Continuous Renormalization Group on the Lattice

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WORK DONE IN COLLABORATION WITH

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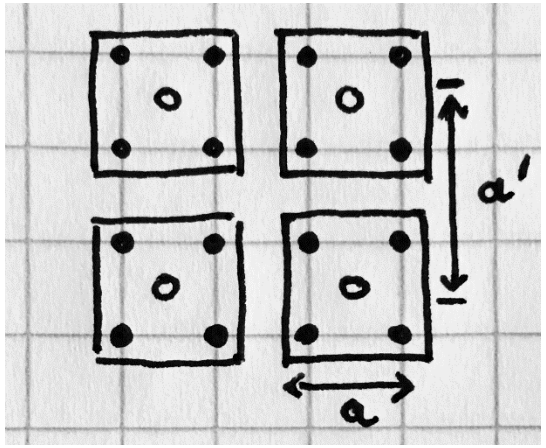
Block-Spin RG

Given initial spins φ, with "bare action" $S_0(\varphi)$, define blocked spins by a local average

$$\varphi_b(x_b) = \frac{b^{\Delta_\phi}}{b^d} \sum_{\varepsilon} \varphi(x+\varepsilon)$$



The lattice spacing (inverse cutoff) changes as

$$a' = ba$$

The blocking transformation gives rise to an *effective* (or "blocked") action $S_b(\phi)$ via

$$\int_{\varphi} \mathrm{e}^{-S_0(\varphi)} = \int_{\varphi} \int_{\phi} \delta(\phi - \varphi_b) \mathrm{e}^{-S_0(\varphi)} = \int_{\phi} \mathrm{e}^{-S_b(\phi)}$$

Iterate the blocking transformation: yields a sequence in the *space of actions*

$$S_0 \to S_1 \to S_2 \to \cdots \to S_n$$

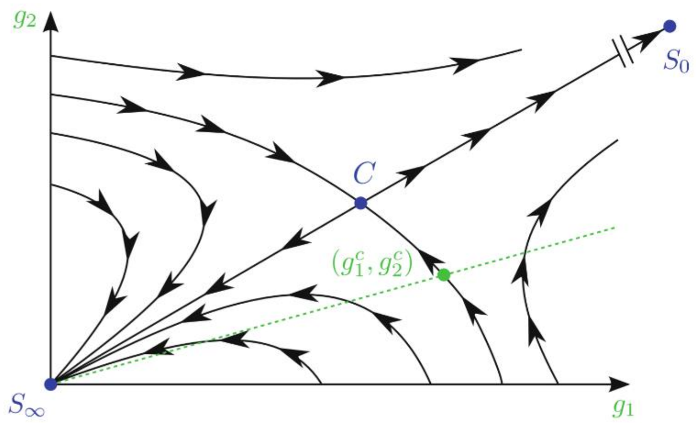
Monte Carlo RG: effective observables can be computed from blocked observables of the bare theory

$$\langle O(\phi) \rangle_{S_b} = \langle O(\varphi_b) \rangle_{S_0}$$

RG Eigenvalues and Scaling Operators

Deviations from a fixed point action behave in a simple way under RG

$$S_* + \sum_a u_a \mathcal{R}_a \longrightarrow S_* + \sum_a b^{y_a} u_a \mathcal{R}_a$$



The operators $\mathcal{R}_a$ are called *scaling operators*, and the u_a are *scaling variables*

The RG eigenvalues y_a determine the importance of $\mathcal{R}_a$ under RG iterations:

- $y_a > 0$ is *relevant*: deformation grows with further iterations
- $y_a = 0$ is *marginal*: constitutes an equivalent fixed point
- $y_a < 0$ is i*rrelevant*: deformation decays with iterations

Scaling operators are generally linear combinations of the familiar monomial operators we use to define our theory, e.g ϕ^2, ϕ^4, $p^2\phi^4$

$$\mathcal{R}_a(\phi) = \sum_k c_{ak} S_k(\phi)$$

Near an RG fixed point, correlators of scaling operators obey simple scaling laws ($z_b = z/b$)

$$\langle \mathcal{R}_a(z_b) \mathcal{R}_a(0) \rangle_{S_b} = b^{2\Delta_a} \langle \mathcal{R}_a(z) \mathcal{R}_a(0) \rangle_{S_0}$$

$$\Delta_a = d - y_a$$

Gradient Flow (GF)

In lattice theory, a tool called "gradient flow" was introduced in 2006/2009, as a smoothing transformation of the fields. The flow is defined by a diffusion-type equation,

$$\partial_t \phi_t(x) = -\frac{\delta \hat{S}(\phi)}{\delta \phi(x)}\Big|_{\phi=\phi_t}$$

where $\hat{S}(\phi)$ is the "flow action," e.g., ϕ^4 for scalars, Yang-Mills for gauge fields, etc..

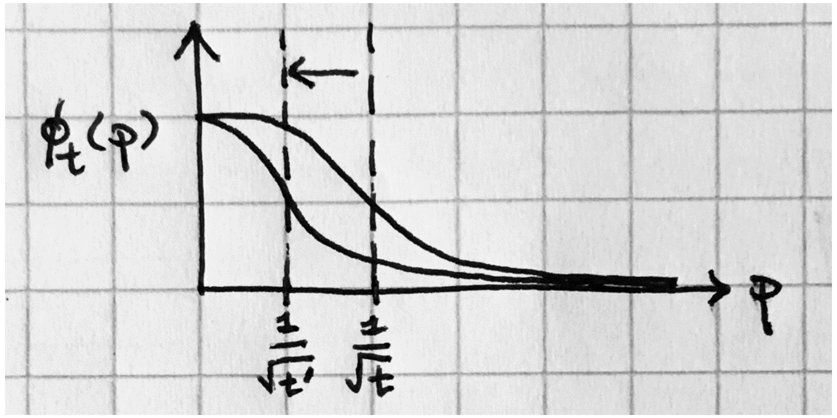
GF has had numerous applications in lattice theory, but we will focus on its smoothing property

The smoothing property of GF damps short-distance fluctuations of the field, much like the blocking transformations we saw earlier

Example: massless free flow is a heat equation,

$$\partial_t \phi_t(x) = \Delta \phi_t(x), \quad \phi_0(x) = \varphi(x)$$

$$\phi_t(x) = \int \mathrm{d}^d z \frac{\mathrm{e}^{-z^2/4t}}{(4\pi t)^{d/2}} \, \varphi(x+z)$$



Gradient Flow Renormalization Group (GFRG)

Let us compare the GF solution to the block-spin definition:

$$\phi_t(x) = \int \mathrm{d}^d z \, \frac{\mathrm{e}^{-z^2/4t}}{(4\pi t)^{d/2}} \, \varphi(x+z) \qquad \text{vs.} \qquad \varphi_b(x_b) = \frac{b^{\Delta_\phi}}{b^d} \sum_\varepsilon \varphi(x+\varepsilon)$$

We see that the quantity $\sqrt{t}$ plays the role of the blocking factor b by determining the mean-squared radius of the heat kernel

$$\langle z^2 \rangle = 2td$$

However, there is no analog of the field rescaling factor b^{Δ_ϕ}, which is necessary for the transformation to have a fixed point, generally

So, we *define* the GFRG transformation by

$$\Phi_t(x_t) = b_t^{\Delta_\phi} \phi_t(x)$$

GFRG observables are then related to bare observables by

$$\langle \mathcal{O}(\Phi_t) \rangle_{S_t} = \langle \mathcal{O}(b_t^{\Delta_\phi} \phi_t) \rangle_{S_0}$$

Ratio Formulas

Assuming that GFRG is a valid "blocking" transformation, we expect the correlator scaling laws to hold

$$\langle \mathcal{R}_a(\Phi_{t'}; z_{t'}) \mathcal{R}_a(\Phi_{t'}; 0) \rangle_{S_{t'}} = (b_{t'}/b_t)^{2\Delta_a} \langle \mathcal{R}_a(\Phi_t; z_t) \mathcal{R}_a(\Phi_t; 0) \rangle_{S_t}$$

Typical monomial operators are linear combinations of the scaling ops,

$$\mathcal{O}_i(\phi) = \sum_a \tilde{c}_{ia} \mathcal{R}_a(\phi)$$

Ratios of correlations of such operators are then dominated at large distances by the *leading* scaling operator (in a given symmetry subspace, e.g., even or odd),

$$\frac{\langle \mathcal{O}_i(\phi_{t'}; z) \mathcal{O}_j(\phi_{t'}; 0) \rangle_{S_0}}{\langle \mathcal{O}_i(\phi_t; z) \mathcal{O}_j(\phi_t; 0) \rangle_{S_0}} = (b_{t'}/b_t)^{2\Delta_a - (n_i + n_j)\Delta_\phi}$$

The LHS can be measured directly in a lattice simulation!

