Graded Hilbert spaces and quantum distillation in QFT

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Statement of the problem-I

- My aim it to generalize the connections between the following three:
- 1) Hilbert space of QFT
- 2) The path integral formulation
- 3) Thermodynamics.
- The motivations comes from the following observation: Hilbert spaces are huge places. e.g. 1) even 500 spin 1/2 particle, its dimension is 2^500, who cares?, 2) for a QFT in volume V, dimension of Hilbert space scales with exp[V}).
- We do not really care about the details of all states in all occasions equally. Sometimes, low energy states are important, and for some other physical phenomena, the growth of density of states of high energy states is important.

Statement of the problem-II

$$Z(\beta) = \mathrm{tr}[e^{-\beta H}]$$

- Thermal partition functions or state sums (with appropriate Boltzmann weights) of a QFT usually exhibit a phase transition (or rapid crossover) as a function of the inverse temperature.

- You can for example show that in order to get the well-known Stefan-Boltzmann law of blackbody radiation, the density of states must grow as:

$$\mathcal{F} = -V_3 \times \frac{\pi^2}{45} N^2 T^4 \qquad\qquad \rho_{SB}(E) \sim e^{E^{3/4} N^{1/2} {V_3}^{1/4}}$$

- So, for such a phenomena, we cannot be ignorant of high energy states or the growth of density of states.

Statement of the problem-II

- On the other hand, many QFTs that are relevant to condensed matter or high energy physics has asymptotic freedom or infrared freedom. That means, there is always a range of temperature T at which these theories become weakly coupled.

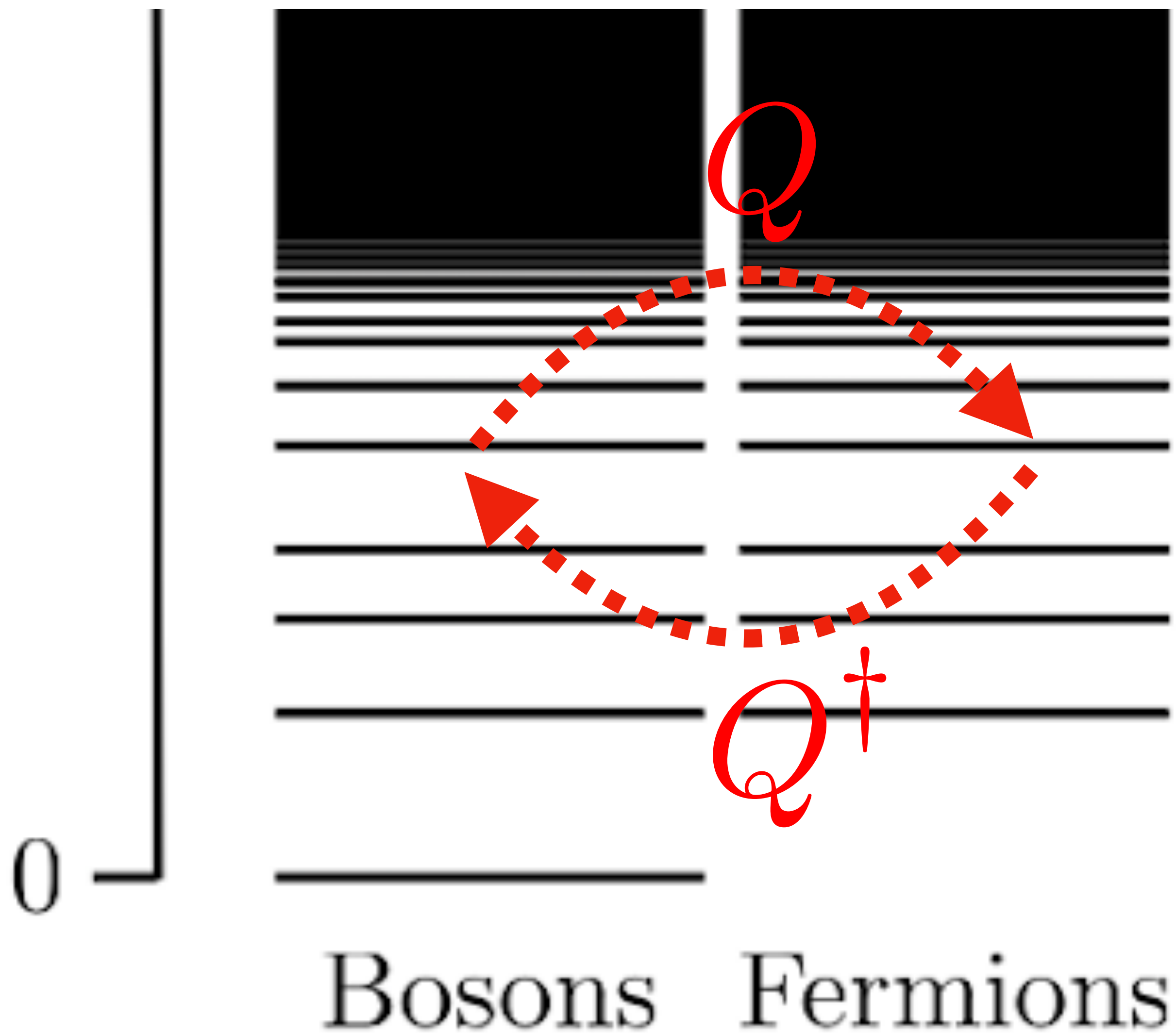
- For example, asymptotic freedom (as in QCD) tells us that the phenomena at the scale of T becomes weakly coupled, and calculable if T is high. But at such T, partition function is extremely contaminated, every state contribute on the same footing.

- Given a general QFT and its Hilbert space, can we construct a different state sum which remains analytic while staying in thermodynamic limit?

- This is my main question for the talk. I believe a positive answer to this question will have a positive impact in understanding QFT.

Simple and not so simple examples of symmetry graded state sums

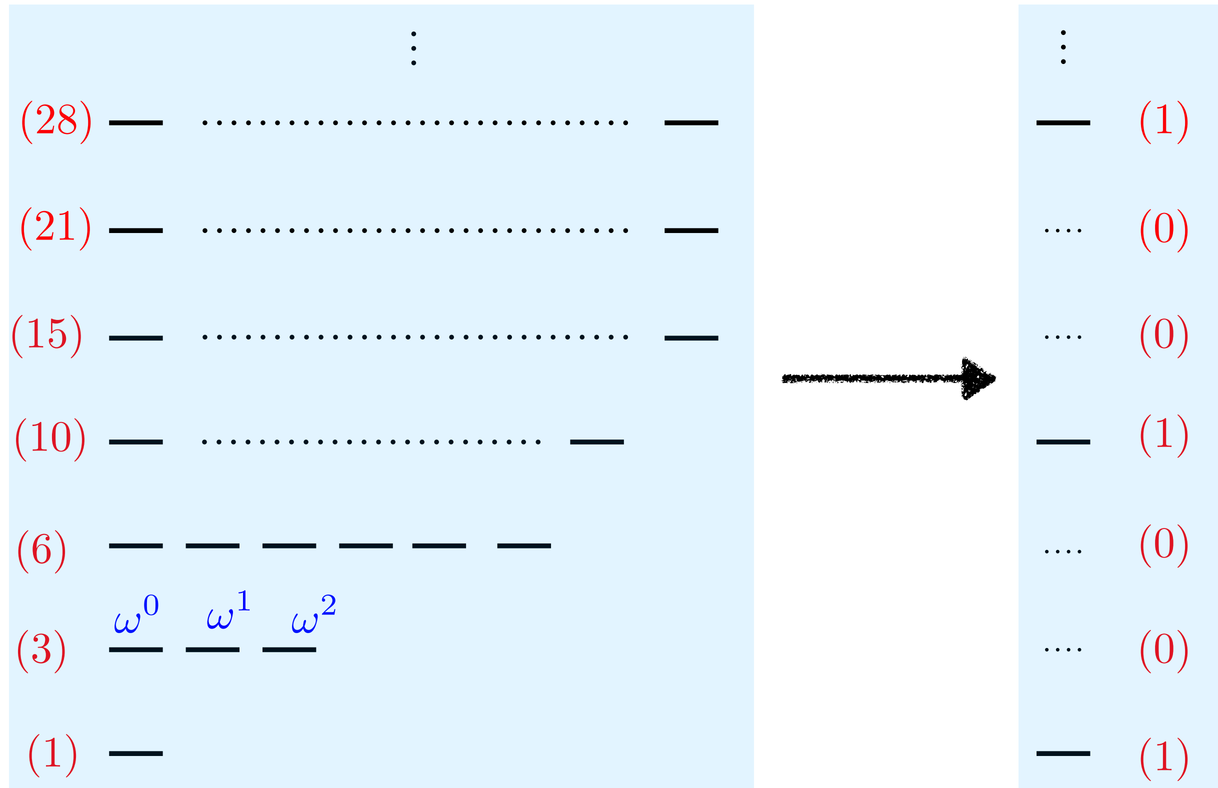
Supersymmetric theories



$$I(\beta) = \mathrm{tr}\left[(-1)^F e^{-\beta H}\right]$$

Just the ground states contribute (assuming the spectrum is discrete), protected by exact supersymmetry. Witten,82

N-dimensional simple harmonic oscillator



$\mathcal{H}$ $\qquad\qquad$ $\mathrm{Distill}[\mathcal{H}]$

$$\mathcal{Z}_{\Omega_F^0}(\beta) = \mathrm{tr}\,\Big(e^{-\beta H}\prod_{j=1}^{N} e^{i\frac{2\pi}{N}j\widehat{Q}_j}\Big), \qquad Q_j = \widehat{a}_j^\dagger \widehat{a}_j$$

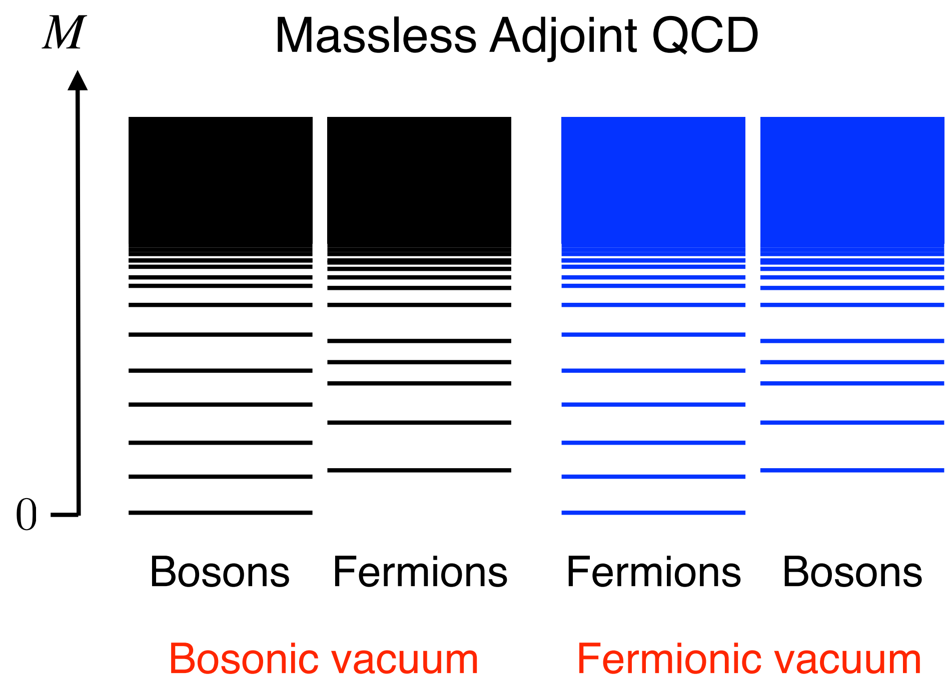
$$\deg(k) = \binom{N+k-1}{k} \longmapsto (\underbrace{1,0,\ldots,0,}_{N}\underbrace{1,0,\ldots,0,}_{N}1,\ldots)$$

Dramatic cancellation in the graded sum.

