, Y. Xin]

$N_f = 4$ QED$_3$ Bootstrap from $M_{1/2}$ Correlators

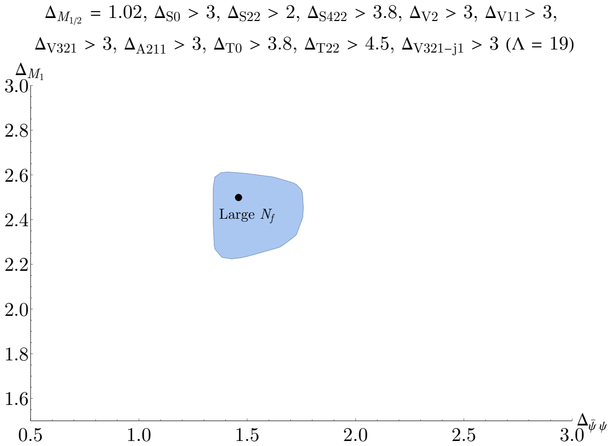




[Chester, Pufu '16]

- Bound on Δ_{M_1} nearly saturated by large N_f predictions, can carve out peninsula by isolating M_1
- However, gap $\Delta_2 = \Delta_{S22} \geq 3$ may be too strong, while other gaps motivated by our knowledge of the spectrum were not used
- We are working to improve the situation by considering mixed $\{\overline{\psi}\psi, M_{1/2}\}$ correlators and using more strategic gap assumptions

$N_f = 4$ QED$_3$ Bootstrap from $\{\overline{\psi}\psi, M_{1/2}\}$ Correlators

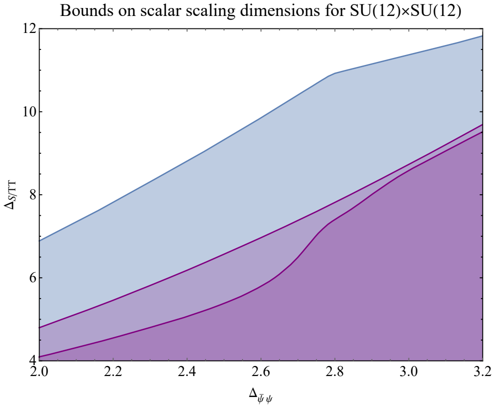


[Albayrak, Erramilli, Z. Li, DP, Y. Xin, in progress]

- We have not yet understood how to isolate $M_{1/2}$, but if we input the large N_f value then other scaling dimensions live in an island

(For progress in scalar QED$_3$, see also: [He, Rong, Su '21; Manenti, Vichi '21])

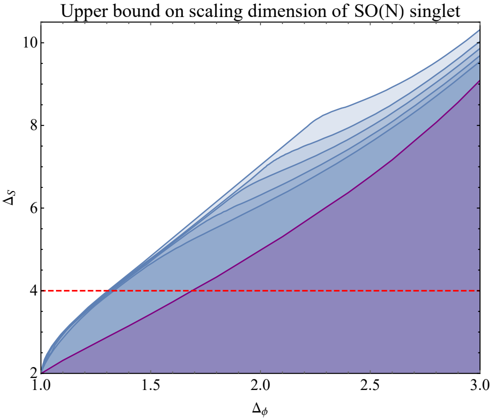
$N_f = 12$ QCD$_4$ Bootstrap from $\overline{\psi}\psi$ Correlators



[Z. Li, DP '20]

- Going to 4d with $SU(12) \times SU(12)$ symmetry, interesting transition around $\Delta_{\overline{\psi}\psi} \sim 2.78$, near estimates $\gamma_{\overline{\psi}\psi}^{SU(3)} \sim [.2, .4]$ from other methods
- Singlet bound is much weaker than physical theory, but could be finding "nearby" solution with leading singlet removed, needs further study...

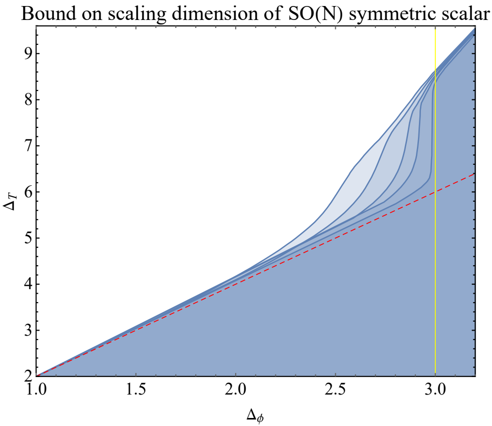
4d $\overline{\psi}\psi$ Bootstrap at Small N_f



[Z. Li, DP '20]

- Singlet bounds for $N = 2N_f^2 = 14, 18, 24, 32, 50$
- Kink in singlet bound smoothes out around $N_f \sim 2.5 - 3$
- Too small to be in conformal window of QCD with fundamental fermions, maybe related to gauge theories with rank 2 representations?

