Sequential Discontinuities of Scattering Amplitudes

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arXiv: 2007.13747 with J. Bourjaily[2,3], A. McLeod[3], M. Schwartz[1] and C. Vergu[3]

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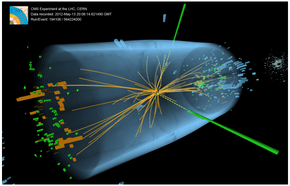
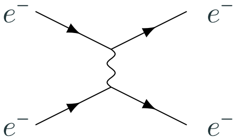
Scattering Amplitudes $\mathcal{M}$

$\mathcal{M} \sim \langle f|\, S \,|i\rangle$: **Probability amplitude** for measuring a final state $|f\rangle$ given an initial state $|i\rangle$

- Used in most **Quantum Field Theory** calculations.
  - Leads to predictions for **collider experiments**.
  - Standard Model observables computed to high precision.
  - Calculated using **Feynman diagrams**.



CMS Experiment 2012

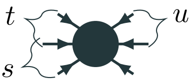


Scattering Amplitudes $\mathcal{M}$

- Properties extensively studied.
  - How to **encode their content**? *Spinors, twistors, amplituhedron?*
  - What are their **symmetries**? *Lorentz invariance, dual conformal invariance, Steinmann relations?*
  - What **functional forms** can they take? *Logarithms, polylogarithms?*
- Often computed in perturbation theory by summing all Feynman diagrams.

Can we exploit constraints to calculate $\mathcal{M}$ more efficiently?

Motivation for Studying Discontinuities of Amplitudes



- Feynman integrals give logarithms and polylogarithms with **branch cuts**.
  - Value of $\mathcal{M}$ depends on whether we use $+i\varepsilon$ or $-i\varepsilon$.
- **Traditional Cutting Rules**: Discontinuities related to cuts of corresponding Feynman diagram.

$$\mathrm{Disc}\mathcal{M} = \mathcal{M}|_{+i\varepsilon} - \mathcal{M}|_{-i\varepsilon} \quad = \quad \sum \mathrm{Cut}\mathcal{M}$$

- **Reconstruct** $\mathcal{M}$ from discontinuities using a basis of functions for Feynman integrals.

Motivation for Studying Discontinuities of Amplitudes

Traditional Cutting Rules:

$$\mathrm{Disc}\mathcal{M} = \mathcal{M}|_{+i\varepsilon} - \mathcal{M}|_{-i\varepsilon} = \sum \mathrm{Cut}\mathcal{M}$$

- *What can we learn from studying* ***sequential discontinuities*** *of $\mathcal{M}$?*

$$Disc Disc \mathcal{M} = ?$$

- *How do we relate sequential discontinuities of $\mathcal{M}$ to* ***cuts****?*
- *What do we gain from a* ***systematic treatment*** *of computing discontinuities?*

Outline

1. Analytic Structure and Discontinuities
    - Need more powerful tools than $\pm i\varepsilon$ for sequential discontinuities.
    - Define discontinuities using **monodromies**.
2. Relations between **Discontinuities** and **Cuts**
    - Use **Time-Ordered Perturbation Theory (TOPT)** to prove results.
    - **New results** for relations between sequential discontinuities and multiple cuts.
    - **New proof** of the **Steinmann** relations.
3. Examples
4. Future Work

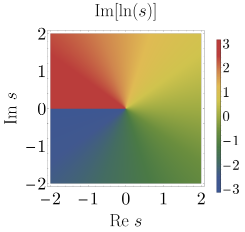
1. Analytic Structure and Discontinuities

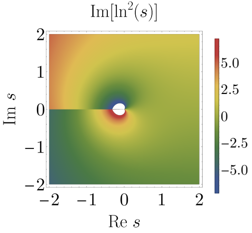
Problems with $i\varepsilon$ Definition of Discontinuity

- $\mathrm{Disc}\mathcal{M} = \mathcal{M}|_{+i\varepsilon} - \mathcal{M}|_{-i\varepsilon}$ **only** defined on the **branch cut**:

$$\mathrm{Disc}_s \ln s = \ln(s + i\varepsilon) - \ln(s - i\varepsilon) = 2\pi i \theta(-s)$$

$$\mathrm{Disc}_s \ln^2 s = \ln^2(s + i\varepsilon) - \ln^2(s - i\varepsilon) = 4\pi i \theta(-s) \ln|s|$$



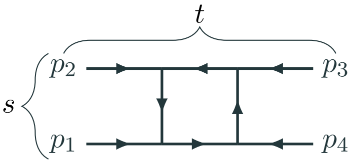


- What is the $i\varepsilon$ prescription of $\mathrm{Disc}_s\mathcal{M}$?

Need a better definition of Disc to take sequential discontinuities.