Large-N limit: **Only ground state contribute**, couter-part of super-symmetric Witten index, but in a boring bosonic QM.

SU(N) 2d QCD(adj)

Fairly non-trivial theory. A number of works in 90s. But its true nature started to reveal itself since 2019. Understanding it fully requires new concepts. But here is one strange fact about N-even case.



$$\mathcal{Z}_{++}(\beta, L) = \mathrm{tr}\left[(-1)^F e^{-\beta H_L}\right] = 0.$$

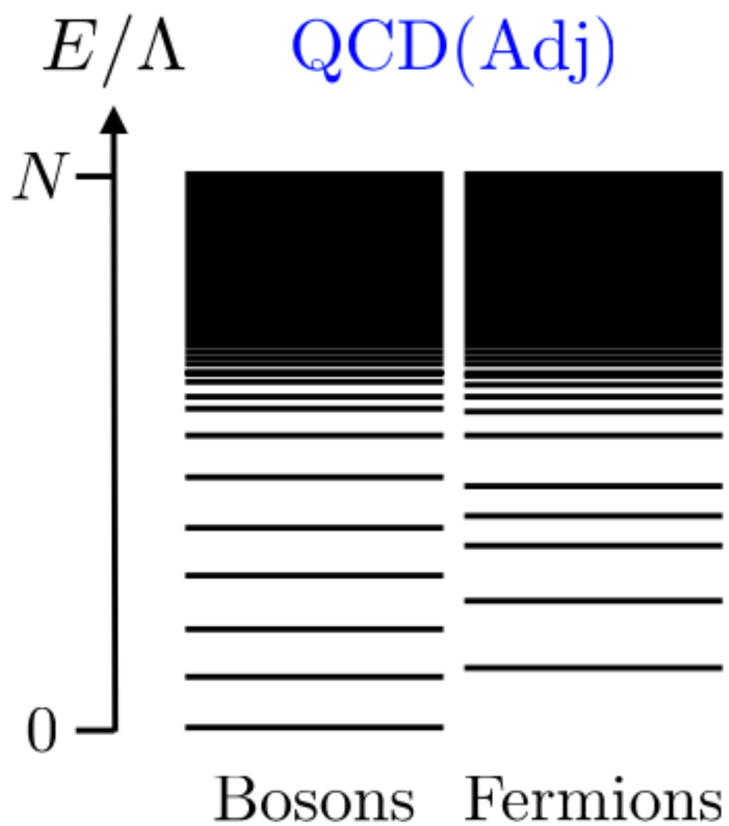
Mixed anomaly between $(\mathbb{Z}_2)_F$ and $(\mathbb{Z}_2)_\chi$ for N even.

Exact Bose-Fermi degeneracy without supersymmetry.

Mixed-anomaly protected spectral degeneracy!

Cherman, Jacobson, Tanizaki, MU 2019, Komargodski et.al. 2020, Klebanov et.al. 2021
Delmastro, Gaiotto, Gomis, 2021

SU(N) 4d QCD(adj)



$$\widetilde{Z}(\beta) = \mathrm{tr}\left[(-1)^F e^{-\beta H}\right]$$

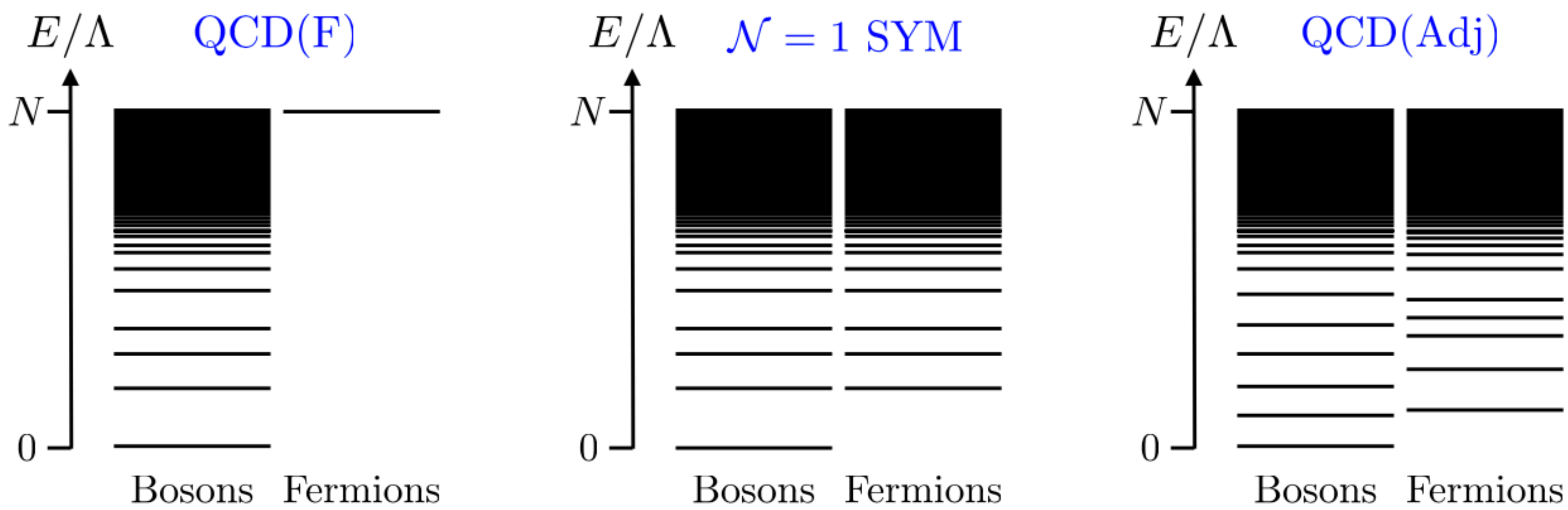
The graded density of states $\rho_B(E) - \rho_F(E)$ for QCD(adj) defined on a curved 3-manifold has the scaling of a 2d QFT. (In the large-N limit.)

$$\rho_B(E) - \rho_F(E) \sim e^{\sqrt{\ell E}}$$

Tremendous spectral cancellation. (No susy, not level by level) Cherman, Shifman, MU 2018

Identical to supersymmetric theories in a similar set-up. De Pietro, Komargodski 2014

How does QCD compare with these QFTs?



Not much resemblance.
At even-Nc, QCD Hilbert space is manifestly bosonic.
For odd-N, at large-N, baryons become heavy with O(N) mass
and finite part of the Hilbert space is again bosonic.

One last example and background story

- In 2012, w/Gerald Dunne, we introduced the idea of resurgence and trans-series in QFT. As an example, we studied CP(N) model by using an SU(N) symmetry-twisted boundary conditions on $S^1 \times \mathbb{R}^1$

- Quite remarkably, almost all interesting non-perturbative properties of the compactified theory matches to the expected properties of infinite volume limit. E.g.

- Mass gap of the order of strong scale

- Renormalons

- Multi-branched theta angle dependence

- These results are in sharp distinction with elegant work of Affleck (82) which study the same theory with thermal compactification, and none of these survive.

- Why small circle theory knows all NP properties?

Quick review of CP(N-1) with tbc.

Point-wise modulus and phase splitting: complexified hyperspherical coordinates

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ \vdots \\ n_N \end{pmatrix} = \begin{pmatrix} e^{i\varphi_1} \cos\frac{\theta_1}{2} \\ e^{i\varphi_2} \sin\frac{\theta_1}{2} \cos\frac{\theta_2}{2} \\ e^{i\varphi_3} \sin\frac{\theta_1}{2} \sin\frac{\theta_2}{2} \cos\frac{\theta_3}{2} \\ \vdots \\ e^{i\varphi_N} \sin\frac{\theta_1}{2} \sin\frac{\theta_2}{2} \sin\frac{\theta_3}{2} \ldots \sin\frac{\theta_{N-1}}{2} \end{pmatrix}$$

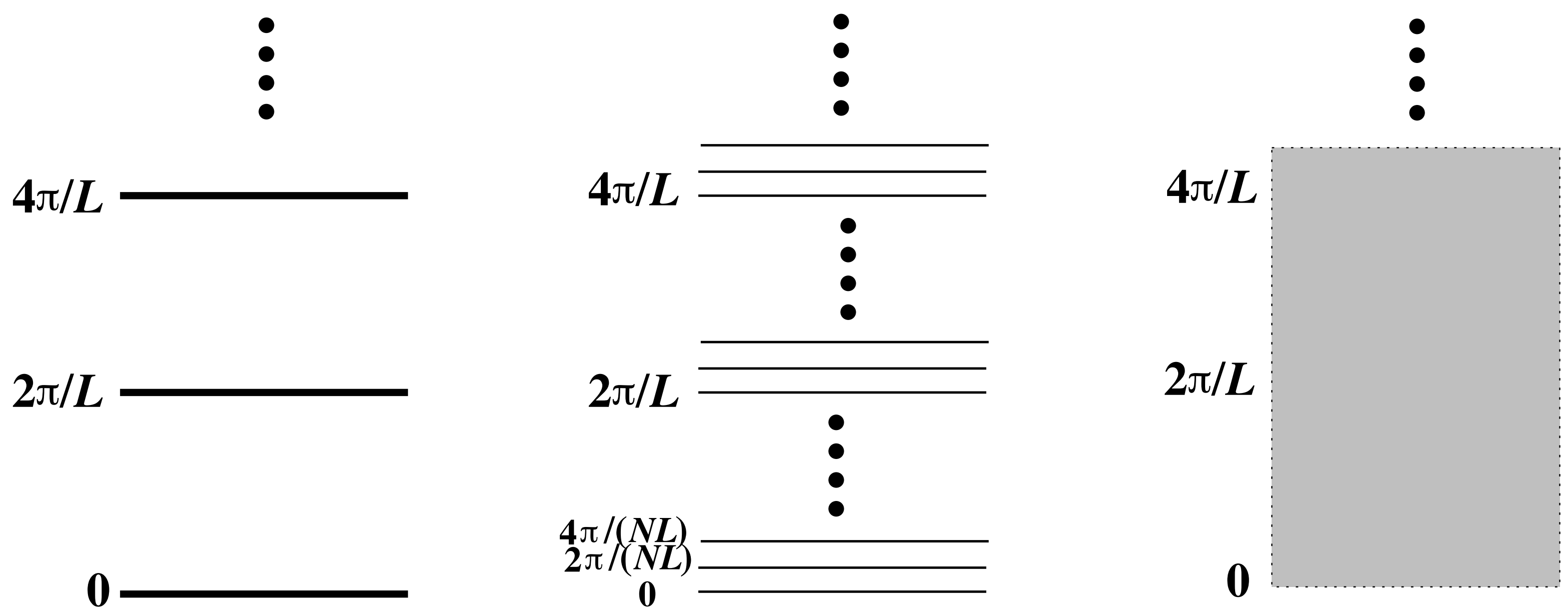
$$\theta_i \in [0,\pi], \qquad \varphi_i \in [0,2\pi).$$

Twisted boundary conditions= Turning on a background SU(N) field

$$n_i(x_1, x_2 + L) = \Omega_{ij} n_j(x_1, x_2)$$

$$\Omega = \mathrm{Diag}\left(1, \omega, \omega^2, \ldots \omega^{N-1}\right), \qquad \omega = e^{i2\pi/N}$$

The dependence of perturbative spectrum to the flavor -holonomy background



$$\Omega = 1_N$$

$$\Omega = \mathrm{Diag}\left(1, \omega, \omega^2, \ldots \omega^{N-1}\right),$$

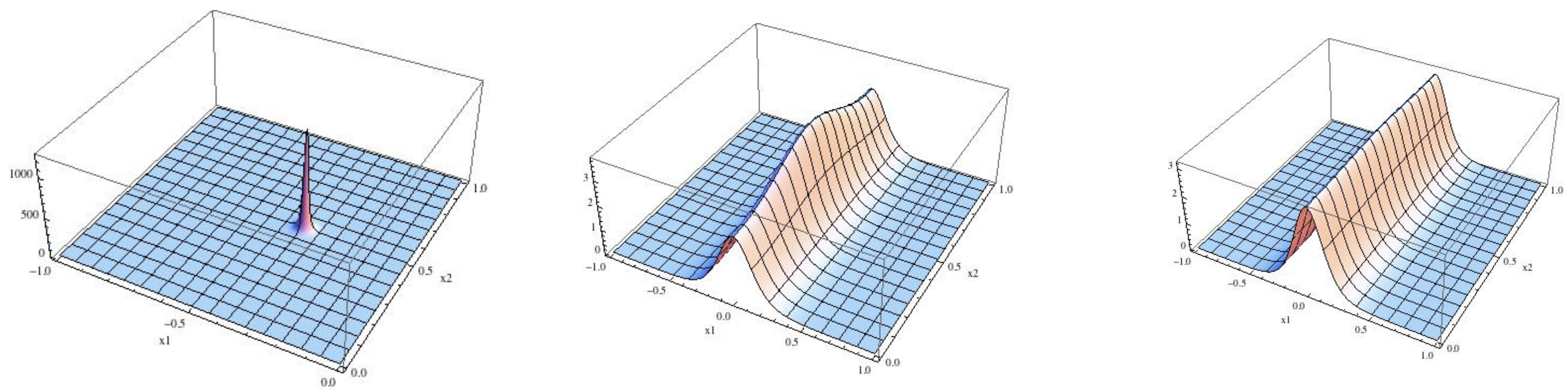
Large-N limit of latter

Same as gauge theory on R3 x S1: Spectrum become dense in the L=fixed, and N-large $\Longrightarrow$ Imprint of the large-N volume independence (large-N or Eguchi-Kawai reduction).