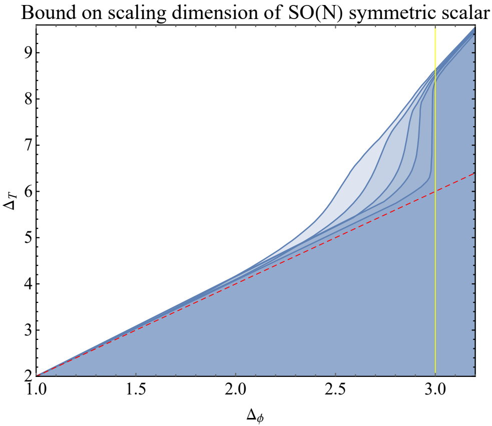
4d $\overline{\psi}\psi$ Bootstrap at Large N_f



[Z. Li, DP '20]

- At large N_f, bound on non-singlet scalar shows a sharp jump: bottom at $(3, 6)$ is free fermion theory while top at $(3, 8)$ is a subtraction of free fermion correlators and generalized free correlators [He, Rong, Su '20].

4d $\overline{\psi}\psi$ Bootstrap at Large N_f



[Z. Li, DP '20]

- Fit to jump location goes roughly like $\Delta_{\overline{\psi}\psi} \sim 3 - \frac{2.5\pm1}{N_f}$ (compare to $\Delta_{\overline{\psi}\psi} = 3 - \frac{22}{25}\frac{n}{N_f}$ for $N_f = \frac{11}{2}N - n$ in Veneziano limit)

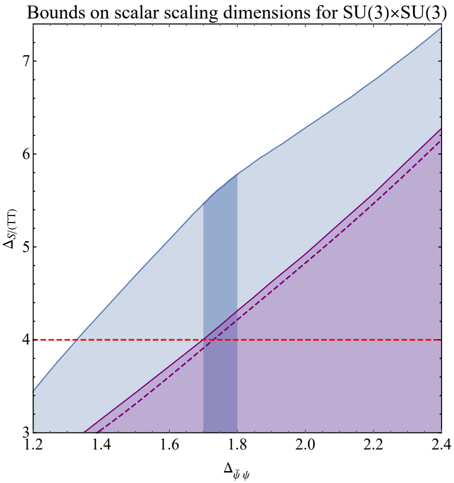
Bootstrap Future

Where do we go from here?

- Find islands for other interesting CFTs
  - 3d Gross-Neveu-Yukawa Models (XY, Heisenberg)
  - Understand how to isolate gauge theories (3d QED, 4d QCD, ...)
  - Superconformal zoo
- Study larger systems of bootstrap equations
  - Mixed correlators with spinning operators (ψ, J^{μ}, $T^{\mu\nu}$)
  - Improve algorithms and software tools
- Improve analytical understanding of bootstrap equations
  - Match to Lorentzian Inversion formula, conformal dispersion relations
  - Incorporate analytical insights into numerical algorithms

Backup Slides

4d $\overline{\psi}\psi$ Bootstrap at $N_f = 3$



[Z. Li, DP '20]

- Taken at face value, $\Delta_{\overline{\psi}\psi} \sim 1.75$, singlet dimension $\Delta_S \sim 5.5$ and bound on symmetric (TT) representation close to marginal $\Delta_{TT} \sim 4$.

Large N GNY predictions

Op.	Parity	$O(N)$	Δ at large N
ψ_i	$+$	V	$1+\frac{4}{3\pi^2 N}+\frac{896}{27\pi^4 N^2}+\frac{c_3}{N^3}+O(1/N^4)$
σ	$-$	S	$1-\frac{32}{3\pi^2 N}+\frac{32(304-27\pi^2)}{27\pi^4 N^2}+O(\frac{1}{N^3})$
$\epsilon\sim\sigma^2$	$+$	S	$2+\frac{32}{3\pi^2 N}-\frac{64(632+27\pi^2)}{27\pi^4 N^2}+O(\frac{1}{N^3})$
$(\sigma_T)_{(ij)}\sim\psi_{(i}\psi_{j)}$	$-$	T	$2+\frac{32}{3\pi^2 N}+O(\frac{1}{N^2})$
$\sigma'\sim\sigma^3$	$-$	S	$3+\frac{64}{\pi^2 N}+\frac{c_2}{N^2}+O(\frac{1}{N^3})$
$\epsilon'\sim\sigma\partial^2\sigma$	$+$	S	$4-\frac{64}{3\pi^2 N}+\frac{c_2'}{N^2}+O(\frac{1}{N^3})$
$\epsilon''\sim\sigma^4$	$+$	S	$4+\frac{448}{3\pi^2 N}+\frac{c_2''}{N^2}+O(\frac{1}{N^2})$
$\psi_i'\sim\sigma^2\psi_i$	$+$	V	$3+\frac{100}{3\pi^2 N}+O(\frac{1}{N^2})$
$\chi_i\sim\sigma^3\psi_i$	$-$	V	$4+\frac{392}{3\pi^2 N}+O(\frac{1}{N^2})$