Problems with $i\varepsilon$ Definition of Discontinuity

Want to study Disc$\mathcal{M}$ in **each Mandelstam separately**



$$= \mathcal{M}\left(p_j^2, s, t, u\right)$$

Intuitively: Define **discontinuity in a channel** s as

$$\mathrm{Disc}_s \mathcal{M} = \mathcal{M}\left(p_j^2, s + i\varepsilon, t, u\right) - \mathcal{M}\left(p_j^2, s - i\varepsilon, t, u\right)$$

- Agrees with cuts in only s?
- **Problem:** Mandelstams are not all independent:

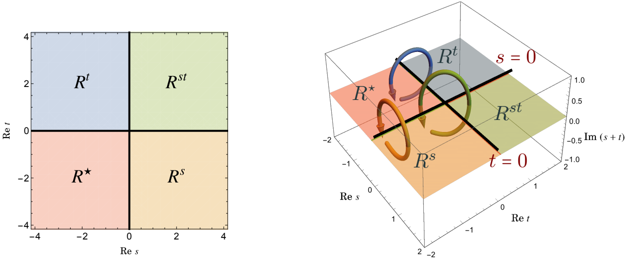
$$s + t + u = \sum p_j^2$$

Disc$_s$ should be invariant under rewriting $\mathcal{M}$.

Definition of Discontinuity

Resolution: Abandon the $\pm i\varepsilon$ notation, take **monodromies**.

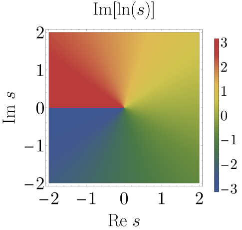
Definition: $\mathrm{Disc}_s\mathcal{M}$ is the **monodromy** of $\mathcal{M}$ around $s = 0$, starting in $\boldsymbol{R^s}$.



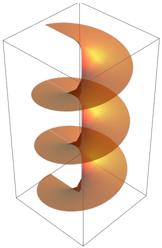
- $\boldsymbol{R^s}$: Region in space of Mandelstams where $s > 0$, all other Mandelstams $s_{i,j,\ldots} < 0$.
- **Monodromy:** How a function changes when analytically continuing around a singularity.

Definition of Discontinuity

Discontinuities with $\pm i\varepsilon$ only account for the **principal branch**:



Monodromies allow for maximal analytic continuation:



Definition of Discontinuity in a Channel

$\mathrm{Disc}_s\mathcal{M}$ is the **monodromy** of $\mathcal{M}$ around $s = 0$, starting in R^s.

- Agrees with the $i\varepsilon$ definition in R^s:

$$[\mathrm{Disc}_s\mathcal{M}]_{R^s} = \left[\mathcal{M}|_{+i\varepsilon} - \mathcal{M}|_{-i\varepsilon}\right]_{R^s}$$

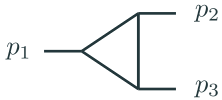
- Results in a **function on complex space**.
- Machinery: **monodromy operator**.

$$[\mathrm{Disc}_s\mathcal{M}]_{R^s} = \left[\left(\mathbb{1} - \mathscr{M}_{\circlearrowleft_0^s}\right)\mathcal{M}\right]_{R^s}$$

- Sequential discontinuities are **natural** and **algebraic**:

$$[\mathrm{Disc}_s\mathrm{Disc}_s\mathcal{M}]_{R^s} = \left[\left(\mathbb{1} - \mathscr{M}_{\circlearrowleft_0^s}\right)\left(\mathbb{1} - \mathscr{M}_{\circlearrowleft_0^s}\right)\mathcal{M}\right]_{R^s}$$

Example of Complex Analytic Structure of $\mathcal{M}$

$\mathcal{M} =$ 

$$\propto \mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) + \frac{1}{2}\ln(z\bar{z})\ln\left(\frac{1-z}{1-\bar{z}}\right)$$

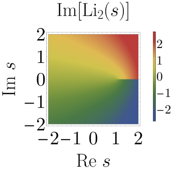
$$\text{with } z\bar{z} = p_2^2/p_1^2, \quad (1-z)(1-\bar{z}) = p_3^2/p_1^2$$

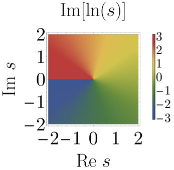
- Dilogarithm $\mathrm{Li}_2(z) = -\int_0^z \frac{\ln(1-s)}{s} ds$ has a branch point at $z = 1$:

$$\left(\mathbb{1} - \mathscr{M}_{\circlearrowleft_1^z}\right) \mathrm{Li}_2(z) = 2\pi i \int_1^z \frac{1}{s} ds = 2\pi i \ln(z)$$