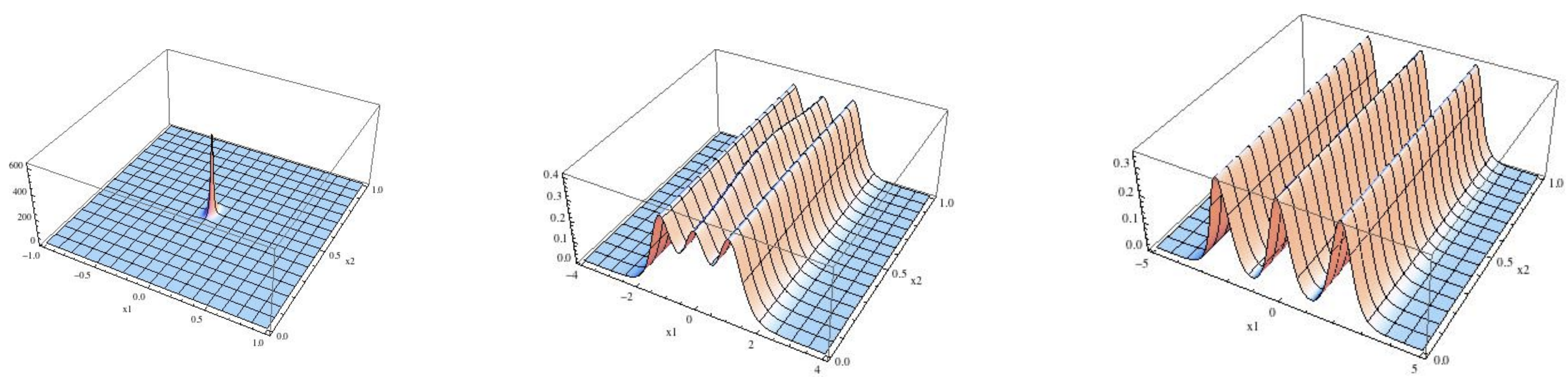
Here, we will study non-pert. effects in the long-distance effective theory within Born-Oppenheimer approx. in case (b) for finite-N.

Topological configurations, 1-defects

In thermal box, and high T, associated with trivial holonomy, the fractionalization does not occur (Affleck, 80s). Plot is for CP(2)



In spatial box, and small-L, associated with non-trivial holonomy, the fractionalization does occur. Large-2d BPST instanton in CP(2) fractionates into 3-types. (Dunne,MÜ, 2012)



Gauge theory counter-part on R_3 x S_1 :
Monopole-instantons or 3d-instanton and twisted instanton.
(caloron constituents) : van Baal, Kraan, (97/98), Lee-Yi (97)

Topological configurations, 1-defects, formally

Fractional instantons: Associated with the N-nodes of the affine Dynkin diagram of SU(N) algebra. The twisted-instanton is present only because the theory is locally 2d! Also derived in Bruckmann et.al.(07, 09)

$$\widetilde{n} \longrightarrow \widetilde{n} + \alpha_i, \qquad \alpha_i \in \Gamma_r^\vee$$

$$\mathcal{K}_k : \qquad S_k = \frac{4\pi}{g^2} \times (\mu_{k+1} - \mu_k) = \frac{S_I}{N} \qquad , \qquad k = 1, \ldots, N$$

$$\mathcal{I}_{2d} \sim e^{-\frac{4\pi}{g^2}} = \left(\frac{\Lambda}{\mu}\right)^{\beta_0} \sim \prod_{j=0}^{\mathfrak{r}} [\mathcal{K}_j]^{k_j^\vee}, \qquad \beta_0 = h^\vee = \sum_{i=0}^{\mathfrak{r}} k_i^\vee, \qquad \mathfrak{r} = \text{rank}[\mathfrak{su}(N)]$$

The mass gap at small-S1: Same as large-N. Due to fractional instantons!
Our small circle still keeps in mind exp[-S_I/N] in long distance dynamics!

In the small-S1 regime, this solves the large-N vs. instanton puzzle!
BPST instantons are unimportant, kink-instantons survive large-N limit!

$$m_g = \frac{C}{\sqrt{\lambda}} \left(1 - \frac{7\lambda}{32\pi} + O(\lambda^2)\right) e^{-\frac{4\pi}{\lambda}} \sim e^{-S_I/N} \qquad \text{for } \mathbb{CP}^{\mathrm{N}-1}$$

Why things work the way they do? Hilbert space distillation idea.

Sulejmanpasic (2016), Dunne, Tanizaki, MU (2018)

$n_i(x_1, x_2 + L) = \Omega_{ij} n_j(x_1, x_2)$ In path integral formalism, becomes:

$$Z_\Omega^{\mathrm{CP(N-1)}} = \mathrm{tr}[e^{-\beta H} \prod_k^{N_f} e^{i \frac{2\pi k}{N_f} Q_k}]$$ in operator formalism,

where Q_k is the number operator for n_k quanta.

Hilbert space distillation in QFT: CP(N-1)

- The global symmetry of CP(N-1) model is actually PSU(N) but not SU(N). Hilbert space constitute reps of PSU(N). There is no gauge invariant fundamental rep of SU(N) in the physical spectrum. There are only meson-like excitations. E.g. CP(1)

$$(nn^\dagger)^k_j(x) \in \mathrm{Adj}_N, \qquad n(x)(e^{i\int_x^y a})(n^\dagger)(y) : \text{singlet}$$

$$\mathcal{Z}(\beta) \supset +(N^2-1)\times e^{-\beta E_{\mathrm{adj}}} + 1\times e^{-\beta E_{\mathrm{singlet}}} + \ldots \underbrace{\rightarrow}_{N\rightarrow\infty} N^2 \times e^{-\beta E_{\mathrm{adj}}}$$

$$\mathcal{Z}_\Omega(L) \supset (-1)\times e^{-\beta E_{\mathrm{adj}}} + 1\times e^{-\beta E_{\mathrm{singlet}}} + \ldots \underbrace{\rightarrow}_{N\rightarrow\infty} 0 \quad \text{HMMM!}$$

Large-N limit: **Only ground state contribute**, couter-part of super-symmetric Witten index in a non-trivial bosonic QFT!

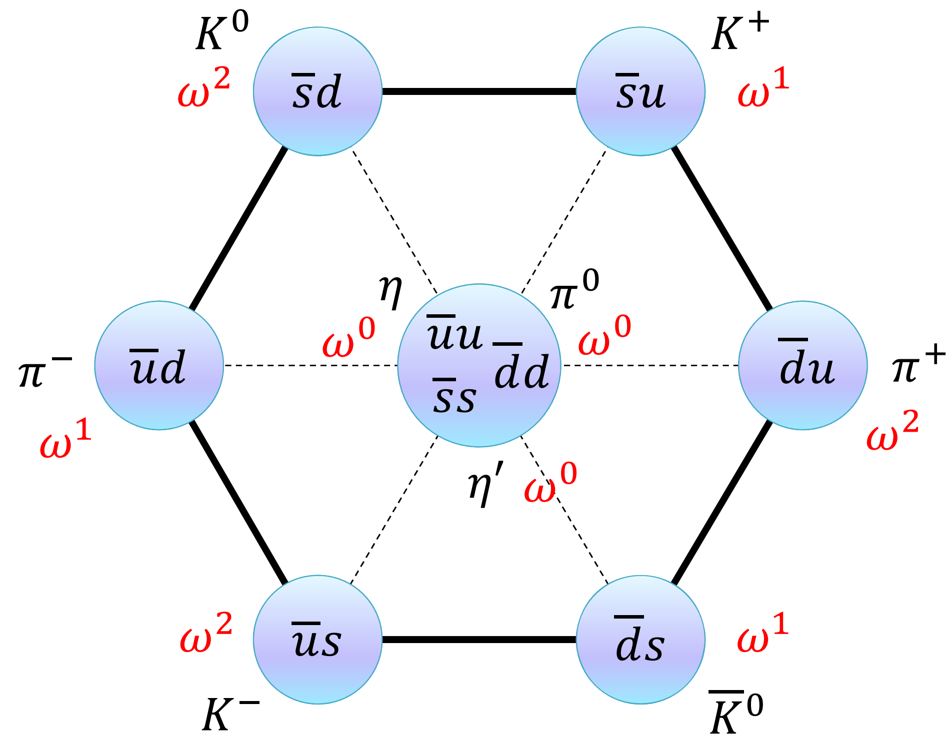
Mixed Anomaly in CP(N-1)

• There is a mixed 't Hooft anomaly between $PSU(N)$ and C at $\theta = \pi$. If we gauge $PSU(N)$, topological charge happens to be quantized in units of $1/N$ and theta angle becomes periodic in units of $2\pi N$. As a result, C ceases to be a symmetry at $\theta = \pi$ implying mixed anomaly.

• Mixed anomaly persists on $\mathbb{R}^1 \times S^1$ if and only if the tbc is $\mathbb{Z}_N$ symmetric.

Can this work in QCD?

- Assume $m_u = m_d = m_s \geq 0$ limit and as an example, consider scalar meson sector.