If we know the form of b_t, we can then extract scaling dimension differences

$$\delta_{ij} = 2\Delta_a - (n_i + n_j)\Delta_\phi$$

From any *pair* of such ratios, we obtain estimates for Δ_a , Δ_ϕ

Application: ϕ^4 theory in 3 dimensions*

*Preliminary results were presented in *Carosso, et al., 2018, Carosso, et al., 2019*

We performed a lattice simulation with bare action

$$S(\varphi) = \sum_x \Big[-\beta \sum_\mu \varphi(x)\varphi(x+\mu) + \varphi^2(x) + \lambda(\varphi^2(x)-1)^2 \Big]$$

using standard Markov Chain Monte Carlo methods

System must be tuned to reach the WFFP

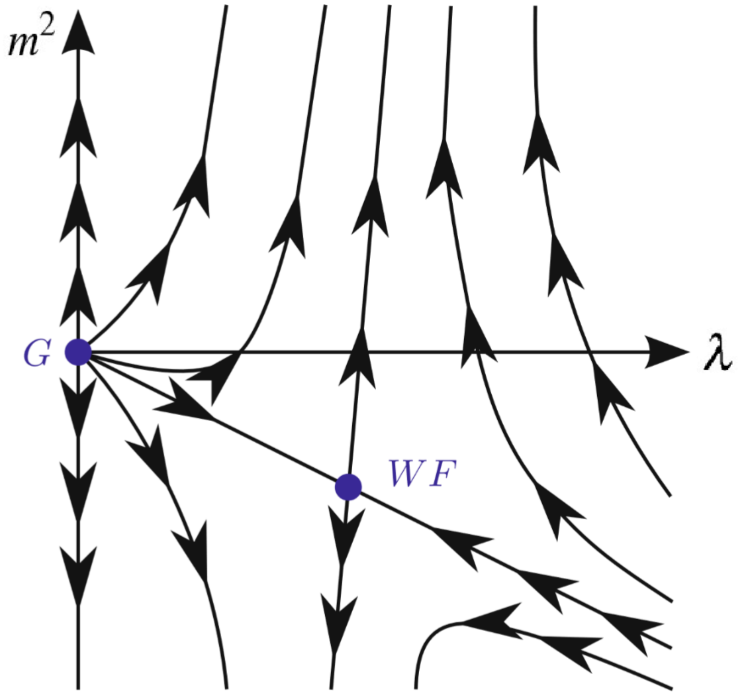
At fixed λ, the critical β values are well-known from separate studies *(Hasenbusch, 1999).* At $\lambda = 1.1$,

$$\beta_c \approx 0.3750966$$

We checked that the system was well-tuned by analyzing the Binder Cumulant (see backups)

Δ	$d=2$	$d=3$
Δ_1	0.125	0.51790(20)
Δ_2	1	1.41169(76)
Δ_3	2.125	2.51790(20)
Δ_4	2	3.845(11)

Exact scaling dimensions of the 2d Ising model, and the most precise lattice determinations of scaling dimensions in 3d phi4 theory, from *Hasenbusch, 1999*.



Adapted from Kopietz *et al.*, *Introduction to the Functional Renormalization Group (Springer 2010)*

Power Law Correlators

We measure mixed correlation functions among operators in the even and odd symmetry subspaces,

$$\{\phi, \phi^3\}, \quad \{\phi^2, \phi^4\}$$

Critical correlators are expected to be power law-like

$$C(z) = \frac{A}{z^{2\Delta}}$$

We detect clear power law signals

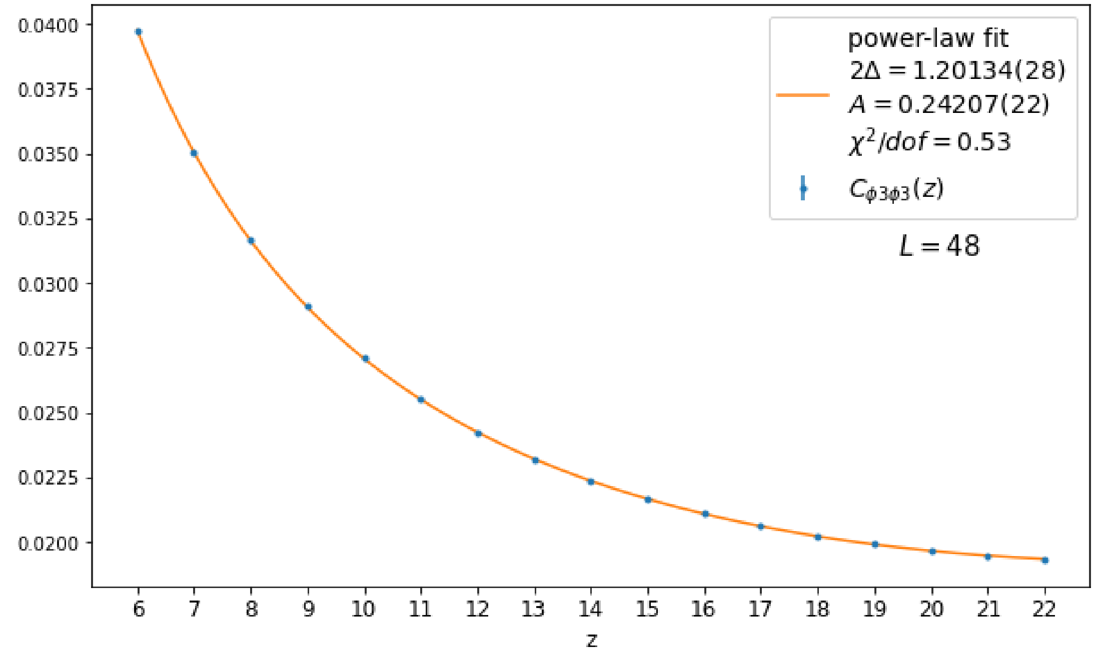
The correlations are dominated by the leading scaling dimensions in each subspace

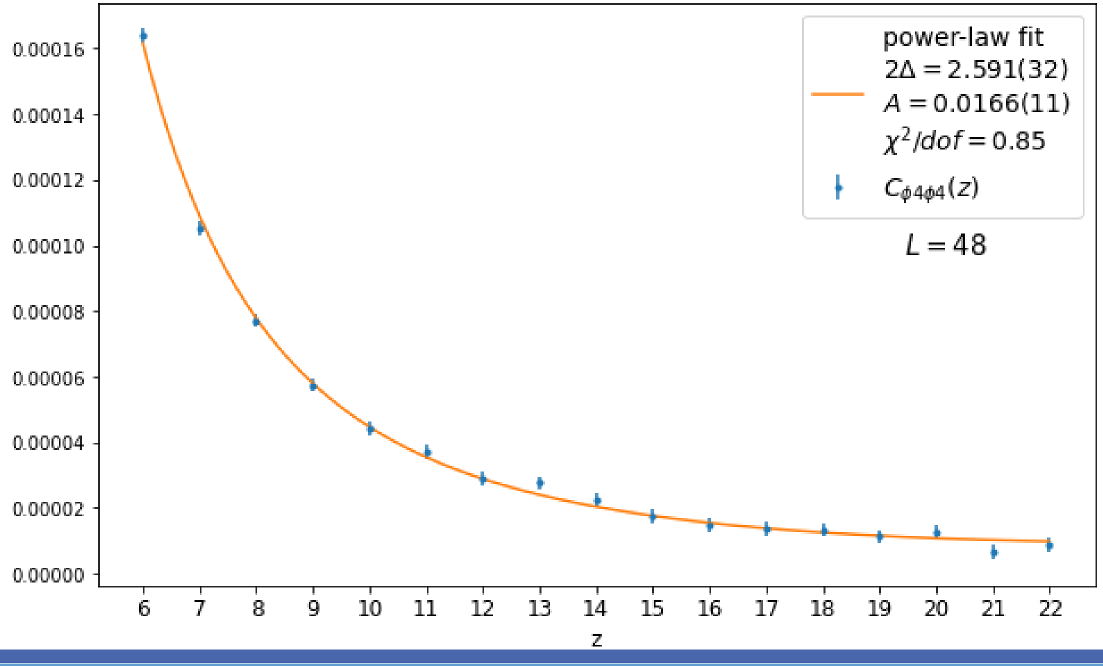
We expect to be able to apply the ratio formula