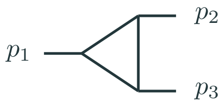
- Logarithm $\ln(z) = \int_1^z \frac{1}{s} ds$ has a branch point at $z = 0$:

$$\left(\mathbb{1} - \mathscr{M}_{\circlearrowleft_0^z}\right) \ln(z) = 2\pi i$$





Example of Complex Analytic Structure of $\mathcal{M}$

$\mathcal{M} =$ 

$$\propto \mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) + \frac{1}{2}\ln(z\bar{z})\ln\left(\frac{1-z}{1-\bar{z}}\right)$$

with $z\bar{z} = p_2^2/p_1^2, \quad (1-z)(1-\bar{z}) = p_3^2/p_1^2$

- Dilogarithm $\mathrm{Li}_2(z) = -\int_0^z \frac{\ln(1-s)}{s} ds$ has a branch point at $z = 1$:

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- Logarithm $\ln(z) = \int_1^z \frac{1}{s} ds$ has a branch point at $z = 0$:

$$\left(\mathbb{1} - \mathscr{M}_{\circlearrowleft_0^z}\right) \ln(z) = 2\pi i$$

Monodromy of Li_2 at $z = 1$ exposes a **new branch point** at $z = 0$.

Useful information in sequential discontinuities.

Summary of Analytic Structure and Discontinuities

- The $\pm i\varepsilon$ definition of discontinuities cannot capture sequential discontinuities.
  - Function defined on a **line**, not on $\mathbb{C}$.
  - Cannot take Disc in each **Mandelstam** separately.
- Resolution: Use **monodromy operators**.
  - Calculations amount to matrix multiplications.
- Discontinuities expose new branch points in $\mathbb{C}$.
  - Useful information in sequential discontinuities.

How do we relate sequential discontinuities to cuts of Feynman diagrams?

2. Cuts

Example: Discontinuities, Monodromies and Cuts

$$\mathcal{M} = $$

$$\propto -\frac{i}{16\pi^2} \ln\left(-p^2 - i\varepsilon\right)$$

$$\left(\mathbb{1} - \mathscr{M}_{\leftrightarrows\bigcirc_{0}^{p^2}}\right)\mathcal{M} \propto -\frac{i}{16\pi^2}\,(-2\pi i) = -\frac{1}{8\pi}$$

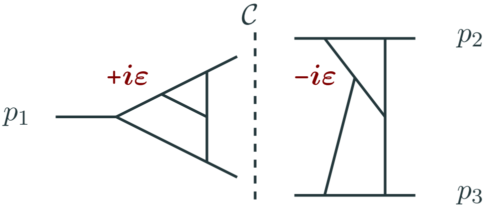
$$[\mathrm{Disc}\mathcal{M}]_{R^{p^2}} \propto -\frac{i}{16\pi^2}\,(-2\pi i)\,\Theta(p^2) = -\frac{1}{8\pi}\Theta(p^2)$$

$$\mathrm{Cut}\mathcal{M} \propto $$

$$= -\frac{1}{8\pi}\Theta(p^2)$$

Traditional Cutting Rules

$$\mathrm{Disc}\mathcal{M} = \mathcal{M}|_{+i\varepsilon} - \mathcal{M}|_{-i\varepsilon} = \sum \mathrm{Cut}\mathcal{M}$$



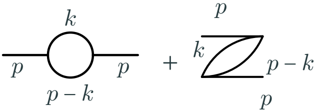
L.h.s. of cut has $+i\varepsilon$, r.h.s. of cut has $-i\varepsilon$.

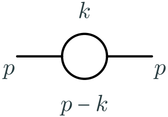
Proofs:

- Cutkosky, using the Landau equations.
- t'Hooft and Veltman, using the largest time equation.
- Time-ordered perturbation theory (TOPT).
  - **Most transparent and easily generalizable.**

Review of Time-Ordered Perturbation Theory (TOPT)

Sum of $v!$ TOPT diagrams $=$ Feynman diagram

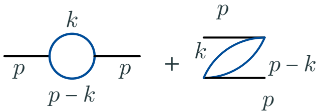


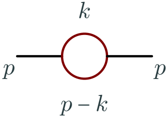


TOPT diagrams	Feynman diagrams
• Time passes from left to right	• Vertices are not ordered
• All particles on-shell: $$E^2 = \vec{p}^2 + m^2$$	• Internal particles virtual: $$E^2 \neq \vec{p}^2 + m^2$$
• $\vec{p}$ conservation at each vertex	• $\vec{p}$ conservation at each vertex
• **Not *E* conservation** at each vertex	• E conservation at each vertex
• Overall E & $\vec{p}$ conservation	• Overall E & $\vec{p}$ conservation
• Individual diagrams not Lorentz invariant	• Manifestly Lorentz-invariant
• Good for proofs & intuition	• Good for calculations