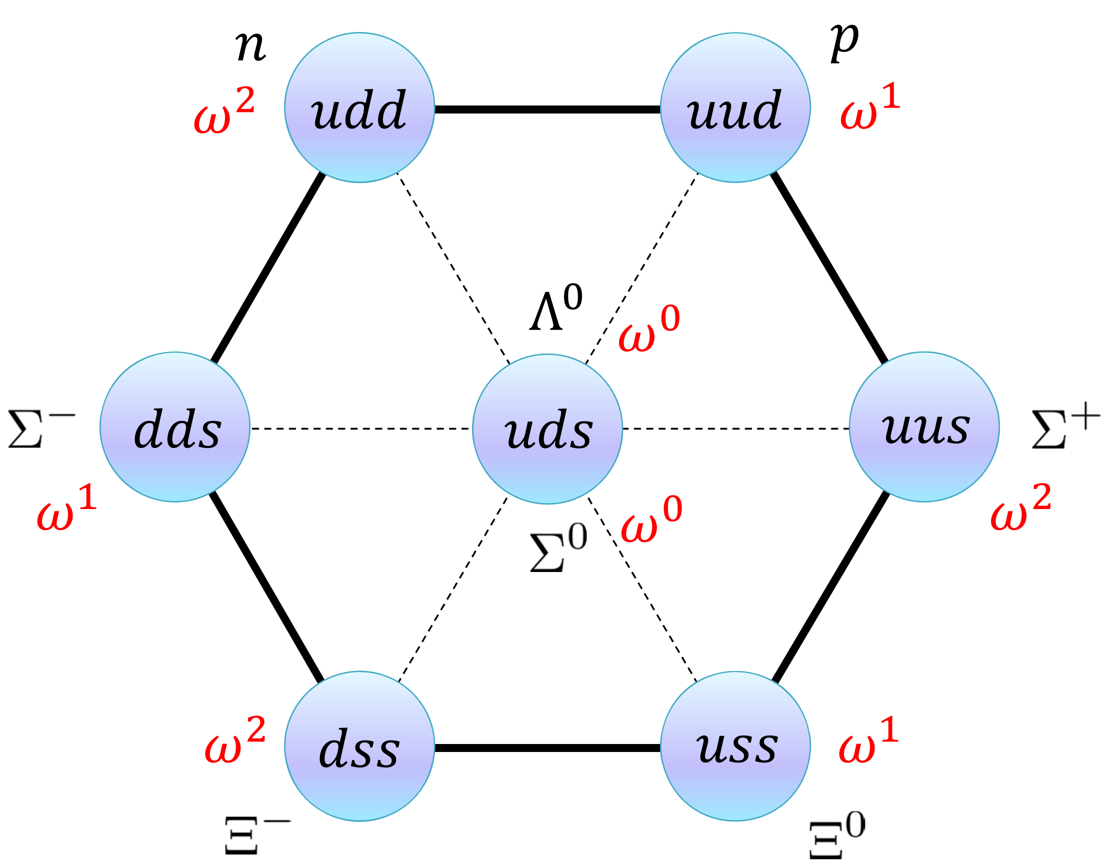


Use $$Z_\Omega^{\rm QCD} = {\rm tr}[e^{-\beta H^{\rm QCD}} \prod_k^{N_f} e^{i\frac{2\pi k}{N_f} Q_k}]$$ What happens?

Morally: $$(N_f^2-1)e^{-\beta E_\pi} + 1e^{-\beta E_{\eta'}} \underbrace{\rightarrow}_{\Omega} (-1)e^{-\beta E_\pi} + 1e^{-\beta E_{\eta'}}$$

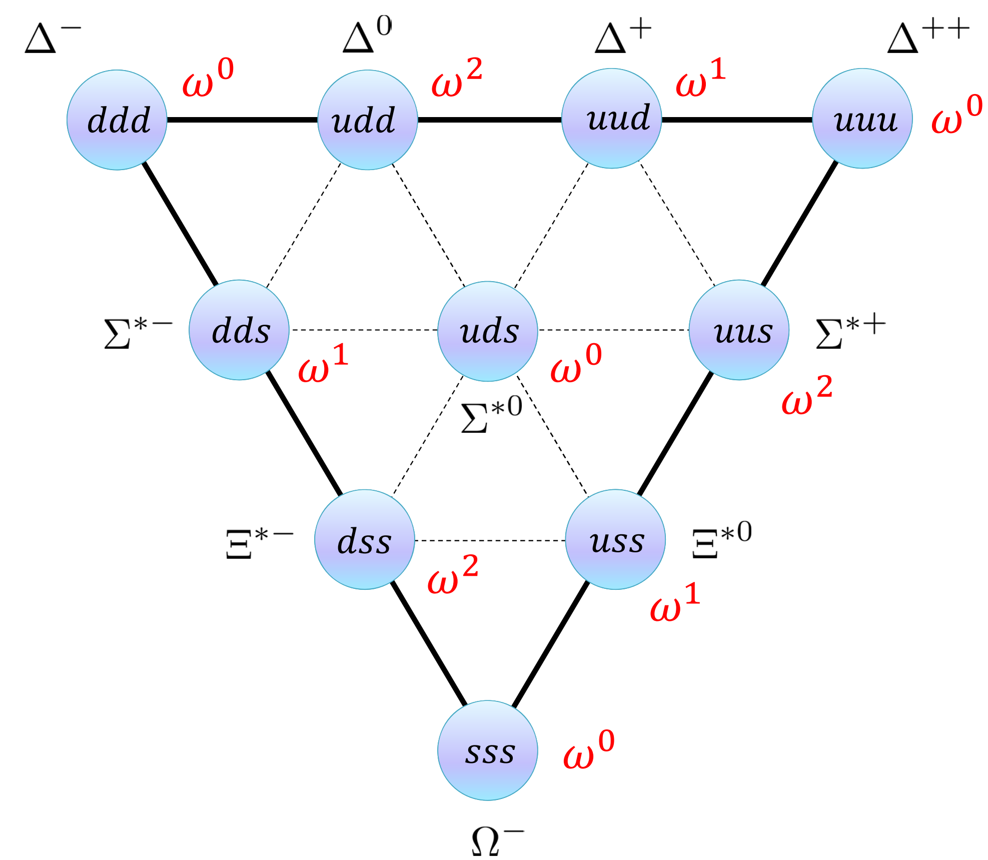
Less than 1 particle contributing to graded P. F. instead of N_f^2

s=1/2 octet graded state-sum



$$16e^{-\beta E_{\rm octet}} \underbrace{\to}_{\Omega} (-2)e^{-\beta E_{\rm octet}}$$

s=3/2 decouplet graded state-sum



$$40e^{-\beta E_{\rm decouplet}} \underbrace{\to}_{\Omega} (+4)e^{-\beta E_{\rm decouplet}}$$

This looks encouraging, but nothing related to pure glue sector cancels!

Modify the theory to QCD(F/adj).

$$\begin{array}{ccccc} \text{SQCD} & \xrightarrow{m_{q^a}\to\infty} & \text{QCD(F/adj)} & \xrightarrow{m_{\psi^a}\to\infty} & \mathcal{N}=1\ \text{SYM} \\ & & \Big\downarrow {\scriptstyle m_\lambda\to\infty} & & \Big\downarrow {\scriptstyle m_\lambda\to\infty} \\ & & \text{QCD(F)} & \xrightarrow{m_{\psi^a}\to\infty} & \text{YM} \end{array}$$

$$\mathcal{L} = \frac{1}{2g^2}\,\mathrm{tr}\,F_{\mu\nu}^2 + \sum_{a=1}^{N_f} \overline{\psi}_a \gamma_\mu D_\mu \psi^a + 2\,\mathrm{tr}\,\overline{\lambda}\overline{\sigma}_\mu D_\mu \lambda$$

$$\mathbf{G} = \frac{SU(N_f)_L \times SU(N_f)_R \times U(1)_V \times U(1)_{A_D} \times \mathbb{Z}_{2\mathrm{gcd}(N_c,N_f)}}{\mathbb{Z}_{N_c} \times (\mathbb{Z}_{N_f})_L \times (\mathbb{Z}_{N_f})_R \times (\mathbb{Z}_2)_\psi}.$$

Extra global U(1)-axial relative to QCD. In SQCD, this is called U(1) R-symmetry.

We consider this theory with vector-like flavor twisted b.c. on R3 x S1.

$$\psi(x_4+\beta) = -\psi(x_4)\overline{\Omega}_F^0$$
$$\lambda(x_4+\beta) = +\lambda(x_4)$$

$$\Omega_F^0 = \mathrm{diag}(1, \omega, \cdots, \omega^{N_f-1}), \qquad \omega = e^{\frac{2\pi i}{N_f}}.$$

B.c. explicitly (but controllably) breaks non-abelian chiral symmetry to its maximal torus.

$$\mathbf{G}_{\mathrm{max-ab}} = \frac{U(1)_L^{N_f-1} \times U(1)_R^{N_f-1} \times U(1)_V \times U(1)_{A_D} \times \mathbb{Z}_{2\mathrm{gcd}(N_c,N_f)}}{\mathbb{Z}_{N_c} \times (\mathbb{Z}_{N_f})_L \times (\mathbb{Z}_{N_f})_R \times (\mathbb{Z}_2)_\psi}.$$

But there are two tremendous gains. I will explain first in detail. Second gain will be just stated. (It tells us that this b.c. is unique in order to preserve certain mixed anomaly polynomial upon compactification.)

Color-Flavor Center (CFC) symmetry

Normally, in the presence of fundamental fermions, we loose center symmetry and Polyakov loops ceases to be a good order parameter.

The reason for this is that a gauge transformation aperiodic up to an element of the center of the group does not respect the original b.c., hence center symmetry is violated. But now, we have

$$\psi^G(x_4+\beta) = -\psi^G(x_4)\overline{\Omega}_F^0 \quad \text{which implies} \quad \psi(x_4+\beta) = -\psi(x_4)\omega^{-1}\overline{\Omega}_F^0.$$

This again changes b.c. but $\omega\Omega_F^0$ is a cyclic permutation of Ω_F^0 and can be undone by flavor rotation generated by the shift matrix

$$S = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & & 0 \end{bmatrix}$$

Cherman, Sen, Unsal, Wagman, Yaffe, 2017 (symmetry realization)
Iritani, Itou, Misumi 2015 (simulations)

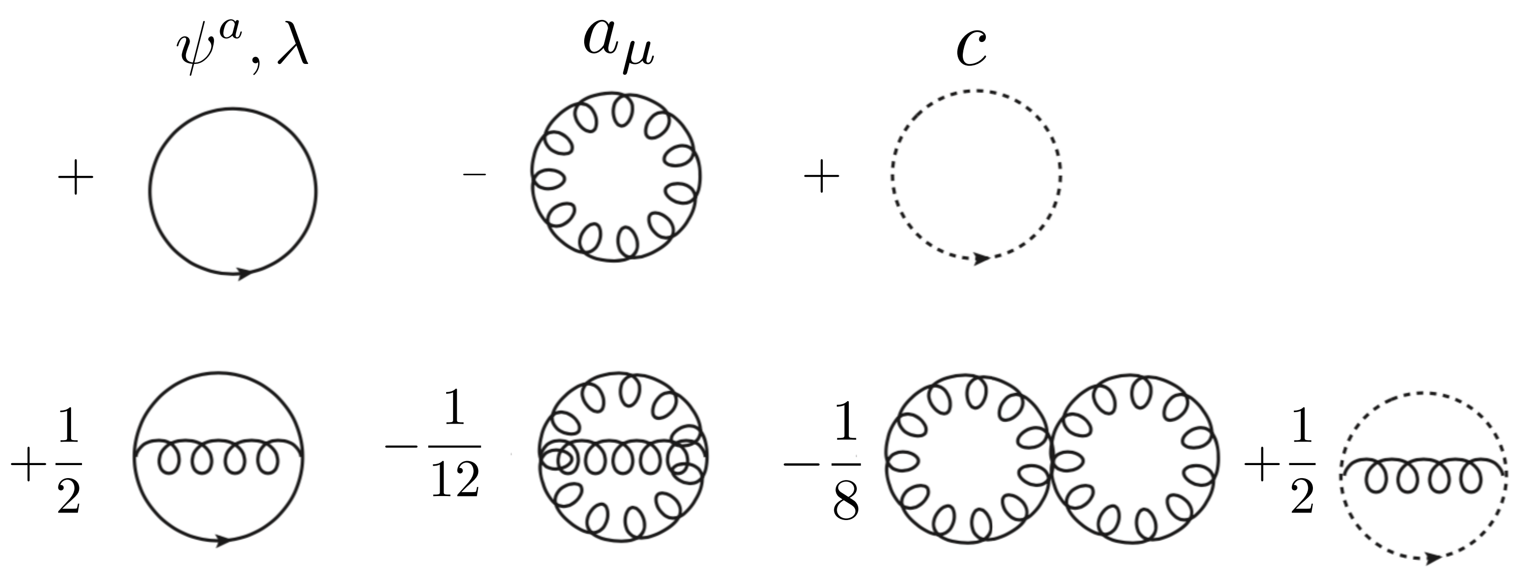
Action of CFC symmetry:

$\mathbb{Z}_{N_c}$: $\quad \mathrm{tr}\,\Omega^p \mapsto \omega^p \; \mathrm{tr}\,\Omega^p$,

$$\widehat{(\psi_R\psi_L)}_p \mapsto \omega^p \, \widehat{(\psi_R\psi_L)}_p \qquad \widehat{(\psi_R\psi_L)}_p :\equiv \frac{1}{\sqrt{N_c}}\sum_{a=1}^{n_{\rm f}} \omega^{-ap}\,\psi_{Ra}\psi_L^a \, :$$

Gauge holonomy potentials

For thermal gauge holonomy potential, we need:



One-loop: Gross, Pisarski, Yaffe 1981

Two-loop:
Korthals Altes, Pisarski and Sinkovics, 99,
Guo and Du, 2018,

Fairly difficult, two results look different.
But turns out to be identical due to non-trivial Bernoulli polynomial identities.
This is proven by Takuya Kanazawa 2019.