$$\frac{\langle \mathcal{O}_i(\phi_{t'}; z)\mathcal{O}_j(\phi_{t'}; 0)\rangle_{S_0}}{\langle \mathcal{O}_i(\phi_t; z)\mathcal{O}_j(\phi_t; 0)\rangle_{S_0}} = (b_{t'}/b_t)^{2\Delta_a-(n_i+n_j)\Delta_\phi}$$

in the odd and even subspaces separately, to extract

$$\delta_{ij} = 2\Delta_a - (n_i + n_j)\Delta_\phi$$





Ratios

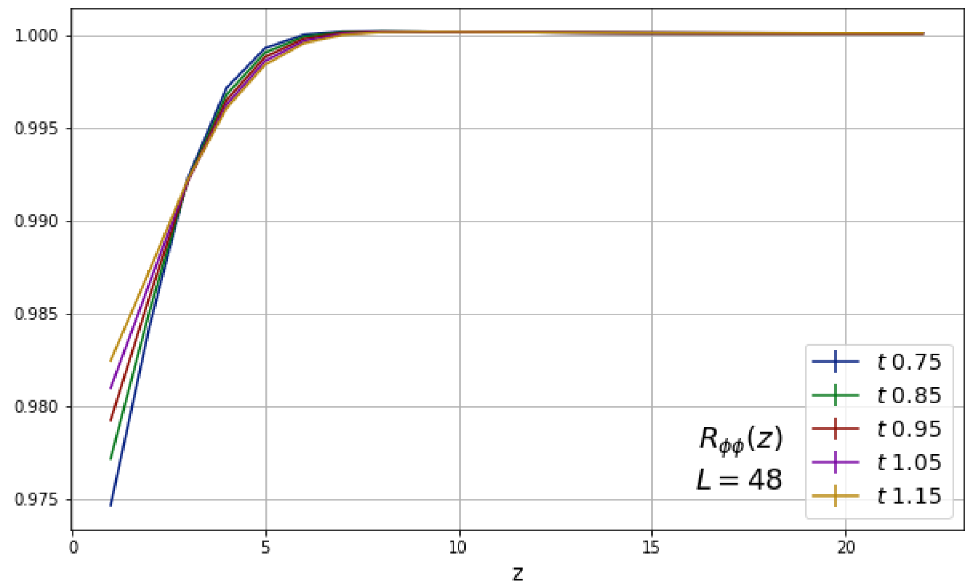
Plateaus in ratios at large distances are expected

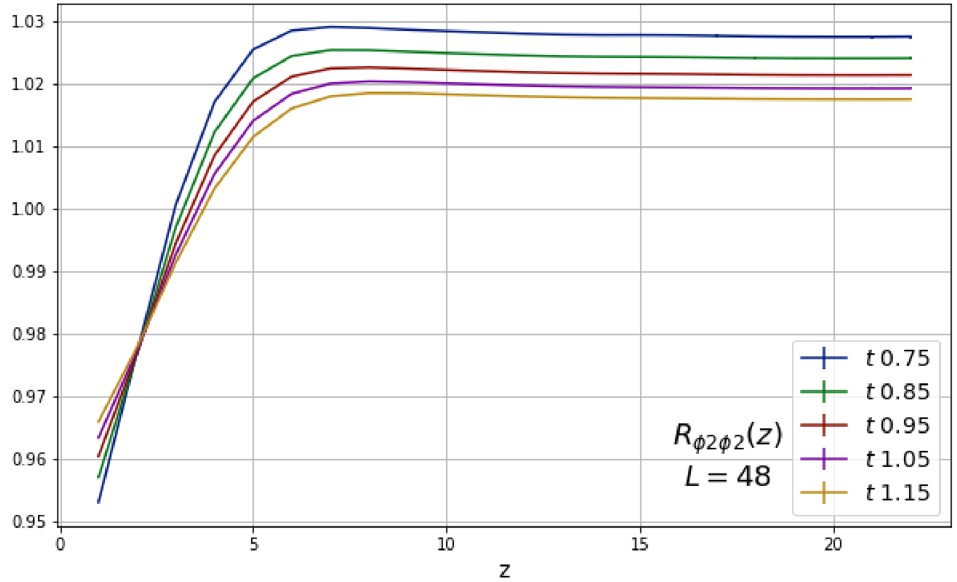
$$R_{\phi^i\phi^j}(t) = \frac{\langle \phi^i_{t+\epsilon}(z)\phi^j_{t+\epsilon}(0)\rangle}{\langle \phi^i_t(z)\phi^j_t(0)\rangle} \sim b_\epsilon(t)^{\delta_{ij}}$$

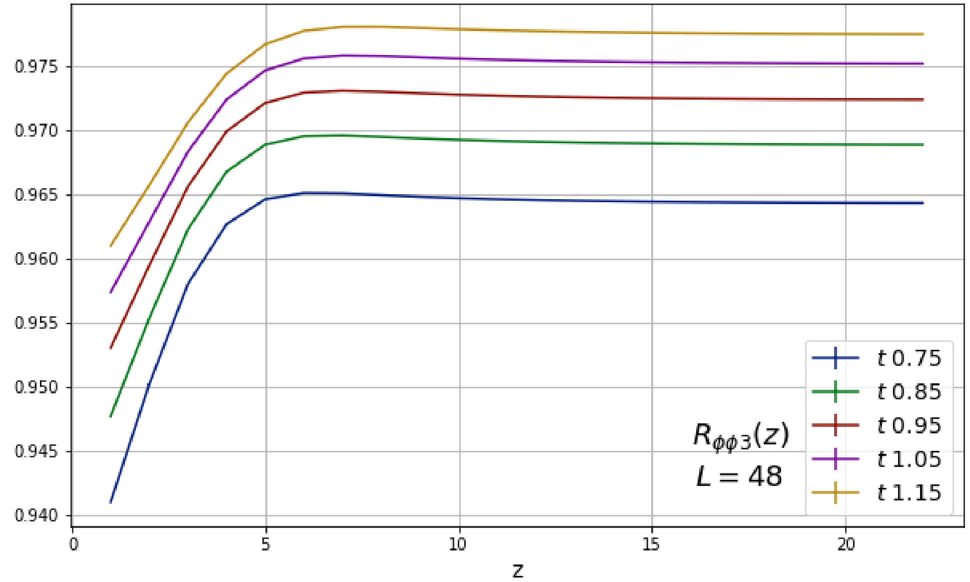
We take $\epsilon = 0.05$ in the rest of our analysis

Right: $\langle\phi\phi\rangle$ and $\langle\phi\phi^3\rangle$ ratios

Below: $\langle\phi^2\phi^2\rangle$ ratios







Analysis of the Ratios

We fit the ratios to an ansatz for the GFRG scale factor

$$R_{\phi^i\phi^j}(t) \sim b_\epsilon(t)^{\delta_{ij}} = \left(1 + \frac{\epsilon}{c^{-1} + t}\right)^{\delta_{ij}/2}$$