[Gracey '92; Derkachov, Kivel, Sepanenko, Vasiliev '93; Gracey '93; Iliesiu, Kos, DP, Pufu, Simmons-Duffin, Yacoby '15; Gracey '17, Manashov, Strohmaier '17; Erramilli, Iliesiu, Kravchuk, Liu, DP, Simmons-Duffin, in progress]

`blocks_3d` Software

- General 3d conformal blocks can be expanded recursively in poles:

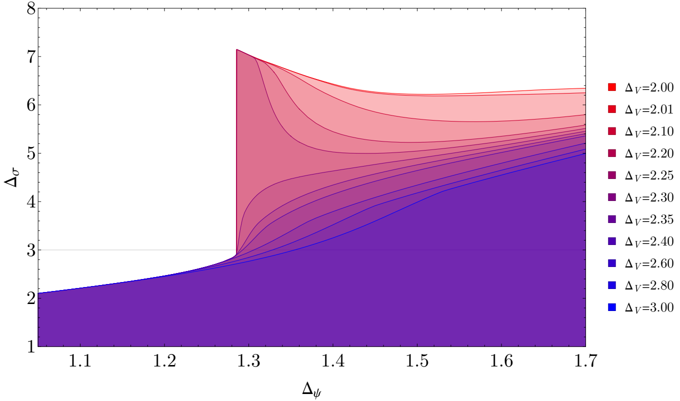
$$g^{ab}_{\Delta,j,I} \sim \frac{1}{\Delta - \Delta_{j,i}} (\mathcal{L}_{j,i})^{a}_{a'} (\mathcal{R}_{j,i})^{b}_{b'} g^{a'b'}_{\Delta'_{j,i},j'_{j,i},I}(z,\bar{z})$$

- `blocks_3d` is efficient, multithreaded, C++ implementation
  [Erramilli, Iliesiu, Kravchuk '19; Erramilli, Iliesiu, Kravchuk, Landry, DP, Simmons-Duffin '20]
- Practical for external fermions ψ, currents J, stress-tensors T, ...

block ($\Lambda = 25$)	j_{12}	j_{43}	Memory (GB)	Time (hr)
$\langle\phi\phi\phi\phi\rangle$	0	0	4	0.014
$\langle\phi\psi\phi\psi\rangle$	$\frac{1}{2}$	$\frac{1}{2}$	7	0.025
$\langle T\phi\phi\phi\rangle$	2	0	11	0.045
$\langle\psi\psi\psi\psi\rangle$	1	1	15	0.068
$\langle T\phi T\phi\rangle$	2	2	36	0.20
$\langle T\psi T\psi\rangle$	$\frac{5}{2}$	$\frac{5}{2}$	48	0.62
$\langle TTT\phi\rangle$	4	2	62	0.94
$\langle TTTT\rangle$	4	4	106	6.9

(See `CFTs4D` package for spinning 4d blocks [Cuomo, Karateev, Kravchuk '17])

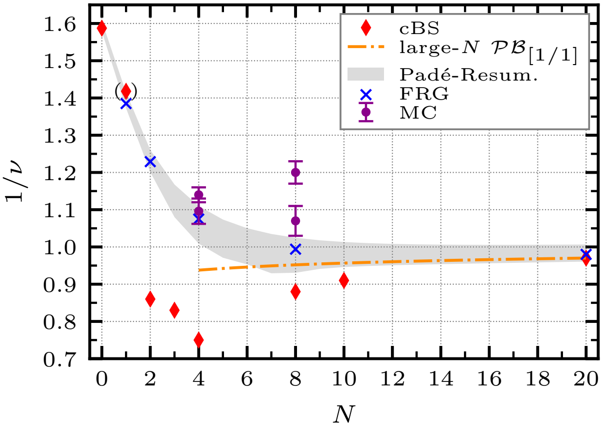
Mysterious jump?



[Erramilli, Iliesiu, Kravchuk, Landry, DP, Simmons-Duffin '20]

- Sharp jump in parity-odd scalar bound from $\langle\psi\psi\psi\psi\rangle$
  [Iliesiu, Kos, DP, Pufu, Simmons-Duffin '17]
- Seems to persist after removing "fake primary effect" ([Karateev, Kravchuk, Serone, Vichi '19]: spin-1 V^μ mimics $\Delta = 3$ scalar)
- Could be evidence for new fermionic CFT w/ no relevant scalars?