In our case, we need invoke appropriate b.c.

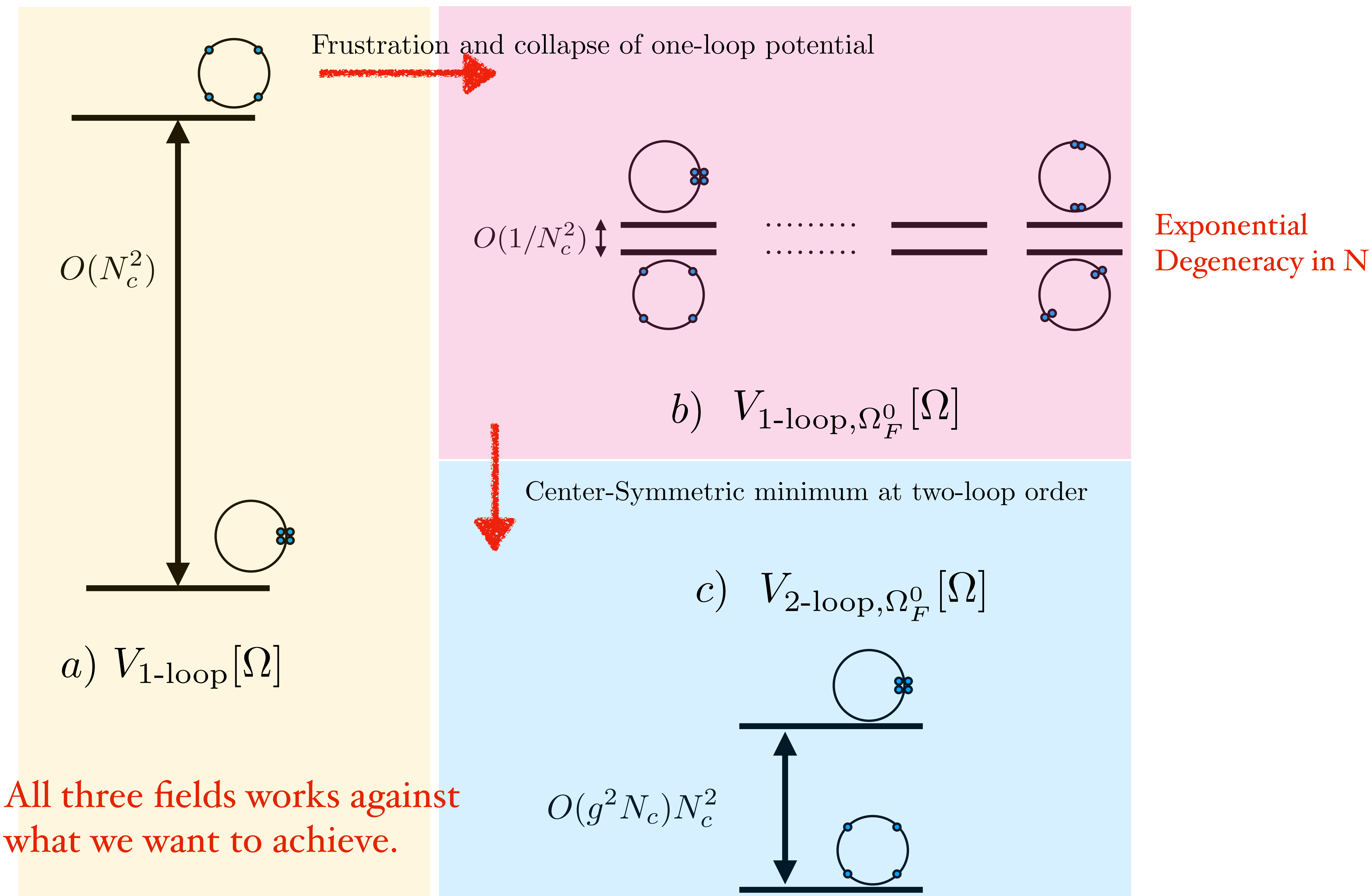
$$\psi(x_4+\beta) = -\psi(x_4)\overline{\Omega}_F^0$$

$$\lambda(x_4+\beta) = +\lambda(x_4)$$

Frustration, collapse and a new governance

Thermal

Flavor and Z2 twisted b.c.



All three fields works against what we want to achieve.

Since this sounds like magic, I want to show you first step of the calculation. Second step is fairly difficult.

$$V_{1-\mathrm{loop,thermal}}(\Omega) = \frac{2}{\beta}\int \frac{d^3p}{(2\pi)^3}\Bigg[+\sum_{i,j}\log(1-e^{-\beta p+iv_{ij}}) - \sum_{i,j}\log(1+e^{-\beta p+iv_{ij}})$$

$$-N_f\sum_{i}\left(\log(1+e^{-\beta p+iv_i})+\mathrm{c.c.}\right)\Bigg]$$

$$= \frac{2}{\pi^2\beta^4}\sum_{n=1}^{\infty}\left[(-1+(-1)^n)\frac{1}{n^4}|\,\mathrm{tr}\,(\Omega^n)|^2 + N_f\frac{(-1)^n}{n^4}\left(\mathrm{tr}\,(\Omega^n)+\mathrm{c.c.}\right)\right]$$

$$-O(N_c^2) \qquad -O(N_f N_c)$$

Explicit center breaking

$$V_{1-\mathrm{loop},\Omega_{\mathrm{F}}}(\Omega) = \frac{2}{\beta}\int \frac{d^3p}{(2\pi)^3}\Bigg[+\sum_{i,j}\log(1-e^{-\beta p+iv_{ij}}) - \sum_{i,j}\log(1+e^{-\beta p+iv_{ij}+i\pi})$$

$$-\sum_{a=1}^{N_f}\sum_{i}\left(\log(1+e^{-\beta p+iv_i-i\epsilon_a})+\mathrm{c.c.}\right)\Bigg]$$

$$= \frac{2}{\pi^2\beta^4}\sum_{n=1}^{\infty}\left[(-1+1)\frac{1}{n^4}|\,\mathrm{tr}\,(\Omega^n)|^2 + \frac{(-1)^n}{n^4}\left(\mathrm{tr}\,(\Omega^n)\,\mathrm{tr}\,(\overline{\Omega}_F^n)+\mathrm{c.c.}\right)\right]$$

0

$$V^{\psi}_{\text{2-loop},\Omega^0_F} \to \frac{g^2}{\beta^4}\frac{3}{\pi^4}\left\{-\frac{N_c^2-1}{8N_cN_f^3}\sum_{n=1}^{\infty}\frac{(-1)^{N_f k}}{k^4}[\mathrm{Tr}(\Omega^{N_f k}) + \text{c.c.}] + \frac{N_f}{24}\sum_{n=1}^{\infty}\frac{\left|\mathrm{Tr}(\Omega^n)\right|^2}{n^4}\right\}$$

Remarkably simple result in Veneziano type large-Nc limit:

$$V_{\text{1-loop},\Omega_F} + V_{\text{2-loop},\Omega_F} = +x\frac{(g^2N_c)}{8\pi^4\beta^4}\sum_{n=1}^{\infty}\frac{\left|\mathrm{Tr}(\Omega^n)\right|^2}{n^4}\,, \qquad x = N_f/N_c$$

Major point: Potential respects CFC (by our construction) and CFC is stable at small circle!

Similar to stability of center symmetry in QCD(adj) on small circle.

What does it mean for thermal and graded density of states?

$$\mathcal{F}_{\text{thermal}}(\beta) = -\frac{\pi^2}{90}\frac{V_3}{\beta^4}\left[\underbrace{2(N_c^2-1)}_{\text{gluons}} + \frac{7}{8}\times\underbrace{2(N_c^2-1)}_{\text{adj.Weyl ferm.}} + \frac{7}{8}\times\underbrace{4N_f N_c}_{\text{fund.D.ferm.}}\right] \qquad \beta\to 0$$

- **Micro interpretation:** Microscopic counting of gluons and quarks. The numbers appearing in free energy such as (N_c^2-1), (N_c^2-1), $4N_fN_c$ count respectively, the number of microscopic bosonic and fermionic degrees of freedom in the QFT.

- **Macro interpretation:** Counting of macroscopic states, the hadrons in physical Hilbert space $\mathcal{H}$. The inverse Laplace transform of the partition function is the density of states of hadrons:

$$\mathcal{Z}(\beta) \sim e^{-\beta\mathcal{F}_{\text{thermal}}} \sim e^{+aN_c^2V_3T^3} \qquad \Longleftrightarrow \qquad \rho_{SB}(E) \sim e^{E^{3/4}N_c^{1/2}(aV_3)^{1/4}}$$

$\rho_{SB}(E)$ is the Stefan-Boltzmann growth. SB growth is special in the sense that it is the largest asymptotic growth in a local finite-N_c QFT. Only at $N_c=\infty$ and string theory, one can obtain a Hagedorn-growth.

$$\mathcal{Z}(\beta) = \operatorname{tr}\left[e^{-\beta H}(-1)^F e^{i\pi Q_0} \prod_{a=1}^{N_f} e^{i\frac{2\pi a}{N_f} Q_a}\right],$$

$$\mathcal{F}_{\text{thermal}}(\beta) = -\frac{\pi^2}{90}\frac{V_3}{\beta^4}\left[\underbrace{2(N_c^2-1)}_{\text{gluons}} + \frac{7}{8}\times \underbrace{2(N_c^2-1)}_{\text{adj.Weyl ferm.}} + \frac{7}{8}\times \underbrace{4N_f N_c}_{\text{fund.D.ferm.}}\right] \qquad \beta \to 0$$

$$\mathcal{F}_{\Omega_F^0}(\beta) = -\frac{\pi^2}{90}\frac{V_3}{\beta\beta^4}\left[\underbrace{2(N_c^2-1)}_{\text{gluons}} \quad \underbrace{-2(N_c^2-1)}_{\text{adj.Weyl ferm.pbc.}} + \frac{7}{2} \underbrace{\frac{1}{N_c^2}}_{\text{fund.D.ferm.tbc.}}\right]$$

$$= -\frac{\pi^2}{90}\frac{V_3}{\beta^4}\left[\frac{7}{2}\frac{1}{N_c^2}\right] \xrightarrow{N_c\to\infty} 0$$

$$\mathcal{Z}_{\Omega_F^0}(\beta) \sim e^{-\beta \mathcal{F}_{\Omega_F^0}(\beta)} \sim e^{+a \frac{1}{N_c^2} V_3/\beta^3} \qquad \Longleftrightarrow \qquad \rho_{\Omega_F^0}(E) \sim e^{\frac{1}{N_c^{1/2}} E^{3/4} (aV_3)^{1/4}}$$

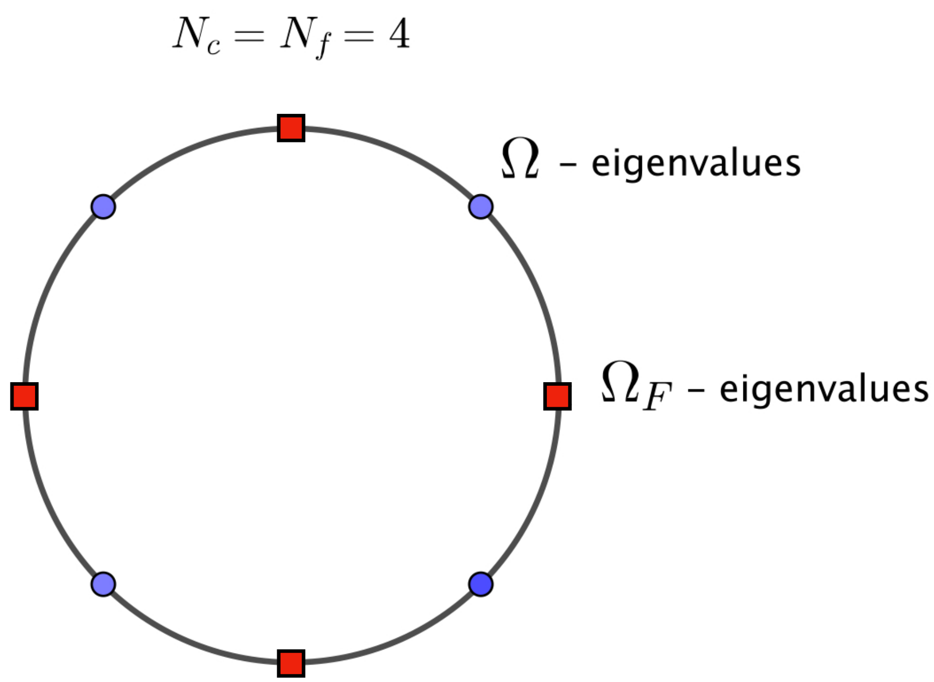
$$\rho_{\Omega_F^0}(E) \sim e^{\sqrt{\ell E}},$$

- Again same as supersymmetric theories and QCD(adj) on M3 x S1.
  (Di Pietro, Komargodski 2014, Cherman, Shifman, MU, 2018)

- Graded density of states is the one of a 2d QFT.