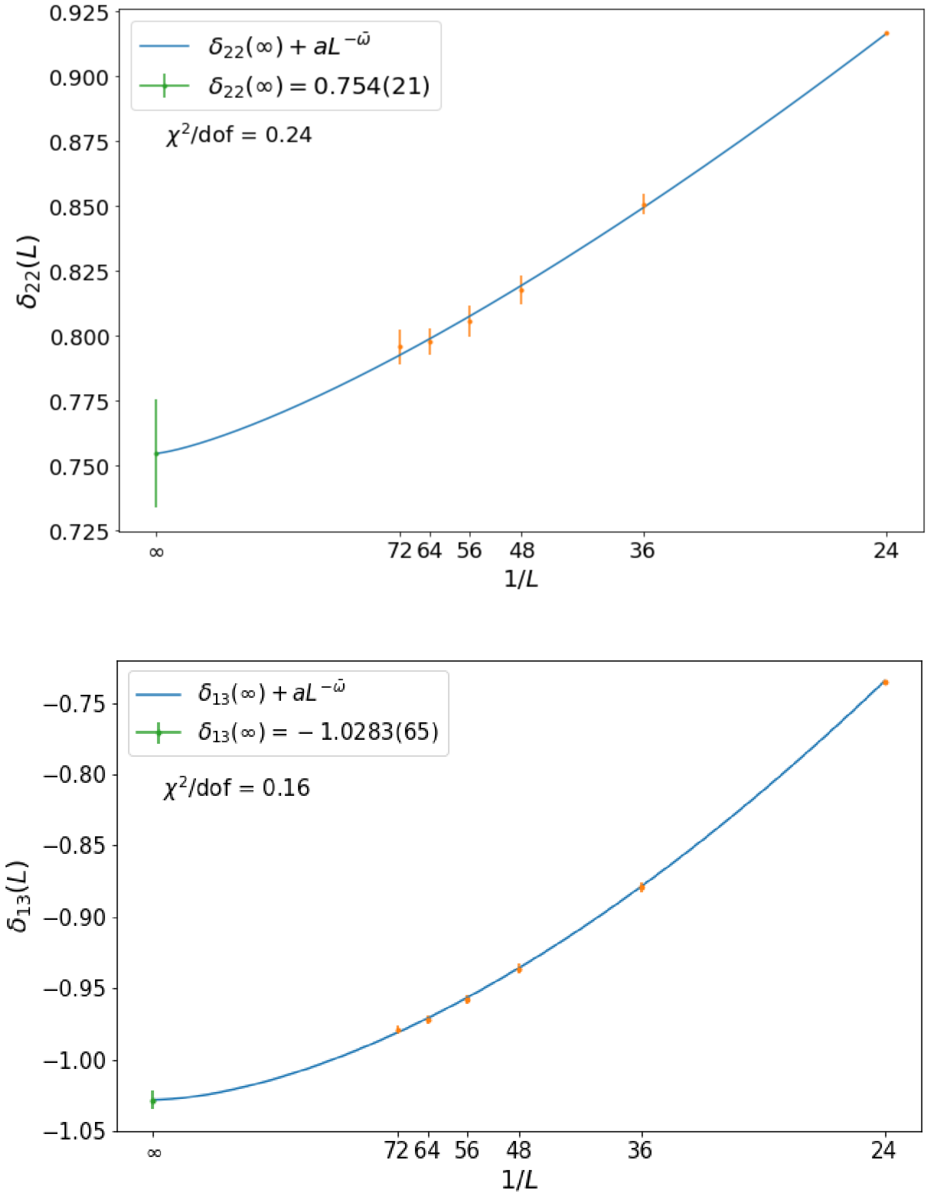
where $b_\epsilon(t) = b_{t+\epsilon}/b_t$ is the *relative* scale factor, and we assume

$$b_t = \sqrt{1 + ct}$$

which is the simplest ansatz with the correct limiting behavior

The δ_{ij} obtained in this way exhibit a notable volume dependence. We extrapolate to infinite volume via

$$\delta_{ij}(L) = \delta_{ij}(\infty) + aL^{-\bar{\omega}}$$



Results for Leading Scaling Dimensions

(i,j)	δ_{ij}(known)	δ_{ij}	$\bar{\omega}$	a	χ^2/dof
$(1,3)$	-1.03580(40)	-1.0283(65)	1.667(80)	59(14)	0.16
$(3,3)$	-2.07160(80)	-2.055(18)	1.73(11)	154(51)	0.30
$(2,2)$	0.7518(17)	0.743(17)	1.17(18)	7.2(3.5)	0.17
$(2,4)$	-0.2840(19)	-0.308(38)	0.96(20)	5.8(2.9)	0.11
$(4,4)$	-1.3198(22)	-1.310(29)	1.56(16)	84(38)	0.20

Infinite volume extrapolations of the exponents according to a corrections-to-scaling ansatz. Corresponding exponents from the precisely known lattice values are tabulated for comparison *(Hasenbusch, 1999).*

$(i,j),(h,k)$	Δ_1	Δ_2
$(1,3)$	0.5141(32)	—
$(3,3)$	0.5138(45)	—
$(2,2),(2,4)$	0.525(21)	1.422(46)
$(2,2),(4,4)$	0.5132(84)	1.398(22)
$(2,4),(4,4)$	0.501(24)	1.349(88)
known	0.51790(20)	1.41169(76)

Scaling dimensions obtained from pairs of the exponent estimates above.

Subleading Scaling Dimensions: Diagonalization

If the subleading operator is significant, or if you want to extract the subleading dimensions, a different approach is needed. Recall that scaling operators satisfy

$$\langle \mathcal{R}_a(\Phi_{t'}; z_{t'})\mathcal{R}_a(\Phi_{t'}; 0)\rangle_{S_{t'}} = (b_{t'}/b_t)^{2\Delta_a}\langle \mathcal{R}_a(\Phi_t; z_t)\mathcal{R}_a(\Phi_t; 0)\rangle_{S_t}$$

Writing the rescaled fields in terms of flowed fields, we obtain a ratio formula

$$\frac{\langle \mathcal{R}_a(b_{t'}^{\Delta_\phi}\phi_{t'}; z)\mathcal{R}_a(b_{t'}^{\Delta_\phi}\phi_{t'}; 0)\rangle_{S_0}}{\langle \mathcal{R}_a(b_{t'}^{\Delta_\phi}\phi_t; z)\mathcal{R}_a(b_{t'}^{\Delta_\phi}\phi_t; 0)\rangle_{S_0}} = (b_{t'}/b_t)^{2\Delta_a}$$

How can we measure these ratios? Conformal invariance at the F.P. implies

$$\langle \mathcal{R}_a\mathcal{R}_b\rangle \propto \delta_{ab}$$

We can construct estimates of the scaling operator ratios as follows:

- Measure the mixed operator correlations $\langle \mathcal{O}_i\mathcal{O}_j\rangle$
- Multiply by appropriate powers of $b_t^{\Delta_\phi}$
- Diagonalize the matrix of correlations numerically
- Form the ratios of the diagonalized correlators

Subleading Dimensions

For the subleading scaling operators, only relatively short distances can be used, where the signal was sufficient and plateaus were observed

(*Right*) Infinite volume extrapolations for Δ_1 and Δ_3 from ratios at distances $z \in [10, 14]$

- The Δ_3 value is consistent with the prediction that (*Rychkov, 2016)*

$$\Delta_3 = 2 + \Delta_1$$

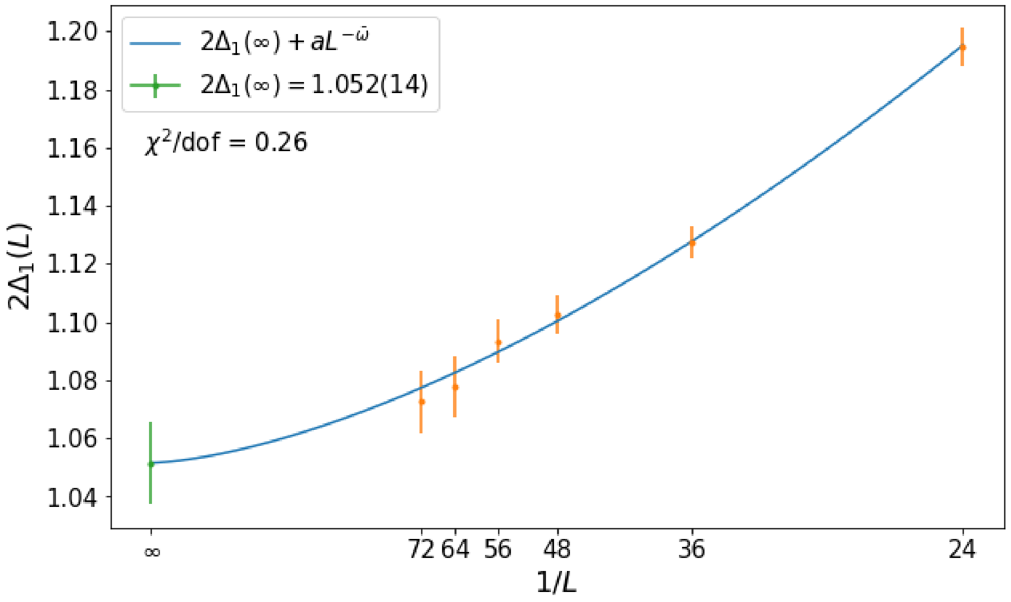
To compare with Hasenbusch values:

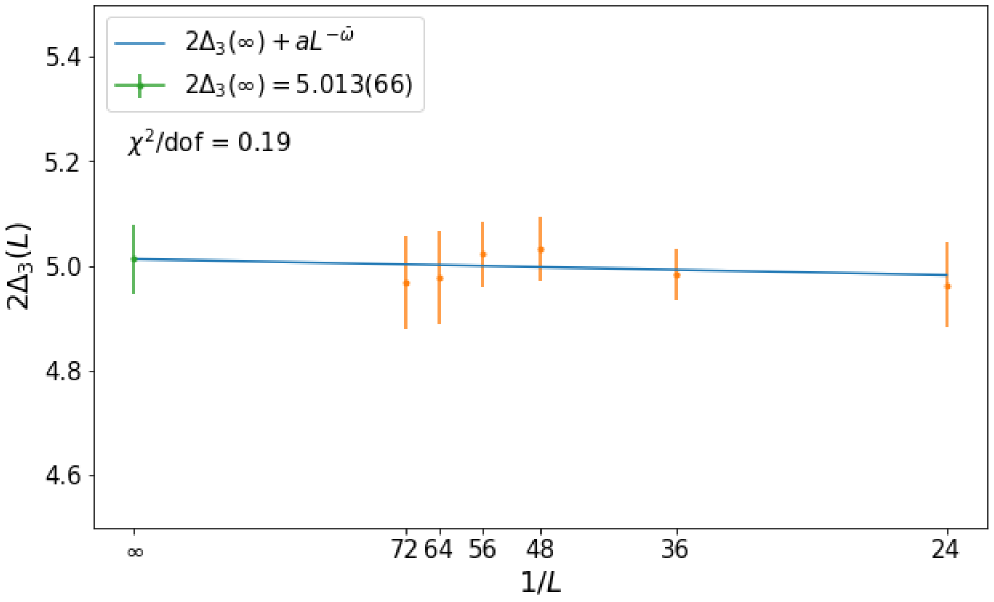
$$2\Delta_3 = 5.03580(40)$$

$$2\Delta_4 = 7.690(22)$$

For Δ_4, the data was insufficient to perform an infinite volume extrapolation, but we report the average across all volumes, for $z \in [10, 13]$:

$$2\Delta_2 = 2.927(27) \qquad 2\Delta_4 = 7.54(63)$$





Other Theories

Gauge theory in 4d: We also applied the method to 12-flavor SU(3) gauge theory in 4 dimensions, a theory related to QCD (*Carosso, et al., 2018)*. The 12-flavor theory differs from QCD in that it's expected to have an IRFP with nontrivial scaling dimensions

- We measured the fermion mass anomalous dimension in the system, finding consistency with other methods
- We also measured the baryon anomalous dimension in this system, constituting a *first* determination from the lattice

Phi4 in 2d: We have computed the δ_{ij}, reported in the table below

(i,j)	δ_{ij}(known)	δ_{ij}	$\bar{\omega}$	a	χ^2/dof
$(1,3)$	-0.25	-0.2616(14)	2.48(21)	127(83)	0.21
$(3,3)$	-0.50	-0.5279(28)	2.35(18)	161(91)	0.55
$(2,2)$	1.50	1.538(20)	1.92(35)	90(93)	0.35
$(2,4)$	1.25	1.299(31)	1.79(31)	103(93)	0.30
$(4,4)$	1.00	1.061(60)	1.62(25)	129(93)	0.39

Deviations may be due to strong subleading scaling operators, suggesting a full diagonalization analysis will be necessary for the leading exponents

GFRG Effective Action

We avoided defining an effective action $S_t(\phi)$ under GFRG by appealing to MCRG, but the question remains:

How can we properly define the GFRG effective action?

The analogy with the blocking transformation suggests

$$\mathrm{e}^{-S_b(\phi)} = \int_\varphi \delta(\phi - \varphi_b)\ \mathrm{e}^{-S_0(\varphi)} \quad \Rightarrow \quad \mathrm{e}^{-S_t(\phi)} = \int_\varphi \delta(\phi - f_t\varphi)\ \mathrm{e}^{-S_0(\varphi)}$$

where f_t is the heat kernel

But this is not sufficient; the integral can be performed exactly, yielding

$$S_t(\phi) = -\mathrm{tr}\ln f_t + S_0(f_t^{-1}\phi)$$

The flowing couplings in this action do not behave as expected; there are no loop contributions

We need a formalism for defining effective actions under continuous RG transformations…

Functional RG (FRG)

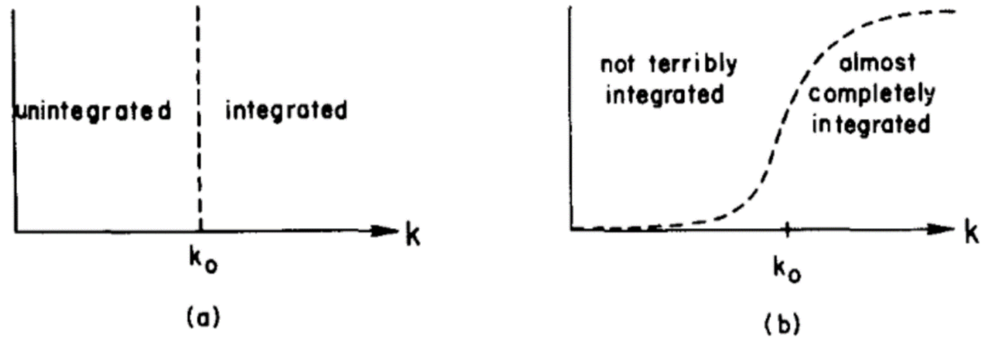
FRG: Define and track the evolution of low-mode actions under continuous RG transformations in continuum (or lattice) field theory (Wilson-Kogut 1973, Wegner-Houghton 1973)

Recall *sharp* high-mode elimination: ($\Lambda < \Lambda_0$)

$$\mathrm{e}^{-S_\Lambda(\phi_<)} = \int_{\phi_>} \mathrm{e}^{-S_{\Lambda_0}(\phi_< + \phi_>)}$$

Wilson & Kogut: Define the low-mode Boltzmann factor by

$$\mathrm{e}^{-S_t(\phi)} = \int_\varphi P_t(\phi, \varphi)\mathrm{e}^{-S_0(\varphi)}$$



Adapted from *Wilson & Kogut (1973)*

where the *constraint functional* is given by

$$P_t(\phi, \varphi) = N_t \exp\left[-\frac{1}{2}\int_p \omega(p) \frac{[\phi(p) - f_t(p)\varphi(p)]^2}{1 - f_t^2(p)} \right]$$

The constraint functional satisfies an “exact RG equation”

$$\frac{\partial P_t(\phi)}{\partial t} = \frac{1}{2}\int_p \left(K_0(p) \frac{\delta^2 P_t(\phi)}{\delta\phi(p)\delta\phi(-p)} + 2\omega(p)\phi(p)\frac{\delta P_t(\phi)}{\delta\phi(p)} \right)$$

… looks like a Fokker-Planck equation!