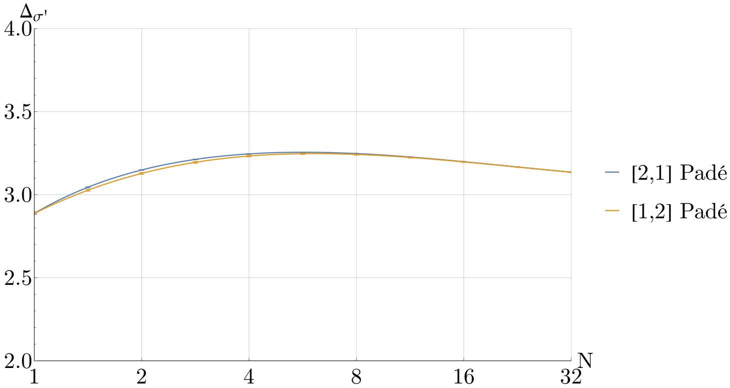
3D Fermion Bootstrap



[Ihrig, Mihaila, Scherer '18]

- However, using a nonrigorous approach called the "extremal functional method" gave $\Delta_\epsilon = 3 - 1/\nu$ which disagreed with other methods

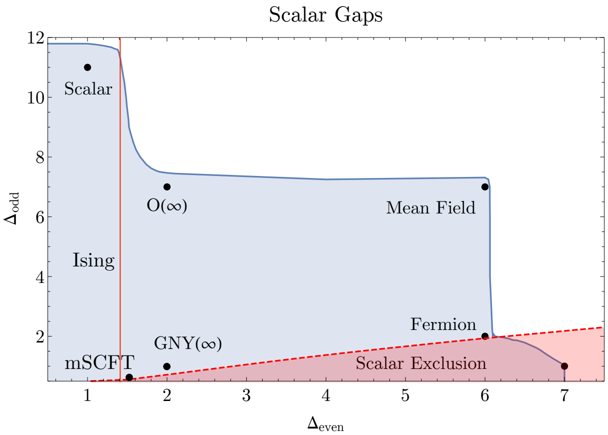
$\Delta_{\sigma'}$ in Gross-Neveu-Yukawa Models



[Erramilli, Iliesiu, Kravchuk, A. Liu, DP, Simmons-Duffin, in progress]

- Our bounds show sensitivity to the scaling dimension of $\sigma' \sim \sigma^3$
- To justify $\Delta_{\sigma'} > 3$, we tried a 2-sided Padé approximation, matching to the large N expansion and known value $2.8869(25)$ at $N = 1$

Map of Allowed Scalar Gaps from $\langle T^{\mu\nu}T^{\rho\sigma}T^{\alpha\beta}T^{\gamma\delta}\rangle$



[Dymarsky, Kos, Kravchuk, DP, Simmons-Duffin, '17]

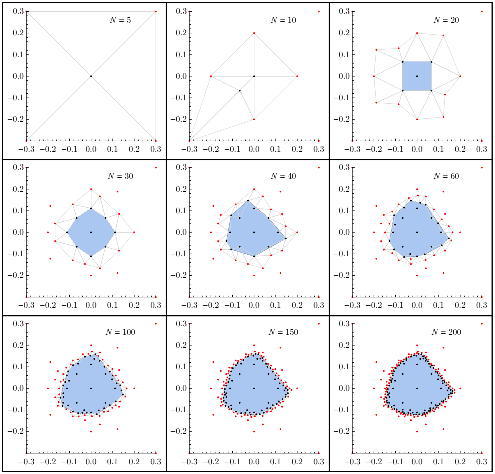
- Allowed $\{\text{scalar}, \text{pseudoscalar}\}$ gaps from stress tensor 4-point functions

$O(2)$ from $\{\phi_i, s, t_{ij}\}$ System

To carry out this 6d search, we employed the following strategy:

1. Use `SDPB 2.0` [Landry, Simmons-Duffin '19], take advantage of parallelization
2. Use hotstarting [Go, Tachikawa '19] to run `SDPB` for fewer iterations
3. Search over Δ's carried out using "Delaunay triangulation" search
4. Search over λ's carried out using "Cutting Surface" algorithm

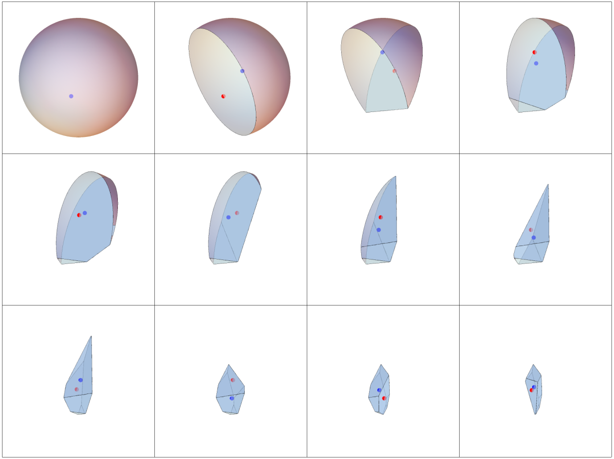
Delaunay Triangulation Search



[Chester, Landry, Liu, DP, Simmons-Duffin, Su, Vichi '19]

- Compute Delaunay triangulation of all tested points, pick midpoint of "biggest" triangle connecting disallowed to allowed