Review of Time-Ordered Perturbation Theory (TOPT)

Sum of $v!$ TOPT diagrams $=$ Feynman diagram





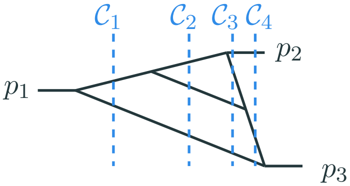
$$\int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\frac{1}{2\omega_{p-k}}\left[\frac{1}{E_p-(\omega_k+\omega_{p-k})+i\varepsilon}+\frac{1}{E_p-(\omega_k+\omega_{p-k}+2\omega_p)+i\varepsilon}\right]$$

$$= -\int \frac{d^4k}{i(2\pi)^4}\frac{1}{k^2-m_1^2+i\varepsilon}\frac{1}{(p-k)^2-m_2^2+i\varepsilon}$$

Cutting Rules in TOPT

Advantages to TOPT:

- Energies are **independent**, Mandelstams are not.
- **One delta** function for **each cut**.
  - Various numbers of on-shell Feynman propagators for each cut through a Feynman diagram.



$$\mathcal{M}|_{+i\varepsilon} \propto \int \frac{1}{E_1 - \omega_1 + i\varepsilon} \frac{1}{E_1 - \omega_2 + i\varepsilon} \frac{1}{E_1 - E_2 - \omega_3 + i\varepsilon} \frac{1}{E_1 - E_2 - \omega_4 + i\varepsilon}$$

Relate Disc$\mathcal{M}$ to cuts using $\frac{1}{E_i + i\varepsilon} - \frac{1}{E_i - i\varepsilon} = -2\pi i \delta\left(E_i\right)$

Results Derived using TOPT

Same channel sequential discontinuities: Equal to a sum of diagrams cut multiple times with a **combinatorial** factor.

$$\begin{aligned}\left[\mathrm{Disc}_s^m \mathcal{M}\right]_{R^s} &= (\mathbb{1} - \mathscr{M}_{\leftrightarrows\bigcirc_0^s})^m \mathcal{M} \\ &= \sum_{k=m} \left\{ \sum_{\ell=1}^{m} (-1)^\ell \binom{m}{\ell} (-\ell)^k \right\} \left[\mathcal{M}_{k\text{-cuts}}\right]_{R_+^s}\end{aligned}$$

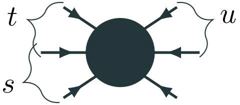
Different channel sequential discontinuities: Equal to a sum of diagrams cut multiple times in a **region** $R^{\{s,t\}}$ where both cuts can be computed.

$$\begin{aligned}\left[\mathrm{Disc}_s \mathrm{Disc}_t \mathcal{M}\right]_{R^{\{s,t\}}} &= (\mathbb{1} - \mathscr{M}_{\leftrightarrows\bigcirc_0^t})(\mathbb{1} - \mathscr{M}_{\leftrightarrows\bigcirc_0^s})\mathcal{M} \\ &= \left[\sum_{k=1} \sum_{\ell=1} (-1)^{k+\ell} \mathcal{M}_{\{k \text{ cuts in } s,\ \ell \text{ cuts in } t\}} \right]_{R_+^{\{s,t\}}}\end{aligned}$$

R_+: $\mathcal{M}$ computed with all $+i\varepsilon$.

Steinmann Relations

$\mathcal{M}$ cannot have sequential discontinuities in partially overlapping channels

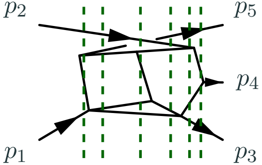


$\mathcal{M}$ cannot contain $\ln(s)\ln(t)$ but can contain $\ln(s)\ln(u)$.

- Important for **bootstrapping** amplitudes.
- Old proof in S-matrix theory [1, 2].
  - Non-perturbative, used unitarity.
- Our new proof in TOPT [3].
  - Applies to **individual Feynman integrals**.

Proof of Steinmann Relations in TOPT

- TOPT denominators have a sequence of energies.



$$-E_5, \quad E_1 - E_5, \quad E_1 - E_5, \quad E_1 - E_5, \quad E_1 - E_5 - E_3, \quad E_1 - E_5 - E_3 + E_2$$

$$p_5^2, \quad (p_1 - p_5)^2, \quad (p_1 - p_5 - p_3)^2, \quad (p_1 - p_5 - p_3 + p_2)^2$$

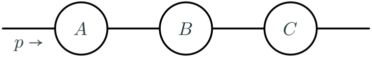
- Each energy is a subset of the sequential ones.

No sequential discontinuities in partially overlapping channels

  - Regions may not exist when some particles are massless.
  - Cannot fix external masses to zero.