Stochastic RG*

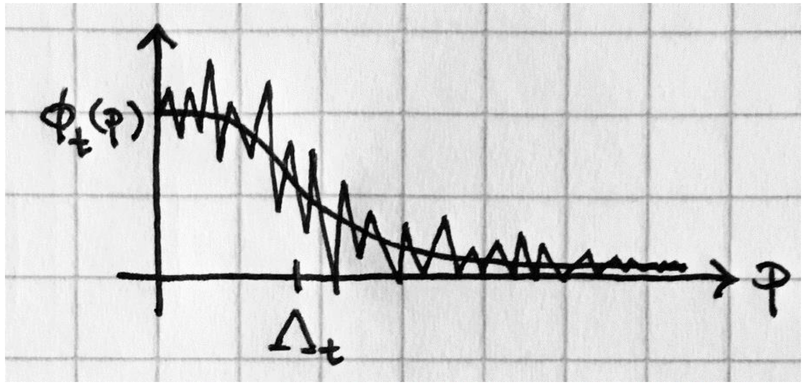
**Carosso, 2020*

Fokker-Planck equations are generated by Langevin equations (LE). Which one generates the FRG equation above? Consider:

$$\partial_t \phi_t(p) = -p^2 \phi_t(p) + \eta_t(p)$$

where the random noise η_t distributed according to

$$\mathrm{d}\mu_0(\eta) = C\exp\Big[-\frac{1}{2\Omega} \int_0^T \mathrm{d}t \; (\eta_t, K_0^{-1} \eta_t) \Big] \mathrm{D}\eta$$



The LE generates a probability distribution of fields ϕ_t at time *t*

$$P(\phi, t; \varphi, 0) = \mathbb{E}_{\mu_0} \big[\delta(\phi - \phi_t[\varphi; \eta]) \big]$$

One can compute the distribution explicitly

$$P(\phi, t; \varphi, 0) = N_t \exp\Big[-\frac{1}{2} (\phi - f_t \varphi, A_t^{-1} (\phi - f_t \varphi)) \Big]$$

Where the kernel is given by

$$A_t(p, k) = (2\pi)^d \delta(p + k) K_0(k) \frac{1 - f_t^2(k)}{2k^2}$$

The distribution has the same form as Wilson and Kogut's constraint functional!

Effective Action and IRFP

The effective action can be written *exactly* in terms of the bare theory's generator of connected Green functions as

$$S_t(\phi) = F_t + \frac{1}{2}(\phi, A_t^{-1}\phi) - W_0^{(t)}(A_t^{-1} f_t\phi)$$

This form allows one to compute terms in the flowing action. The vertices correctly implement high-mode loops!

For Schwinger regularization, $K_0(p) = \mathrm{e}^{-p^2/\Lambda_0^2}$, one can explicitly compute the scale factor of the RG transformation

$$b_t = \frac{\Lambda_0}{\Lambda_t} = \sqrt{1 + 2t\Lambda_0^2}$$

For ϕ^4 in 3d, we have checked that there is a Gaussian F.P. and a WFFP of the effective action when written in terms of the rescaled fields

$$\phi(p) = \Lambda_0^{d_\phi} b_t^{-\Delta_\phi} \Phi(\bar{p}) \qquad\qquad p = \Lambda_t \bar{p}$$

But how does this relate to the GFRG transformation we described earlier?

Stochastic MCRG and Gradient Flow

Observables of the effective theory can be written as double averages

$$\langle \mathcal{O}(\phi) \rangle_{S_t} = \langle \mathbb{E}_{\mu_0} \left[\mathcal{O}(\phi_t[\varphi;\eta]) \right] \rangle_{S_0}$$

This constitutes MCRG because they may be computed without knowledge of the effective action

Numerically implementable: generate an ensemble of bare fields with usual lattice Monte Carlo, and integrate the Langevin equation on every configuration

Since we know the solution of the Langevin equation, we can compute, for example

$$\langle \phi(x)\phi(y) \rangle_{S_t}^{\mathrm{conn}} = \langle f_t\varphi(x) f_t\varphi(y) \rangle_{S_0}^{\mathrm{conn}} + A_t(x-y)$$

$$\langle \phi(x_1)\cdots\phi(x_n) \rangle_{S_t}^{\mathrm{conn}} = \langle f_t\varphi(x_1)\cdots f_t\varphi(x_n) \rangle_{S_0}^{\mathrm{conn}}$$

The function $A_t(z)$ is exponentially suppressed for large separations $z \gg \Lambda_t^{-1}$

Composite operator correlations satisfy similar relations:

$$\langle \phi^2(x)\phi^2(y) \rangle_{S_t}^{\mathrm{conn}} = \langle (f_t\varphi)^2(x)(f_t\varphi)^2(y) \rangle_{S_0}^{\mathrm{conn}} + A_t(x-y)\langle f_t\varphi(x) f_t\varphi(y) \rangle^{\mathrm{conn}} + 2A_t(x-y)^2$$

Thus: GF correlations are the asymptotic limits of stochastic RG correlations!

One can also derive ratio formulas like those of GFRG

Summary

In this work we have presented a few new approaches to continuous RG transformations:

Gradient Flow RG

GF can be supplemented by a field rescaling to define an RG transformation

Ratio formulas allow for measurement of scaling dimensions of the fundamental field and composite operators on the lattice

Generally applicable, with lattice results in
- ϕ^4 in 2d, 3d
- 12-flavor SU(3) gauge theory in 4d

Stochastic RG

Use Langevin equations to define RG transformations

SRG implies GFRG for long-distance observables

SRG can be implemented on the lattice, constituting a new (continuous) approach to MCRG

…Thanks for listening!

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Generalization to Gauge-Fermion Systems

Define the RG transformations of the gauge and (staggered) fermion fields with the simplest diffusion equations that preserve their symmetry:

- Gauge fields evolve according to Wilson flow (*Lüscher, 2009*)

$$\partial_t U_\mu(x,t) = -g_0^2(\partial S_W[U])(x,t)U_\mu(x,t)$$

- Fermions evolve with a gauge-covariant heat equation

$$\partial_t \psi(x,t) = \Delta[U]\psi(x,t)$$

At long distances, flowed-correlators should exhibit RG scaling of the fixed point if the system is tuned towards criticality

Nf=12, SU(3) gauge theory is expected to be conformal or near-conformal, so the ratio formula should be applicable

The mass and pseudoscalar anomalous dimensions are related: $\gamma_m = -\gamma_{\mathrm{ps}}$

$$P_t(x) = \overline{\psi}_t(x)\varepsilon(x)\psi_t(x)$$

Can also try measuring the baryon anomalous dimension, γ_N

$$B_t(x) = \epsilon_{abc}\psi_t^a(x)\psi_t^b(x)\psi_t^c(x)$$

Super Ratios

An issue with the ratio formula is that it includes the (usually unknown) anomalous dimension of the fundamental field, e.g.

$$R_P(t) \propto t^{\gamma_{\rm ps}-2\gamma_\psi}$$

And we cannot measure γ_ψ directly from the ratio $R_\psi(t)$

Note: if an operator *A* has no anomalous dimension, then its ratio formula is

$$R_A(t) \propto t^{-n_A\gamma_\psi}$$

This could be used to measure γ_ψ, or to cancel it's effect in another ratio. Thus we may form the *super-ratio*