- Remarkable degree of cancellation.

Gauge dynamics on small R3 x S1



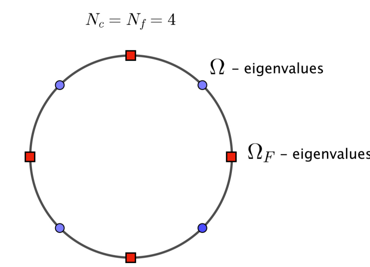
$$\Omega = \mathrm{Diag}\left(e^{iv_1}, e^{iv_2}, \ldots, e^{iv_{N_c}}\right)$$

$$\Omega_F = \mathrm{Diag}\left(e^{i\epsilon_1}, e^{i\epsilon_2}, \ldots, e^{i\epsilon_{N_f}}\right)$$

$$SU(N_c) \to U(1)^{N_c - 1} \,.$$

Fractional instantons (or monopole-instantons)

Self-duality eq. $$Da_4 = *_3 F$$



Actions of the leading saddles

$$S_i = \frac{4\pi}{g^2} v \cdot \alpha_i = \frac{4\pi}{g^2}(v_{i+1} - v_i) = \frac{8\pi^2}{g^2 N}$$

Monopole operators (if fermion zero modes were not there)

$$\mathcal{M}_i = e^{-S_0} e^{-\frac{4\pi}{g^2}\alpha_i \cdot \phi + i\alpha_i \cdot \sigma}, \qquad i = 1, \ldots N_c$$

Gauge holonomy Dual photon

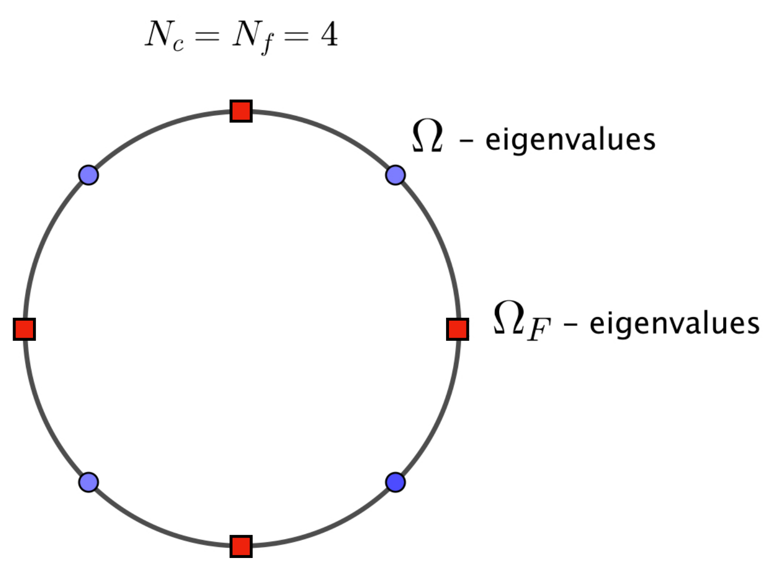
$$a_4^i \beta \equiv v^i + \phi^i, \qquad F^i_{\mu\nu} = g^2/(2\pi\beta)\epsilon_{\mu\nu\alpha}\partial^\alpha \sigma^i, \qquad i = 1, \ldots N_c$$

Index theorem and fermion zero modes

$$\mathcal{I}_{\alpha_i} = 2 \qquad \text{adjoint}$$

$$\mathcal{I}_{\alpha_i} = \sum_{a=1}^{N_f} \left(\text{sign}[\epsilon_a - v_i] - \text{sign}[\epsilon_a - v_{i+1}]\right)$$

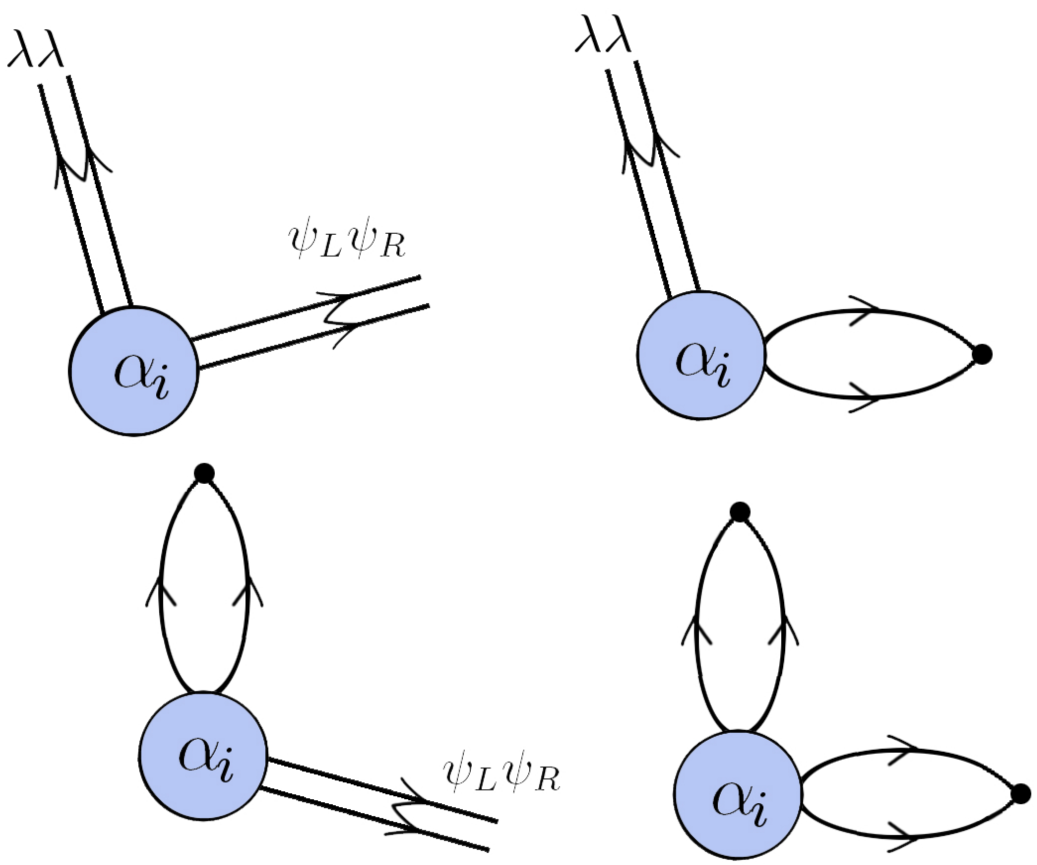
- Generalization of APS to R3 x S1,
- Nye-Singer, 2000, Poppitz-MU 2008



Thanks to flavor twist, otherwise all fundamental zero modes would be localized to one monopole.

$$\left[\mathcal{I}_{\alpha_1}, \mathcal{I}_{\alpha_2}, \ldots, \mathcal{I}_{\alpha_{N_c}}\right] = \underbrace{[2, 2, \ldots, 2]}_{\text{adj. fermion}} + \underbrace{[2, 2, \ldots, 2]}_{\text{fund. fermion}}$$

Monopole operators



$$\mathcal{M}_i = \begin{cases} e^{-S_i} e^{-\frac{4\pi}{g^2}\alpha_i\cdot\phi + i\alpha_i\cdot\sigma}(\psi_{Ri}\psi_L^i)(\alpha_i\cdot\lambda)^2, & m_\lambda = 0,\ m_\psi = 0 \quad \text{QCD(F/Adj)} \\ e^{-S_i} f_\lambda\, e^{-\frac{4\pi}{g^2}\alpha_i\cdot\phi + i\alpha_i\cdot\sigma}(\psi_{Ri}\psi_L^i), & m_\lambda > 0,\ m_\psi = 0 \quad \text{QCD(F)} \\ e^{-S_i} f_\psi e^{-\frac{4\pi}{g^2}\alpha_i\cdot\phi + i\alpha_i\cdot\sigma}(\alpha_i\cdot\lambda)^2, & m_\lambda = 0,\ m_\psi > 0 \quad \text{N=1 SYM} \\ e^{-S_i} f_{\lambda\psi} e^{-\frac{4\pi}{g^2}\alpha_i\cdot\phi + i\alpha_i\cdot\sigma}, & m_\lambda > 0,\ m_\psi > 0 \quad \text{YM} \end{cases}$$

Grand canonical ensemble and EFT on $\mathbb{R}^3 \times S^1$

$$\mathcal{T} = \{\mathcal{M}_i,\ \ \overline{\mathcal{M}}_i,\ \ [\mathcal{M}_i\overline{\mathcal{M}}_{i\pm1}],\ \ [\mathcal{M}_i\overline{\mathcal{M}}_i],\ \ [\mathcal{M}_i\overline{\mathcal{M}}_j\mathcal{M}_k],\ldots\}$$