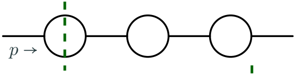
3. Examples

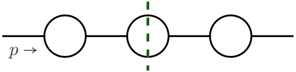
Each uncut bubble gives a log:

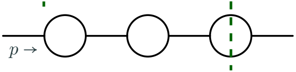
$\mathcal{M} =$ 

$\propto \ln^3\left(-p^2 - i\varepsilon\right)$

1 cut

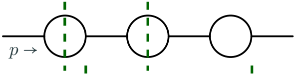


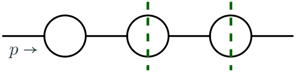
$=$ 

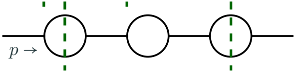
$=$ 

$\propto (-2\pi i)\ln^2\left(-p^2 - i\varepsilon\right)$

2 cuts

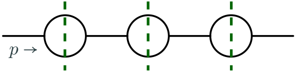


$=$ 

$=$ 

$\propto (-2\pi i)^2\ln\left(-p^2 - i\varepsilon\right)$

3 cuts



$\propto (-2\pi i)^3$

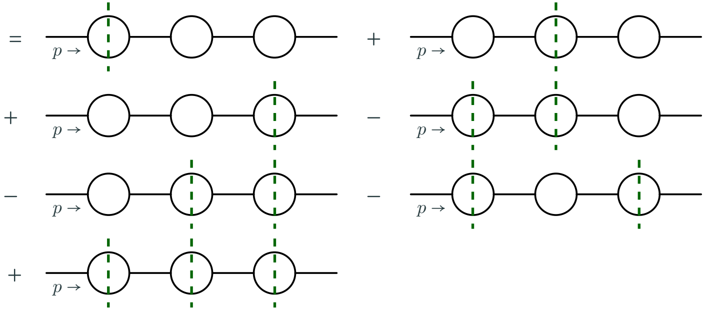
Example: Disc$\mathcal{M}$ for Chain of Bubbles

- **Discontinuity** calculated using monodromy matrices:

$$\left[\mathrm{Disc}_{p^2}\mathcal{M}\right]_{R^{p^2}} \propto (-2\pi i)\ln^2\left(-p^2 - i\varepsilon\right) - 3(-2\pi i)^2\ln\left(-p^2 - i\varepsilon\right) + (-2\pi i)^3$$

- **Cuts** calculated by putting particles on shell, using $+i\varepsilon$:

$\sum \mathrm{Cut}\mathcal{M} =$



- Agreement with formula:

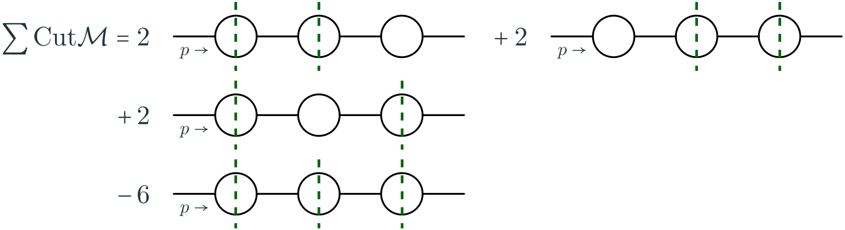
$$\boxed{\left[\mathrm{Disc}_{p^2}\mathcal{M}\right]_{R^{p^2}} = \mathcal{M}^{(\text{1-cuts})} - \mathcal{M}^{(\text{2-cuts})} + \mathcal{M}^{(\text{3-cuts})}}$$

Example: $\mathrm{Disc}^2\mathcal{M}$ for Chain of Bubbles

- **Discontinuity:**

$$\left[\mathrm{Disc}^2_{p^2}\mathcal{M}\right]_{R^{p^2}} \propto 6\left(-2\pi i\right)^2 \ln\left(-p^2 - i\varepsilon\right) - 6(-2\pi i)^3$$

- **Cuts:**



- Agreement with formula:

$$\boxed{\left[\mathrm{Disc}^2_{p^2}\mathcal{M}\right]_{R^{p^2}} = 2\mathcal{M}^{(\text{2-cuts})} - 6\mathcal{M}^{(\text{3-cuts})}}$$

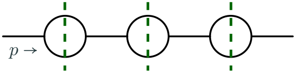
Example: $\mathrm{Disc}^3\mathcal{M}$ for Chain of Bubbles

- **Discontinuity**:

$$\left[\mathrm{Disc}^3_{p^2}\mathcal{M}\right]_{R^{p^2}} \propto 6(-2\pi i)^3$$

- **Cuts**:

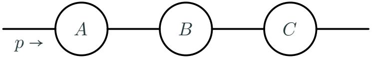
$$\sum \mathrm{Cut}\mathcal{M} = 6$$



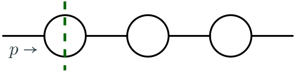
- Agreement with formula:

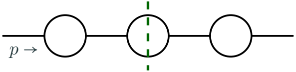
$$\boxed{\left[\mathrm{Disc}^3_{p^2}\mathcal{M}\right]_{R^{p^2}} = 6\mathcal{M}^{(\text{3-cuts})}}$$

Example: Chain of Bubbles, Summary

$\mathcal{M} =$ 

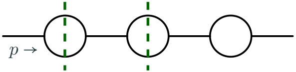
$\propto \ln^3\left(-p^2 - i\varepsilon\right)$

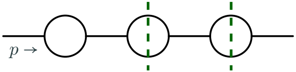
$\mathrm{Disc}\mathcal{M} =$ 

$+$ 

$+ \cdots$

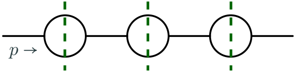
$$\propto 3\,(-2\pi i)\ln^2\left(-p^2 - i\varepsilon\right) - 3(-2\pi i)^2\ln\left(-p^2 - i\varepsilon\right) + (-2\pi i)^3$$

$\mathrm{Disc}^2\mathcal{M} = 2$ 

$+\,2$ 

$+ \cdots$

$$\propto 6\,(-2\pi i)^2\ln\left(-p^2 - i\varepsilon\right) - 6(-2\pi i)^3$$

$\mathrm{Disc}^3\mathcal{M} = 6$ 

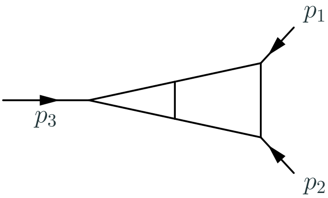
$\propto 6\,(-2\pi i)^3$

Discontinuities $= \sum$ **multiple cut** diagrams

Example: Two-loop Triangle

$$z\bar{z} = \frac{p_2^2}{p_1^2}$$

$$(1-z)(1-\bar{z}) = \frac{p_3^2}{p_1^2}$$



$$\mathcal{M} \propto 6[\mathrm{Li}_4(z) - \mathrm{Li}_4(\bar{z})] - 3\ln(z\bar{z})[\mathrm{Li}_3(z) - \mathrm{Li}_3(\bar{z})] + \frac{1}{2}\ln^2(z\bar{z})[\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z})]$$

Compare the following **discontinuities** and **cuts**:

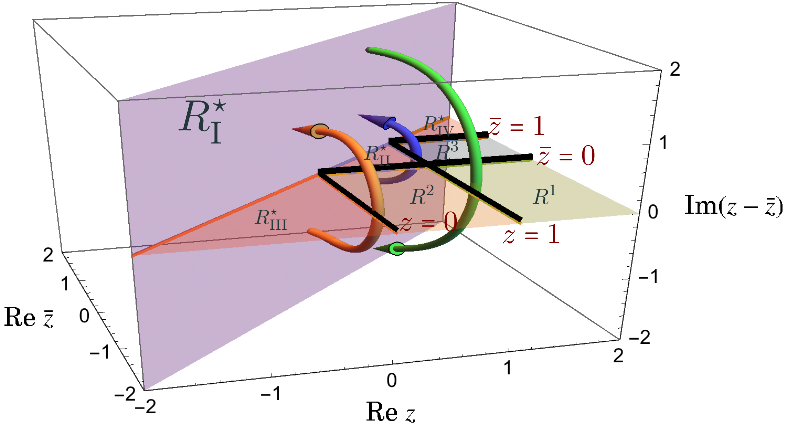
$$\mathrm{Disc}_{p_2^2}\mathrm{Disc}_{p_2^2}\mathcal{M} \qquad \mathrm{Disc}_{p_1^2}\mathrm{Disc}_{p_2^2}\mathcal{M}$$

Energy Rotations in z, $\bar{z}$ Plane

$$z\bar{z} = \frac{p_2^2}{p_1^2}$$

$$(1-z)(1-\bar{z}) = \frac{p_3^2}{p_1^2}$$

$$\mathcal{M} \propto 6[\mathrm{Li}_4(z) - \mathrm{Li}_4(\bar{z})] - 3\ln(z\bar{z})[\mathrm{Li}_3(z) - \mathrm{Li}_3(\bar{z})] + \frac{1}{2}\ln^2(z\bar{z})[\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z})]$$



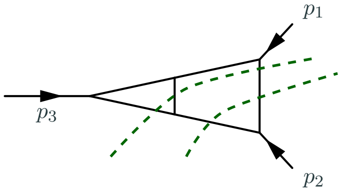
2-loop Triangle: Same Channel

- **Discontinuity** calculated using monodromy matrices:

$$\Big[\mathrm{Disc}_{p_2^2}\mathrm{Disc}_{p_2^2}\mathcal{M}(z,\bar{z})\Big]_{R^2} \propto \mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z})$$