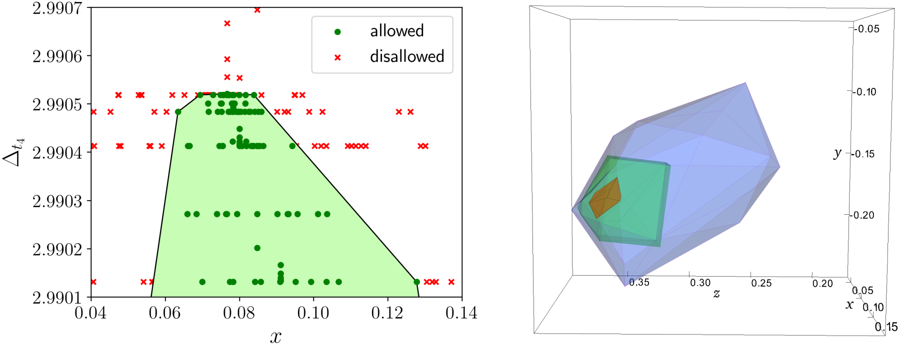
Cutting Surface Algorithm



[Chester, Landry, Liu, DP, Simmons-Duffin, Su, Vichi '19]

- Each computation excludes a *region* of $\vec{\lambda}$-space: $\vec{\lambda} \cdot \alpha[F_{\vec{\Delta}}] \cdot \vec{\lambda} > 0$
- After $\sim 10 - 30$ tests either find allowed point or rule out entire region

$O(3)$ from $\{\phi_i, s, t_{ij}\}$ System



[Chester, Landry, Liu, DP, Simmons-Duffin, Su, Vichi, '20]

- `tiptop` search: map out approximate island at a given gap $\Delta_{t_4} - 3$, shrink bounding box of island by factor of 2, increase gap (via binary search) until it can no longer accommodate smaller box, then iterate

Lorentzian Inversion Review

- The basic idea is to decompose CFT 4-pt functions $\langle\sigma\sigma\sigma\sigma\rangle \propto g(z,\overline{z})$ in a basis of "principal series" ($\Delta = d/2 + i\alpha$) partial waves

$$g(z,\overline{z}) = \sum_{\ell=0}^{\infty} \int_{d/2-i\infty}^{d/2+i\infty} \frac{d\Delta}{2\pi i} c(\Delta,\ell) g_{\Delta,\ell}(z,\overline{z}) + (\text{non-norm.})$$

where the physical spectrum is encoded in the poles of $c(\Delta,\ell)$.

Lorentzian Inversion Review

- Using orthogonality and Lorentzian continuation one "inverts" the formula to obtain $c(\Delta, \ell) = c^t(\Delta, \ell) + (-1)^\ell c^u(\Delta, \ell)$:

$$c^t(\Delta, \ell) \;=\; \frac{\kappa_{\Delta+\ell}}{4} \int_0^1 dz d\bar{z} \mu(z, \bar{z}) g_{\ell+d-1, \Delta+1-d}(z, \bar{z}) \mathsf{dDisc}\left[g(z, \bar{z})\right]$$

with

$$\begin{aligned} \mu(z, \bar{z}) &= \left|\frac{z-\bar{z}}{z\bar{z}}\right|^{d-2} \frac{1}{(z\bar{z})^2} \\ \mathsf{dDisc}\left[g(z, \bar{z})\right] &= g(z, \bar{z}) - \frac{1}{2} g(z, \bar{z}e^{2\pi i}) - \frac{1}{2} g(z, \bar{z}e^{-2\pi i}) \end{aligned}$$

See [Caron-Huot '17; Simmons-Duffin, Stanford, Witten '17; Kravchuk, Simmons-Duffin '18]

Lorentzian Inversion Review

- Expanding $g(z,\bar{z}) = \left(\frac{z\bar{z}}{(1-z)(1-\bar{z})}\right)^{\Delta_\sigma} \sum \lambda^2_{\sigma\sigma\mathcal{O}} g_{\Delta,\ell}(1-z, 1-\bar{z})$ in a finite number of known contributions, we can compute the integrals

- Matching identity operator reveals poles $\frac{1}{\Delta-(2\Delta_\sigma+\ell)}$ corresponding to "double-twist" operators: $\sigma\partial_{\mu_1}\ldots\partial_{\mu_\ell}\sigma$

- Other exchanged operators give anomalous dimensions ($\log(z)$ terms) and correct their OPE coefficients (regular terms)

Approach developed in various works: [Sleight, Taronna '18; Kravchuk, Simmons-Duffin '18; Cardona, Sen '18; Karateev, Kravchuk, Simmons-Duffin '18; Cardona, Guha, Kanumilli, Sen '18; Albayrak, Meltzer, DP '19, '20; Caron-Huot, Gobeil, Zahraee '20]