$$R_P(t)R_A(t)^{-n_P/n_A} \propto t^{\gamma_{\rm ps}}$$

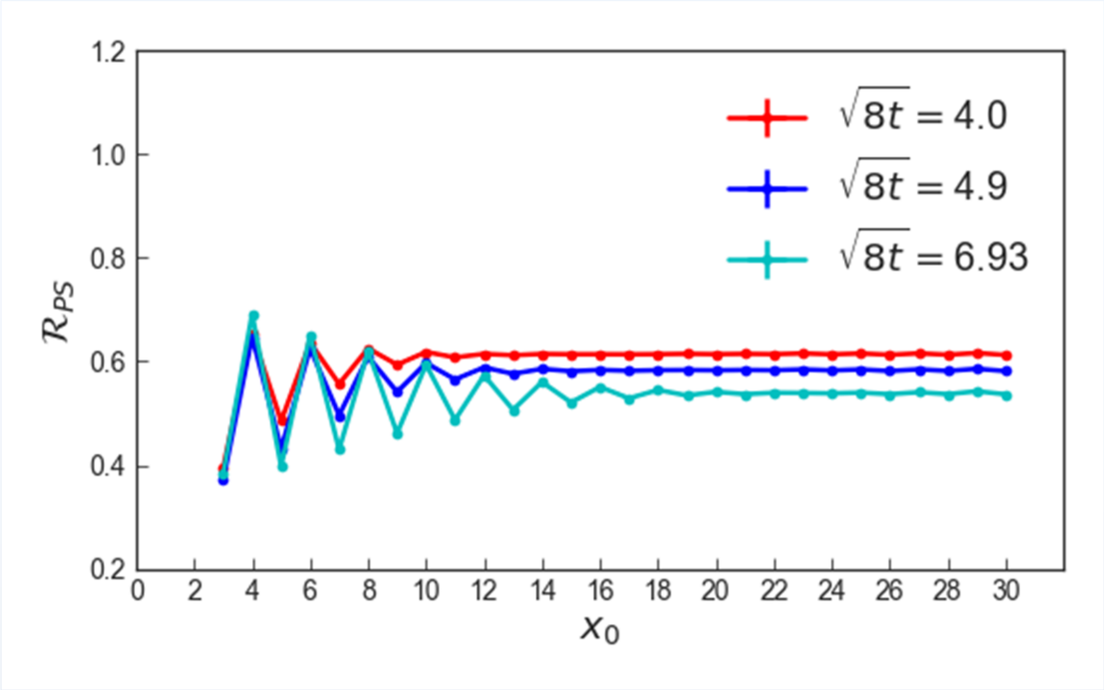
We choose the axial vector A_4 as our conserved operator

Pseudoscalar Ratios

$$R_P(t) \propto t^{\gamma_{\text{ps}} - 2\gamma_\psi}$$

The ratios of P-P correlators exhibit the expected plateaus at large distance

Short-distance smearing effects oscillate due to averaging nearby staggered fermions



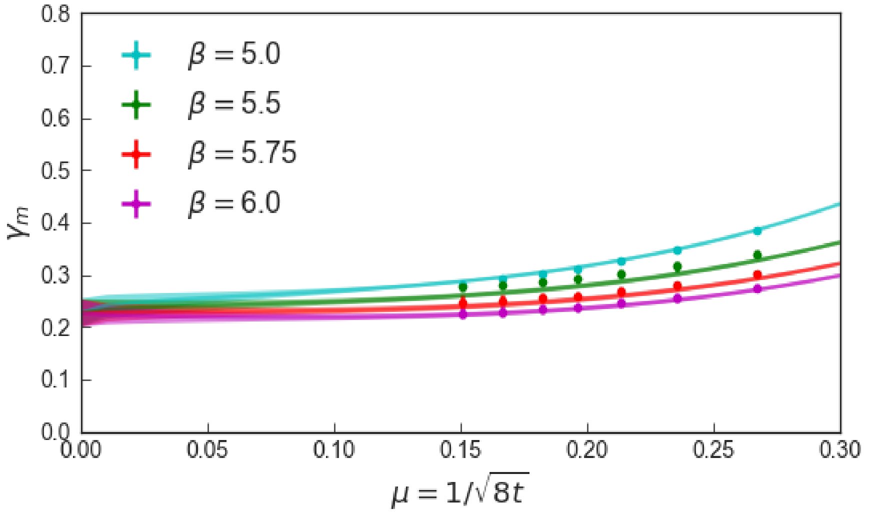
Carosso, et al., *Phys. Rev. Lett.* ***121***, *201601*

Anomalous Dimensions

Infinite volume, infinite time extrapolation yields

$$\gamma_m = 0.23(6)$$

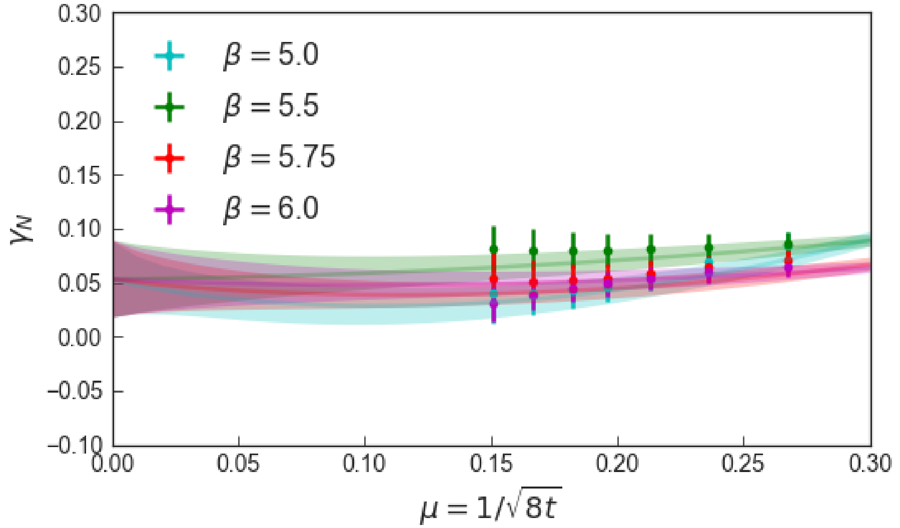
Consistent with several previous studies, both lattice and perturbative



Extrapolation of the nucleon anomalous dimension

$$\gamma_N = 0.05(5)$$

First non-perturbative prediction of γ_N for this system!



Carosso, et al., *Phys. Rev. Lett.* ***121***, *201601*

Tuning

Binder Cumulant

$$U_4 = 1 - \frac{\langle M^4 \rangle}{3\langle M^2 \rangle^2}$$

extrapolates to a universal value as

$$U_4 = U_4^* + c_1(\lambda) L^{-\omega}$$

Hasenbusch found that $c_1(\lambda)$ was smallest at $\lambda = 1.1$

He estimated the critical value (in 3d)

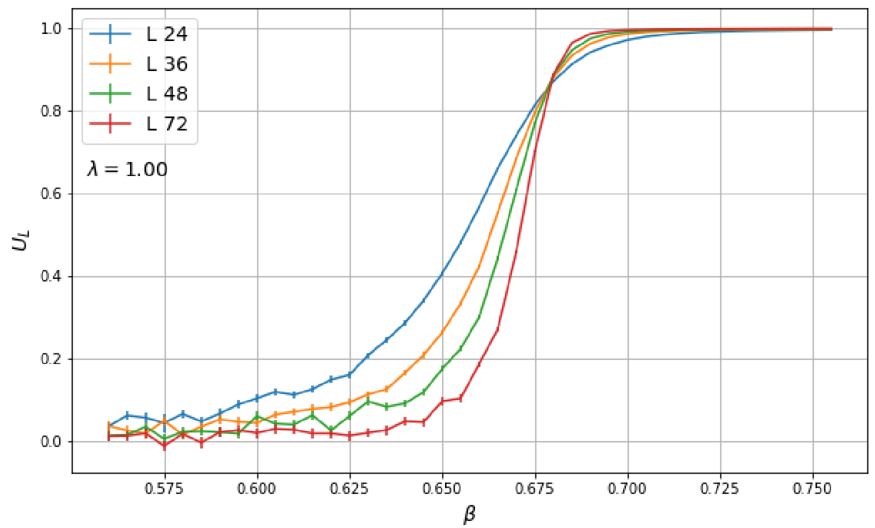
$$U_4^* = 0.69819(12)$$

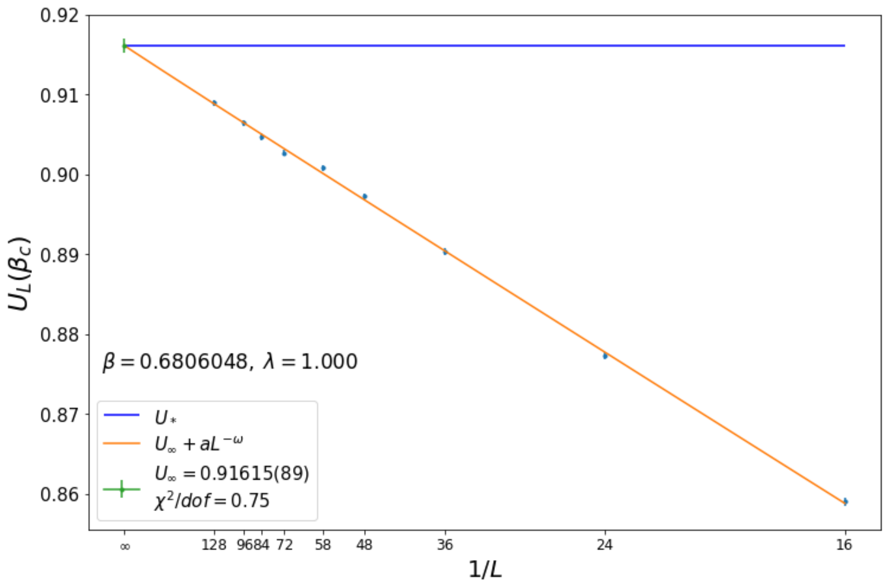
and at $\lambda = 1.1$, $\beta = 0.3750966$