- **Cut** calculated by putting particles on shell:

$$\Big[\mathcal{M}^{(2\text{-cuts})}\Big]_{R^2} =$$



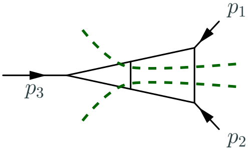
$$\propto \frac{\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z})}{2}$$

- As predicted, agree up to the **combinatorial factor**:

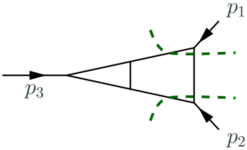
$$\boxed{\Big[\mathrm{Disc}_{p_2^2}\mathrm{Disc}_{p_2^2}\mathcal{M}(z,\bar{z})\Big]_{R^2} = 2\Big[\mathcal{M}^{(2\text{-cuts})}\Big]_{R^2}}$$

2-loop Triangle: Different Channel Cuts

$$\left[\mathcal{M}^{(\text{2-cuts})}\right]_{R^{12}} =$$



$+$



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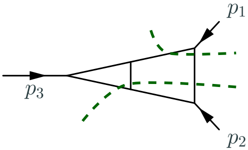
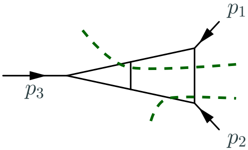
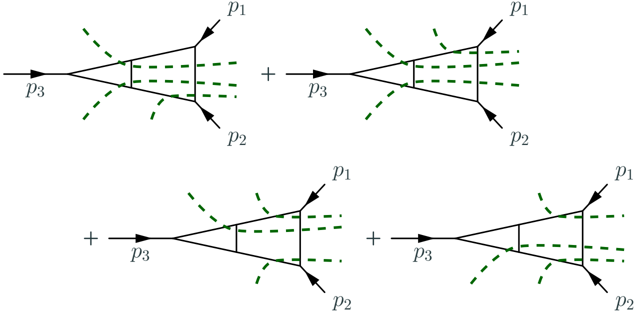


$+$



$$\propto (2\pi i)^2 \Big\{ \mathrm{Li}_2(\bar{z}) - \mathrm{Li}_2(z - i\varepsilon) - \frac{1}{2}\ln^2 z + \ln z \ln \bar{z} + i\pi \ln z - 2\pi i \ln \bar{z} \Big\}$$

$$\left[\mathcal{M}^{(3\text{-cuts})}\right]_{R^{12}} =$$



$$\propto (2\pi i)^3 \left\{\ln z - \ln \bar{z}\right\}$$

2-loop Triangle: Different Channels

- Discontinuity:

$$\left[\mathrm{Disc}_{p_2^2}\mathrm{Disc}_{p_1^2}\Phi_2\right]_{R^{12}} \propto (2\pi i)^2\Big\{\mathrm{Li}_2(\bar{z}) - \mathrm{Li}_2(z - i\varepsilon) - \frac{1}{2}\ln^2 z + \ln z \ln \bar{z} - i\pi \ln z\Big\}$$

- Cuts:

$$\mathcal{M}^{(2\text{-cuts})} \propto (2\pi i)^2\Big\{\mathrm{Li}_2(\bar{z}) - \mathrm{Li}_2(z - i\varepsilon) - \frac{1}{2}\ln^2 z + \ln z \ln \bar{z} + i\pi \ln z - 2\pi i \ln \bar{z}\Big\}$$

$$\mathcal{M}^{(3\text{-cuts})} \propto (2\pi i)^3\left\{\ln z - \ln \bar{z}\right\}$$

- Agreement with formula:

$$\boxed{\left[\mathrm{Disc}_{p_2^2}\mathrm{Disc}_{p_1^2}\mathcal{M}_2\right]_{R^{12}} = \mathcal{M}^{(2\text{-cuts})} - \mathcal{M}^{(3\text{-cuts})}}$$

4. Future Work

Future Work

- Extend analysis to **massless external particles**?
  - Steinmann relations used without proof for bootstrapping in $\mathcal{N} = 4$ super Yang-Mills.
- Apply new results to **bootstrapping**?
  - What constraints can be obtained by the form of the monodromy matrix?
- Bootstrap **Finite $\boldsymbol{S}$-matrix**?
  - IR finite operator defined [4, 5] in theories with massless particles using Soft-Collinear Effective Theory (SCET).
  - Encodes hard dynamics of scattering amplitudes.
  - Can be interpreted as:
    1. Wilson Coefficients in SCET.
    2. Remainder functions in $\mathcal{N} = 4$ super Yang-Mills.
    3. Coherent states.