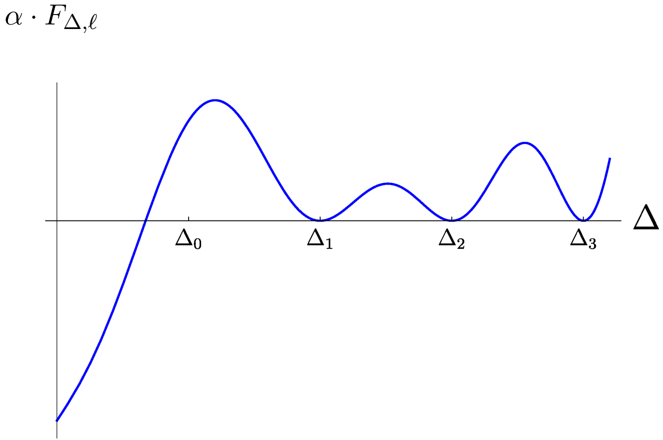
Anomalous Dimensions from Scalar Exchange

$$\delta\tau_{[\sigma\sigma]_0}(\bar{h})_{\mathsf{pert}} = -\frac{\lambda^2_{\sigma\sigma\epsilon}}{1+\delta P_{[\sigma\sigma]_0}(\bar{h})}\frac{2\Gamma(\Delta_\sigma)^2\Gamma\left(1+\frac{\Delta_\epsilon-2\Delta_\sigma}{2}\right)^2\Gamma(\Delta_\epsilon)}{\Gamma\left(1-\frac{\Delta_\epsilon-2\Delta_\sigma}{2}\right)^2\Gamma\left(\frac{\Delta_\epsilon-2\Delta_\sigma}{2}\right)^2\Gamma\left(\frac{\Delta_\epsilon}{2}\right)^2}$$
$$\times\frac{\Gamma(\bar{h}-\Delta_\sigma+1)\Gamma\left(\bar{h}-\frac{\Delta_\epsilon-2\Delta_\sigma}{2}-1\right)}{\Gamma(\bar{h}+\Delta_\sigma-1)\Gamma\left(\bar{h}+\frac{\Delta_\epsilon-2\Delta_\sigma}{2}+1\right)}$$
$$\times\,{}_4F_3\left(\begin{matrix}\frac{\Delta_\epsilon-d+2}{2},\frac{\Delta_\epsilon-2\Delta_\sigma}{2}+1,\frac{\Delta_\epsilon-2\Delta_\sigma}{2}+1,\frac{\Delta_\epsilon}{2}\\ \frac{\Delta_\epsilon-2\Delta_\sigma}{2}-\bar{h}+2,\frac{\Delta_\epsilon-2\Delta_\sigma}{2}+\bar{h}+1,\Delta_\epsilon-\frac{d-2}{2}\end{matrix};1\right)$$

$$\delta\tau_{[\sigma\sigma]_0}(\bar{h})_{\mathsf{np}} = -\frac{\lambda^2_{\sigma\sigma\epsilon}}{1+\delta P_{[\phi\phi]_0}(\bar{h})}\frac{2\Gamma(\Delta_\sigma)^2\Gamma\left(\Delta_\epsilon-\frac{d-2}{2}\right)\Gamma(\Delta_\epsilon)}{\Gamma\left(1-\frac{\Delta_\epsilon-2\Delta_\sigma}{2}\right)^2\Gamma\left(\frac{\Delta_\epsilon-2\Delta_\sigma}{2}\right)^2\Gamma\left(\frac{\Delta_\epsilon}{2}\right)^3\Gamma\left(\frac{\Delta_\epsilon-d-2}{2}\right)}$$
$$\times\frac{\Gamma(\bar{h})^2\Gamma(\bar{h}-\Delta_\sigma+1)\Gamma\left(\bar{h}+\Delta_\sigma-\frac{d}{2}\right)\Gamma\left(\frac{\Delta_\epsilon-2\Delta_\sigma}{2}-\bar{h}+1\right)}{\Gamma(2\bar{h})\Gamma\left(\bar{h}+\Delta_\sigma+\frac{\Delta_\epsilon}{2}-\frac{d}{2}\right)}$$
$$\times\,{}_4F_3\left(\begin{matrix}\bar{h},\bar{h},\bar{h}+\Delta_\sigma-1,\bar{h}+\Delta_\sigma-\frac{d}{2}\\ 2\bar{h},\bar{h}-\frac{\Delta_\epsilon-2\Delta_\sigma}{2},\bar{h}+\Delta_\sigma+\frac{\Delta_\epsilon}{2}-\frac{d}{2}\end{matrix};1\right)$$

- E.g., for exchange of a scalar ϵ, integral yields two pieces: $\delta\tau_{[\sigma\sigma]_0}(\bar{h}) = \delta\tau_{[\sigma\sigma]_0}(\bar{h})_{\mathsf{pert}} + \delta\tau_{[\sigma\sigma]_0}(\bar{h})_{\mathsf{np}}$ (here $\bar{h}\equiv\frac{\Delta+\ell}{2}$)