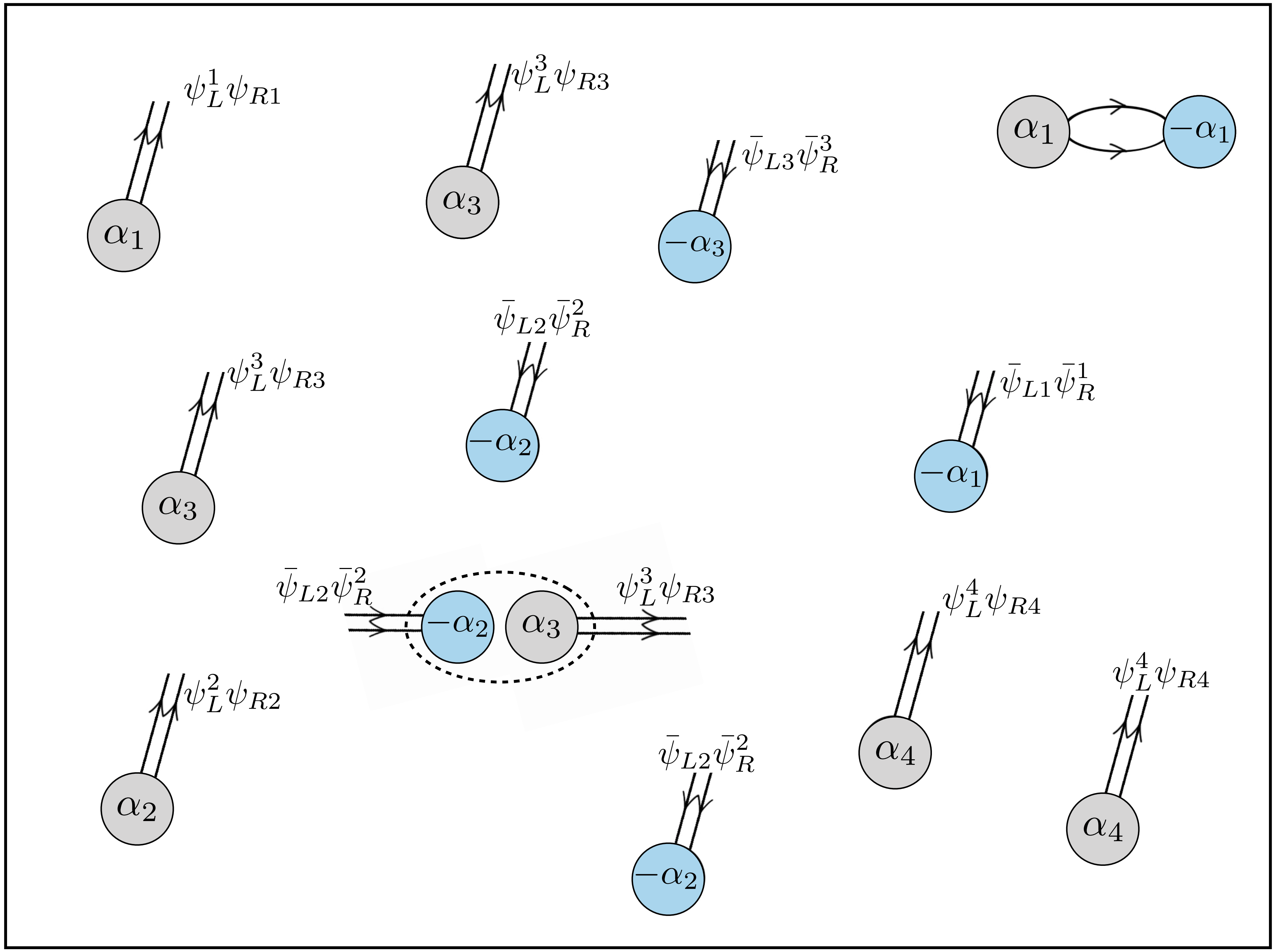
Non-perturbative contributions of critical points (inluding critical points at infinity).

$$\prod_{\mathcal{T}}\sum_{n_{\mathcal{T}}=0}^{\infty}\frac{1}{n_{\mathcal{T}}!}\left[\int d^3x\ \mathcal{T}\right]^{n_{\mathcal{T}}}$$

$$=\prod_{i=1}^{N_c}\left(\sum_{n_{\mathcal{M}_i}=0}^{\infty}\frac{1}{n_{\mathcal{M}_i}!}\left[\int d^3x\ \mathcal{M}_i\right]^{n_{\mathcal{M}_i}}\right)\left(\sum_{n_{\overline{\mathcal{M}}_i}=0}^{\infty}\frac{1}{n_{\overline{\mathcal{M}}_i}!}\left[\int d^3x\ \overline{\mathcal{M}}_i\right]^{n_{\overline{\mathcal{M}}_i}}\right)\cdots$$

$$=\exp\left[\sum_{i=1}^{N_c}\int d^3x\ \ (\mathcal{M}_i+\overline{\mathcal{M}}_i+\mathcal{B}_{ii}+\mathcal{B}_{i,i+1}\ldots)\right]$$

Snapshot of Euclidean vacuum



This is the result of reliable semi-classics. It is unrelated to the so called dilute instanton gas (which is not reliable semi-classics.)

A thought provoking question:

Can gluons acquire a chiral charge?

Sounds absurd?

Wait and see...

Mixing of topological shift symmetry and chiral symmetry

Topological shift symmetry

$$[U(1)_J]^{N_c-1} : \sigma \to \sigma + \varepsilon, \qquad \mathcal{J}_\mu = \partial_\mu \sigma$$

Protects gaplessness of dual photons to all orders in perturbation theory.

$$\mathcal{L}_{kin.} = \frac{g^2}{16\pi^2\beta}\left((\partial_\mu\phi)^2 + (\partial_\mu\sigma)^2\right)$$

Non-perturbatively, monopoles violate it.

$$\partial_\mu \mathcal{J}_\mu = \partial_\mu B_\mu = \rho_m(x) \neq 0$$

RHS is monopole density

What charges are violated or conserved at the monopole event?

Consider a collection of n_i monopoles of type α_i for $i = 1, \ldots, N_c$ sprinkled in between two asymptotic time slice. The magnetic charge non-conservation is

$$
\begin{aligned}
\Delta \mathbf{Q}_m &= \mathbf{Q}_m(t=\infty) - \mathbf{Q}_m(t=-\infty) = \int d^2x F_{12}\Big|_{t=-\infty}^{t=+\infty} \\
&= \int_{S^2_\infty} F_{12} \\
&= \frac{4\pi}{g} \sum_{i=1}^{N_c} n_i \alpha_i \\
&= \frac{4\pi}{g} \left(n_1 - n_{N_c}, n_2 - n_1, n_3 - n_2, \ldots, n_{N_c-1} - n_{N_c}\right)
\end{aligned}
$$

These charges violate *emergent* $[U(1)_J]^{N_c-1}$ explicitly and completely. However, the non-conservation of magnetic charge is not whole story in theories with dynamical fermions.

A puzzle about chiral symmetry

The axial current associated with non-abelian chiral symmetry in 4d: $J_\mu^{5A} = \overline{\psi}\gamma_\mu\gamma_5 T^A\psi$ where T^A are generators of $SU(N_f)$. Charges are: $\mathbf{Q}^{5A} = \int d^3x\, \psi^\dagger\gamma_5 T^A\psi$. These charge commutes with the Hamiltonian $[H, \mathbf{Q}^{5A}] = 0$ for all A.

In the the graded partition function, we have $SU(N_f)_V$ charges along Cartan sub-algebra turned on, and the operator $H' = H - i\sum_{a\in\text{Cartan}} \frac{\epsilon_a}{\beta} Q_a$ only commutes with the Cartan generators of the axial charges, $[H', \mathbf{Q}^{5A}] = 0$ for all $A \in$ Cartan.

Consider again collection of n_i monopoles of type α_i for $i = 1, \ldots, N_c$ sprinkled in between two asymptotic time slice. Then, the apparent axial charge non-conservation will be

$$\begin{aligned}
\Delta\mathbf{Q}^5 &= \mathbf{Q}^5(t=\infty) - \mathbf{Q}^5(t=-\infty) \\
&= \sum_{A=1}^{N_f} n_A\alpha_A \\
&= 2\left(n_1 - n_{N_f}, n_2 - n_1, n_3 - n_2, \ldots, n_{N_f-1} - n_{N_f}\right)
\end{aligned}$$

But non-abelian chiral symmetry is non-anomalous and all non-perturbative effects must respect it. What is going on?

Assume $N_f = N_c$. It is clear that magnetic charge non-conservation and chiral charge non-conservation are exactly proportional to each other for any background. In fact, we can construct a linear combination of these two-charges which is respected by all non-perturbative and topological configurations:

$$\tilde{\mathbf{Q}} = \frac{g}{4\pi}\mathbf{Q}_m - \mathbf{Q}^5, \qquad \text{such that} \qquad \Delta\tilde{\mathbf{Q}} = 0$$

What does this mean? Here, chiral symmetry is a symmetry in microscopic theory. While topological shift symmetry is a emergent symmetry in EFT, valid to all orders in perturbation theory. So, the genuine microscopic symmetry here is only chiral symmetry, and this whole mechanism is present so that the chiral charge of the fermion bilinear can be transferred to gauge fluctuations!

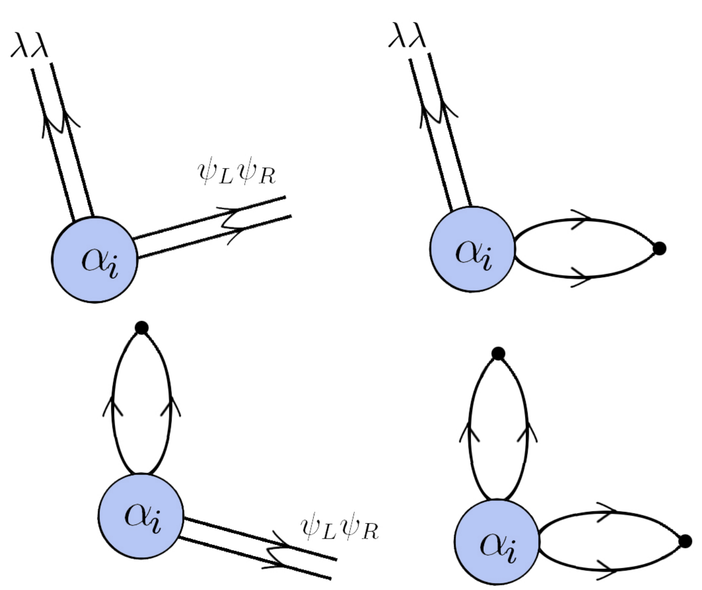
i.e. gauge fluctuations acquire a chiral charge!

$$[U(1)_A]^{N_f-1}: \qquad (\psi_{Ri}\psi_L^i) \to e^{i\varepsilon_i}(\psi_{Ri}\psi_L^i), \qquad \lambda\lambda \to \lambda\lambda$$

$$U(1)_{A_D}: \qquad (\psi_{Ri}\psi_L^i) \to e^{-2i\gamma}(\psi_{Ri}\psi_L^i), \qquad \lambda\lambda \to e^{2i\gamma}\lambda\lambda$$

$$e^{\alpha_i \cdot z} \to e^{-i\varepsilon_i} e^{\alpha_i \cdot z}, \qquad \varepsilon_i = \alpha_i \cdot \varepsilon$$

Hence, monopole operator is invariant under continuous chiral symmetry. But to achieve this, the dual photon must acquire a chiral charge. This is inevitable consequence of the index theorem.



$$\mathcal{M}_i = \begin{cases} e^{-S_i} e^{-\frac{4\pi}{g^2}\alpha_i\cdot\phi + i\alpha_i\cdot\sigma}(\psi_{Ri}\psi_L^i)(\alpha_i\cdot\lambda)^2, & m_\lambda = 0,\ m_\psi = 0 \\ e^{-S_i} f_\lambda\, e^{-\frac{4\pi}{g^2}\alpha_i\cdot\phi + i\alpha_i\cdot\sigma}(\psi_{Ri}\psi_L^i), & m_\lambda > 0,\ m_\psi = 0 \\ e^{-S_i} f_\psi e^{-\frac{4\pi}{g^2}\alpha_i\cdot\phi + i\alpha_i\cdot\sigma}(\alpha_i\cdot\lambda)^2, & m_\lambda = 0,\ m_\psi > 0 \\ e^{-S_i} f_{\lambda\psi} e^{-\frac{4\pi}{g^2}\alpha_i\cdot\phi + i\alpha_i\cdot\sigma}, & m_\lambda > 0,\ m_\psi > 0 \end{cases}$$

Chiral symmetry order parameters: Because of the topological shift and chiral symmetry mixing, in gauge theories in general there are two types of chiral order parameters:

$$\text{Monopole (magnetic flux) operators :} \qquad e^{\alpha_i \cdot z}$$

$$\text{Fermion bilinears, multilinears :} \qquad \psi_{Ra}\psi_L^b, \quad \mathrm{tr}\,\lambda\lambda, \quad (\psi_{Ra}\psi_L^b\,\mathrm{tr}\,\lambda\lambda)$$

In all calculable examples in semi-classical domain on $\mathbb{R}^3 \times S^1$, χSB occurs due to condensation of the magnetic flux operators.