Results

- Discontinuities defined as **monodromies** around singularities.
  - Start in **kinematic region** where cut can be performed.
  - Monodromy matrices make calculations of monodromies algebraic.
- TOPT used to prove:
  1. **Same channel discontinuities:** Equal to a sum of diagrams cut multiple times with a **combinatorial** factor.
  2. **Different channel discontinuities:** Equal to a sum of diagrams cut multiple times in a **kinematic region** where all cuts can be computed.
  3. **Steinmann Relations:** $\mathcal{M}$ cannot have sequential discontinuities in partially overlapping channels.

[1] O Steinmann. “Über den Zusammenhang Zwischen den Wightmanfunktionen und den Retardierten Kommutatoren”. In: *Helvetica Physica Acta* 33 (1960), pp. 257–298.

[2] O Steinmann. “Wightman-Funktionen und Retardierte Kommutatoren. II”. In: *Helvetica Physica Acta* 33 (1960), pp. 347–362.

[3] Jacob L. Bourjaily, Holmfridur Hannesdottir, Andrew J. McLeod, Matthew D. Schwartz, and Cristian Vergu. "Sequential Discontinuities of Feynman Integrals and the Monodromy Group". In: (July 2020). arXiv: 2007.13747 [hep-th].

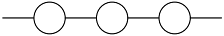
[4] Holmfridur Hannesdottir and Matthew D. Schwartz. "A Finite S-Matrix". In: (June 2019). arXiv: 1906.03271 [hep-th].

[5] Holmfridur Hannesdottir and Matthew D. Schwartz. "S-Matrix for massless particles". In: *Phys. Rev. D* 101.10 (2020), p. 105001. DOI: 10.1103/PhysRevD.101.105001. arXiv: 1911.06821 [hep-th].

Backup Slides

Example of Monodromy Matrix - $\ln^3(s)$

$$\mathcal{M} = $$



$$\propto \ln^3(s)$$

- Collect total differentials into a vector.

$$d\left(\frac{\ln^n s}{n!}\right) = \left(\frac{\ln^{n-1} s}{(n-1)!}\right)\frac{ds}{s}\,,$$

$$\mathcal{V} \equiv \begin{pmatrix} 1 & \ln s & \frac{1}{2}\ln^2 s & \frac{1}{3!}\ln^3 s \end{pmatrix}$$

- Solve differential equation.

$$d\mathcal{V} = \mathcal{V}\cdot\omega$$

with

$$\omega = \begin{pmatrix} 0 & \frac{ds}{s} & 0 & 0 \\ 0 & 0 & \frac{ds}{s} & 0 \\ 0 & 0 & 0 & \frac{ds}{s} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Example of Monodromy Matrix - $\ln^3(s)$

- Collect solutions in a normalized matrix.

$$\mathscr{M}_{\gamma_0} = \begin{pmatrix} 1 & \ln s & \frac{1}{2}\ln^2 s & \frac{1}{3!}\ln^3 s \\ 0 & 1 & \ln s & \frac{1}{2}\ln^2 s \\ 0 & 0 & 1 & \ln s \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with $d\mathscr{M}_{\gamma_0} = \mathscr{M}_{\gamma_0} \cdot \omega$.

- Calculate monodromies around $s = 0$.

$$\mathscr{M}_{\circlearrowleft_0^s} = \begin{pmatrix} 1 & 2\pi i & \frac{1}{2}(2\pi i)^2 & \frac{1}{3!}(2\pi i)^3 \\ 0 & 1 & 2\pi i & \frac{1}{2}(2\pi i)^2 \\ 0 & 0 & 1 & 2\pi i \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Example of Monodromy Matrix - $\ln^3(s)$

- Compute any sequence of discontinuities by multiplying matrices.

$$
\begin{aligned}
&(\mathbb{1} - \mathscr{M}_{\circlearrowleft_0^s}) \cdot \mathscr{M}_{\gamma_0}(s) \\
&= \begin{pmatrix}
0 & 2\pi i & 2\pi i \ln s + \frac{(2\pi i)^2}{2} & \frac{2\pi i}{2} \ln^2 s + \frac{(2\pi i)^2}{2} \ln s + \frac{(2\pi i)^3}{3!} \\
0 & 0 & 2\pi i & 2\pi i \ln s + \frac{(2\pi i)^2}{2} \\
0 & 0 & 0 & 2\pi i \\
0 & 0 & 0 & 0
\end{pmatrix}
\end{aligned}
$$

$$\mathbf{Disc}_s \frac{\ln^3(s)}{3!} = \frac{2\pi i}{2} \ln^2 s + \frac{(2\pi i)^2}{2} \ln s + \frac{(2\pi i)^3}{3!}$$