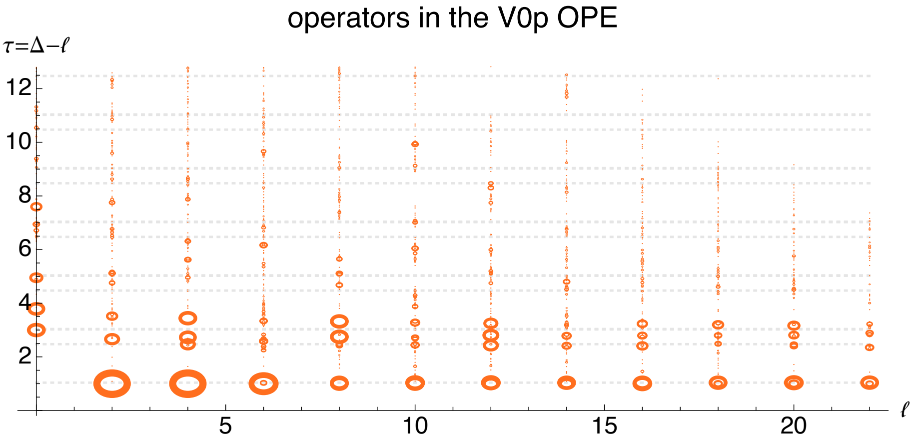
Extremal Functional



[DP, Simmons-Duffin '10; Paulos, El-Showk '12; plot from Paulos, Zan '20]

- By going to a boundary of the allowed region (e.g., extremizing $\lambda_{\phi\phi s}$), we can extract extremal spectra corresponding to the zeros of $\alpha \cdot F_{\Delta,\ell}$

$O(2)$ from $\{\phi_i, s, t_{ij}\}$ System



[Chester, Landry, Liu, DP, Simmons-Duffin, Su, Vichi, '19]

- Can be done in practice using `spectrum-extraction` Python script (`spectrum.py`), which uses `sdpb` output

Analytical Bootstrap

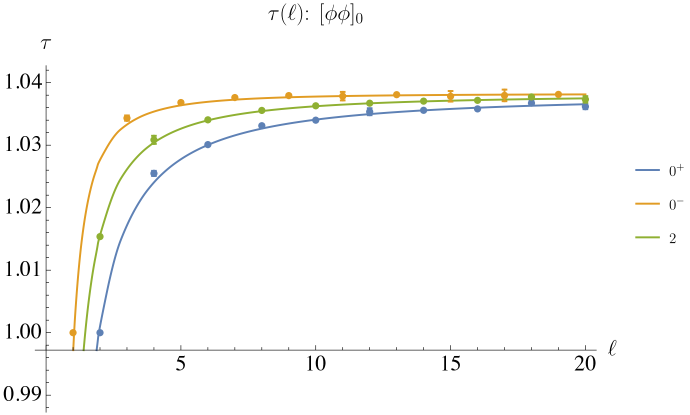
Extremal spectra can be compared with analytical bootstrap predictions:

- Lightcone Bootstrap ($z \to 0$): $\exists$ trajectories of "double-twist" operators $\sim \sigma\partial^\ell\sigma$ with twist asymptoting to $\tau(\ell \to \infty) = 2\Delta_\sigma - \frac{\#}{\ell} - \frac{\#}{\ell^{\Delta_\epsilon}} + \ldots$
  [Fitzpatrick, Kaplan, DP, Simmons-Duffin '12; Komargodski, Zhiboedov '12]

- Lorentzian Inversion $\to$ All-orders analytic function [Caron-Huot '17]

  $\tau(\ell) \sim \int \mathrm{dDisc}[g] \sim \sum_{\mathcal{O}} {}_4F_3(\ldots)$

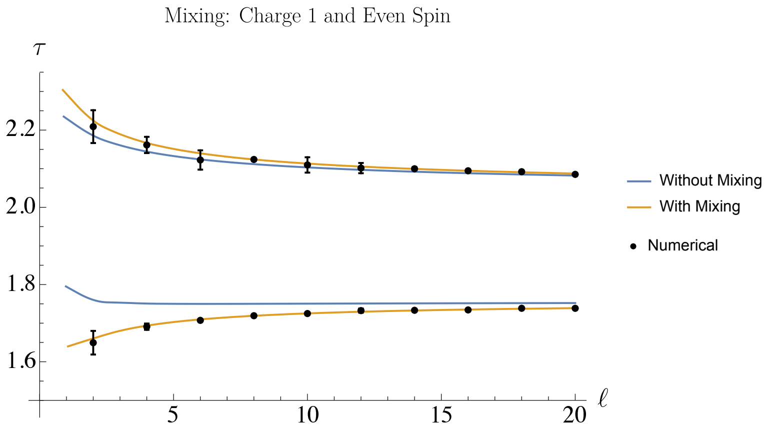
$O(2)$: Comparison with Analytics



[Albayrak, Meltzer, DP '19; J. Liu, Meltzer, DP, Simmons-Duffin '20]

- Excellent agreement between leading-twist extremal spectra and analytics after including exchange of $\{s, t, J^\mu, T^{\mu\nu}\}$