Chiral symmetry breaking

QCD(F/adj)

$$\langle \mathrm{VAC}|e^{-\alpha_i \cdot z}|\mathrm{VAC}\rangle = e^{-\frac{4\pi}{g^2}(v_{i+1}-v_i)}\langle e^{-\frac{4\pi}{g^2}\alpha_i\cdot\phi + i\alpha_i\cdot\sigma}\rangle = e^{-S_0}e^{i\delta_i},$$

$$\mathrm{Diag}[e^{i\delta_1}, \ldots, e^{i\delta_{N_f}}] \in \mathbf{T}^{N_f-1}$$

$$[U(1)_A]^{N_f-1} \times U(1)_{A_D} \longrightarrow U(1)_{A_D}$$

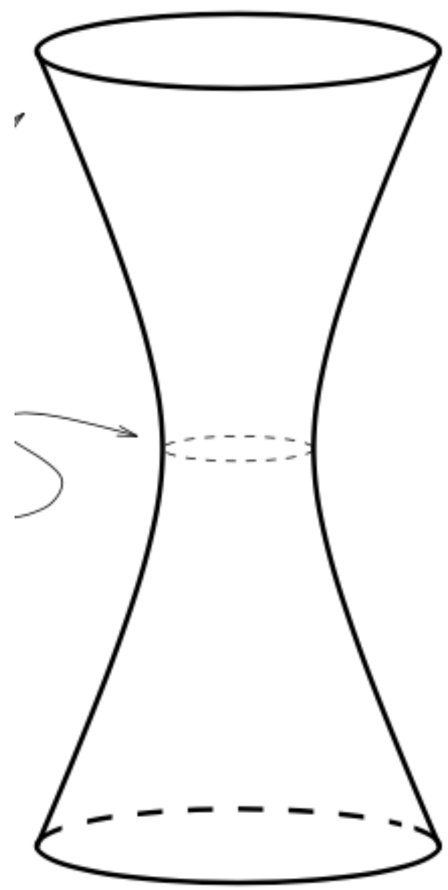
There is a U(1) part of chiral symmetry that is not broken.
Is this sensible?

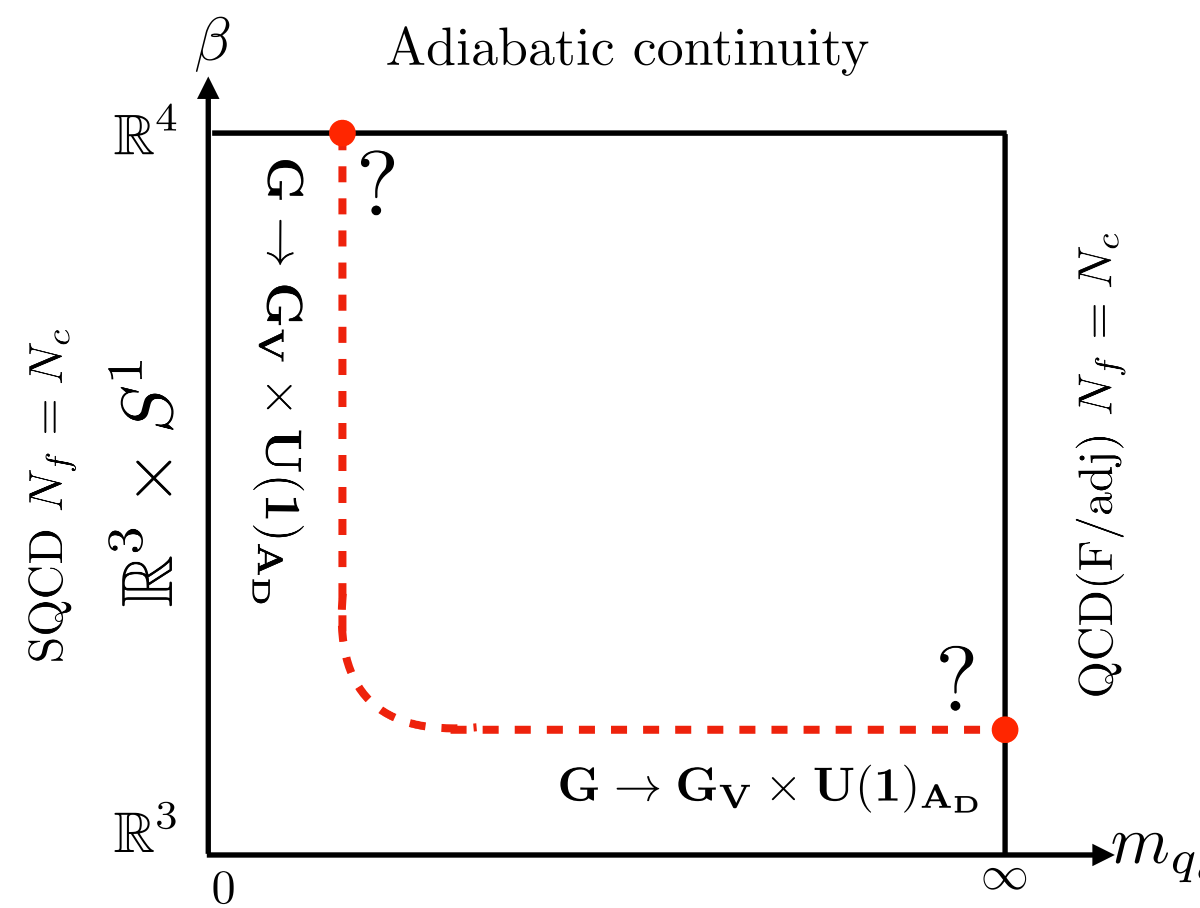
Nf= Nc SQCD vs. QCD(F/adj)

Recall that QCD(F/adj) can be obtained from SQCD with mass deformation for scalars.

$$\det M - B\overline{B} = \Lambda^{2N_c}.$$ Seiberg 1994

$$M^a_b = \Lambda^2 \delta^b_a$$

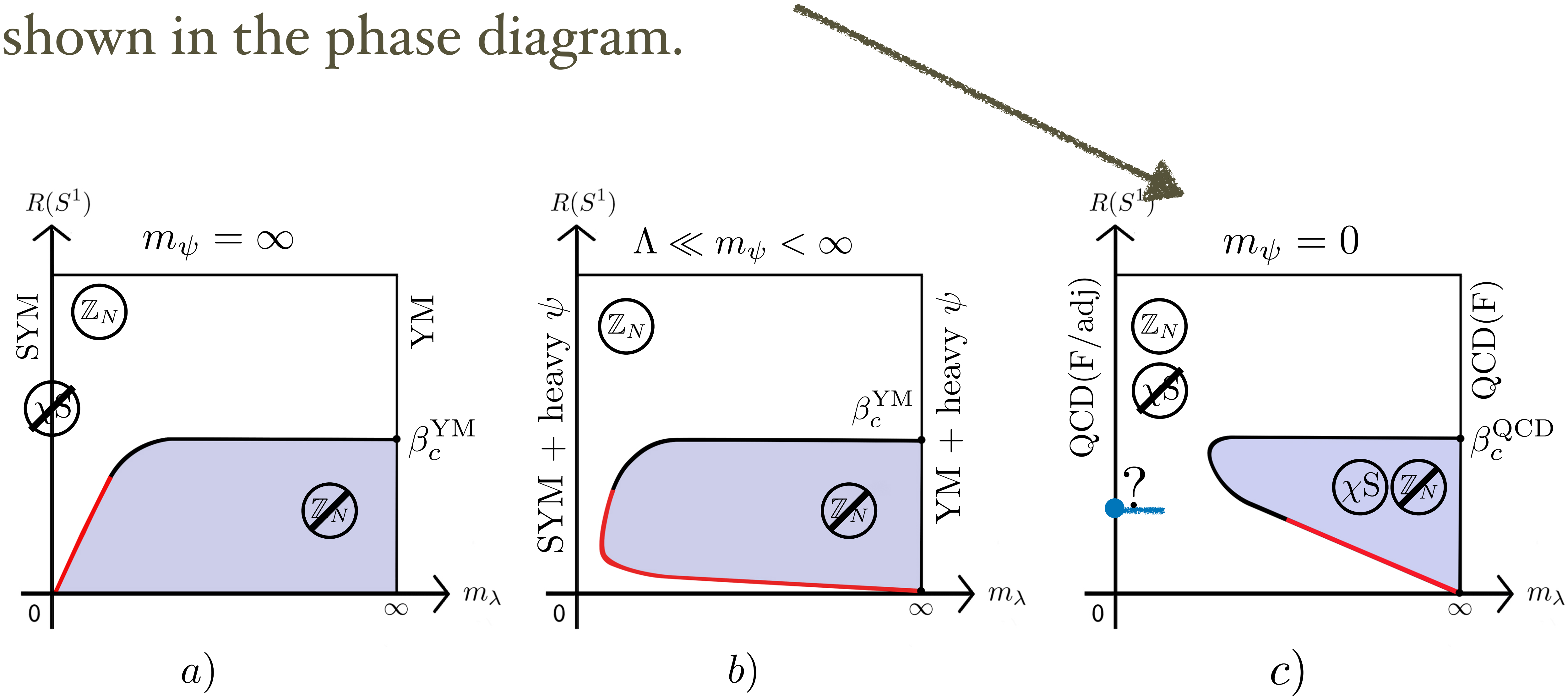




Adiabatic continuity between SQCD and QCD(F/adj).

QCD(F)

To obtain QCD(F), turn on a mass for adjoint. In this case, we obtain adiabatic continuity between small and large circle as shown in the phase diagram.

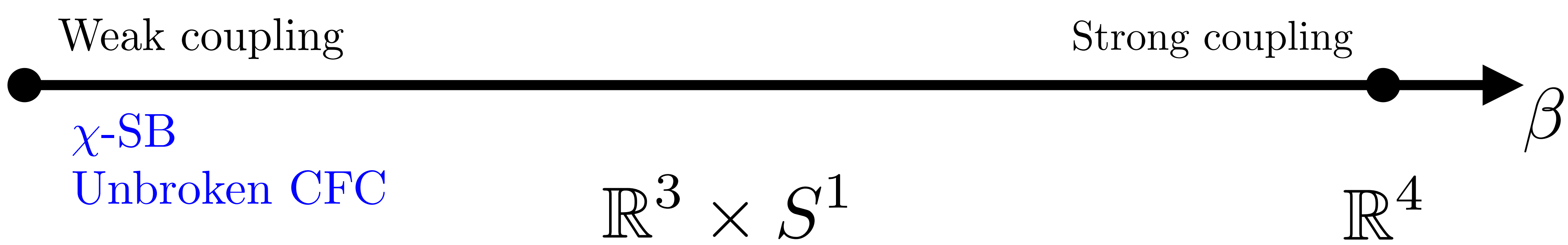


CFC unbroken, and chirally broken small and large circle regime. The ground state is maximal torus of chiral Lagrangian.

Persistent mixed anomaly

A mixed 't Hooft anomaly that exists on R4 persists upon compactification if and only if the flavor twisted b.c. are used.

Adiabatic continuity?



Persistent mixed anomaly: $\frac{2\mathrm{lcm}(N_f, N_c)}{2\pi} \int B^{(2)} \wedge B^{(1)}$

Different phases are consistent with the mixed anomaly.

Chiral Lagrangian in flavor holonomy background

$$S_{\Omega_F} = \int_{\mathbb{R}^3 \times S^1} \left[\frac{f_\pi^2}{4} \operatorname{tr} |D_\mu \Sigma|^2 \right]$$

Only Nf-1 meson remain exactly gapless at large circle.

From microscopic point of view, there are exactly Nf-1 dual photons in small circle theory that remain gapless.

$$\Sigma(x) = \begin{bmatrix} e^{i\alpha_1 \cdot \sigma} & 0 & & \\ 0 & e^{i\alpha_2 \cdot \sigma} & & \\ & & \ddots & \\ & & & e^{i\alpha_{N_f} \cdot \sigma} \end{bmatrix}$$

In the Nf=Nc chirally broken vacuum, we can also calculate fermion bilinear.

$$\langle \psi_L^i \psi_{Rj} \rangle \sim \delta_j^i \beta^{-3} e^{-S_0} e^{i\delta_i} = \delta_j^i \Lambda^3 e^{i\delta_i}$$

using the one loop beta function of Nf=Nc QCD. (remarkably, this is the same mechanism with N=1 SYM (Hollowood, Khoze,Davies,Mattis 99) , but after going through many intermediate steps!