Lorentzian Inversion for O(2) Model

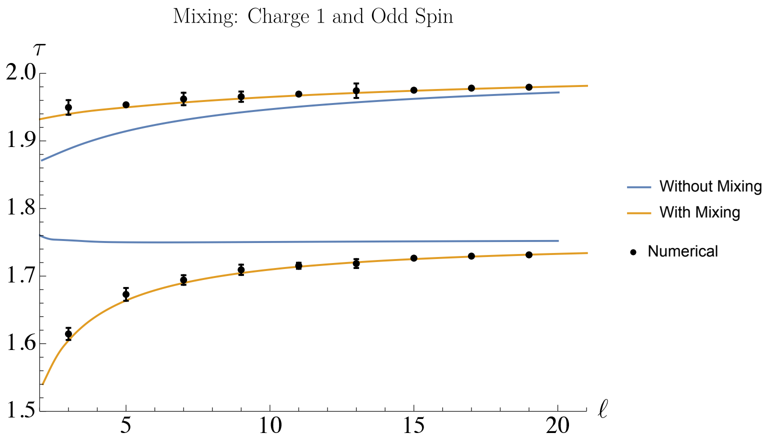


[J. Liu, Meltzer, DP, Simmons-Duffin '20]

In some cases (e.g., charge 1 tower), to get best agreement:

- Add t-channel contributions from tower of leading double-twists
- Resolve mixing effects (diagonalize "twist Hamiltonian"): $[\phi s]$ and $[\phi t]$

Lorentzian Inversion for O(2) Model



[J. Liu, Meltzer, DP, Simmons-Duffin '20]

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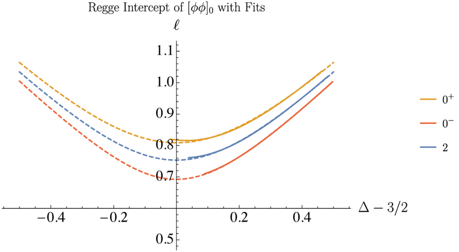
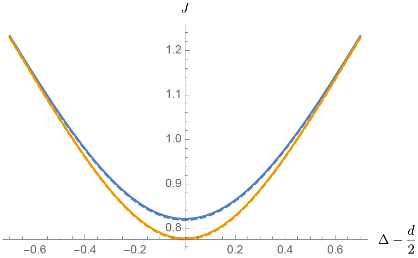
Regge Intercepts: $O(2)$ (left) and Ising (right)

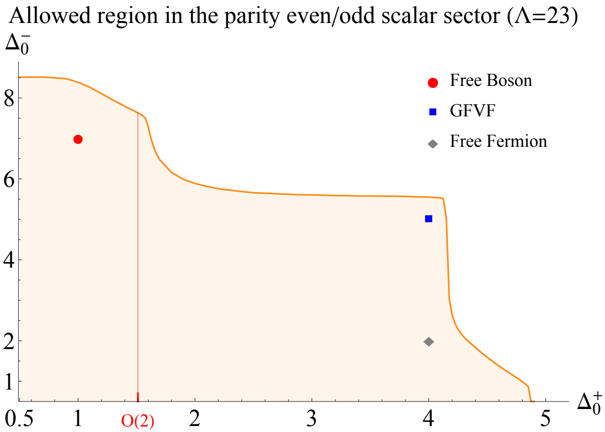




[J. Liu, Meltzer, DP, Simmons-Duffin '20; Caron-Huot, Gobeil, Zahraee '20]

- Can also extrapolate down to low spin and predict leading Regge intercepts, which control behavior of correlator in Regge limit (in AdS, $\sim$ high energy, fixed impact parameter scattering)

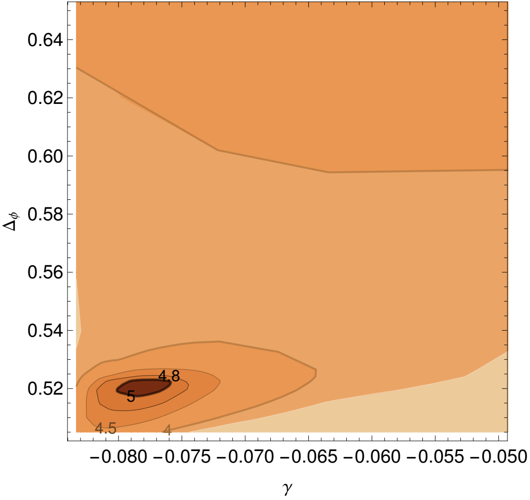
Map of Allowed Scalar Gaps from $\langle J^\mu J^\nu J^\rho J^\sigma \rangle$



[Dymarsky, Penedones, Trevisani, Vichi '17]

- Allowed {scalar, pseudoscalar} gaps from current 4-point functions

Map of Allowed Couplings from $\{J^\mu, \phi\}$ System



[Reehorst, Trevisani, Vichi '19]

- Allowed couplings $\langle JJT \rangle \propto \gamma$ in $O(2)$ model after imposing $T'_{\mu\nu}$ gap