Right: Binder Cumulant in 2d

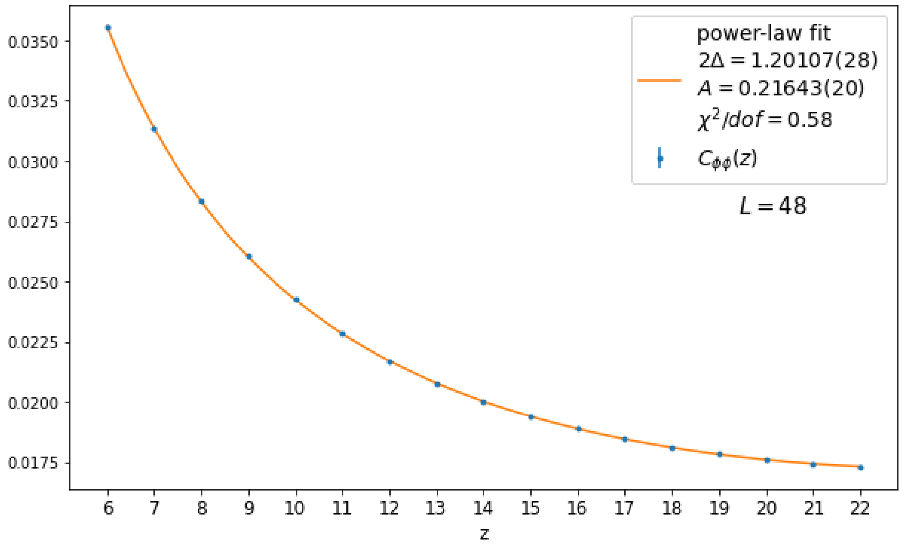
My extrapolations:

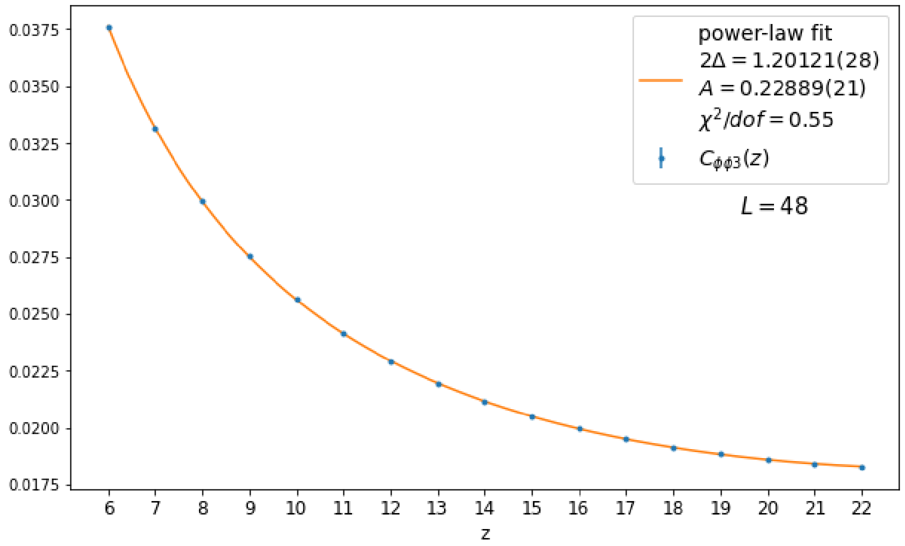
d	U_∞	ω	a	χ^2/dof
2	0.91615(89)	0.989(38)	-0.890(86)	0.75
3	0.6971(20)	0.845(10)	0.036(44)	0.43

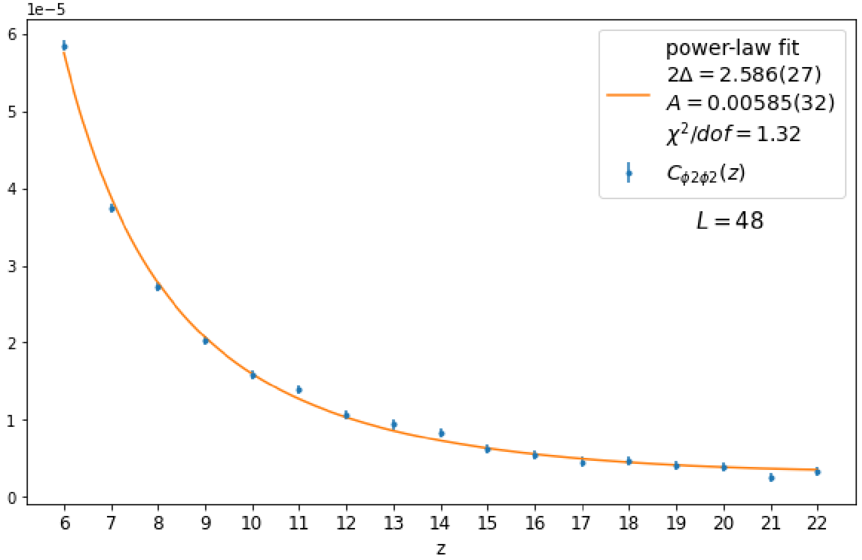


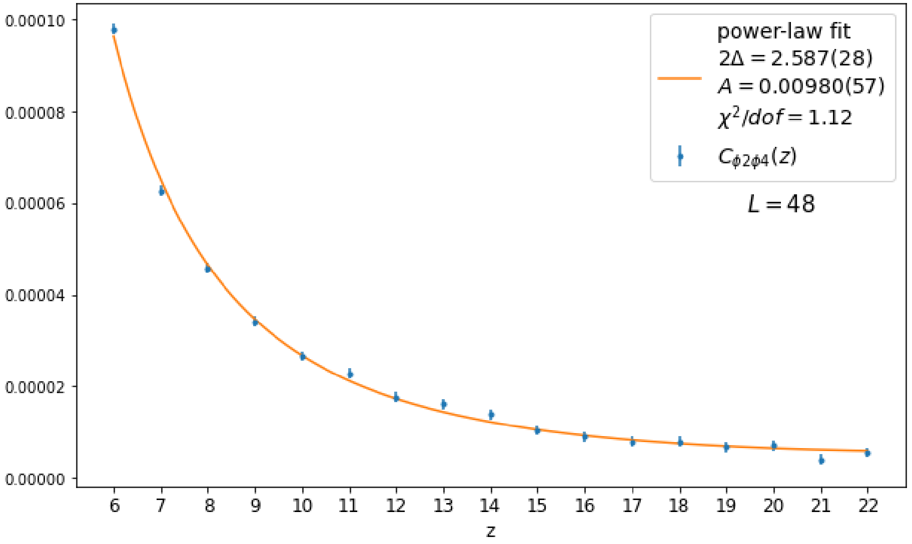


More Power Law Fits









MCMC Simulation Details

1 MC sweep = 50 Wolff cluster updates + 5 Metropolis radial updates. 10M MC sweeps per volume (largest two vols: 1.5M)

We conservatively took 10k sweeps as thermalization cut

Binning analysis to estimate true errors. Autocorrelations estimated with

$$\tau_{\text{int}} = \epsilon_*^2 / 2\epsilon_0^2$$

Autocorrelations in the range 4 – 6 sweeps

Measurements performed every 5 sweeps

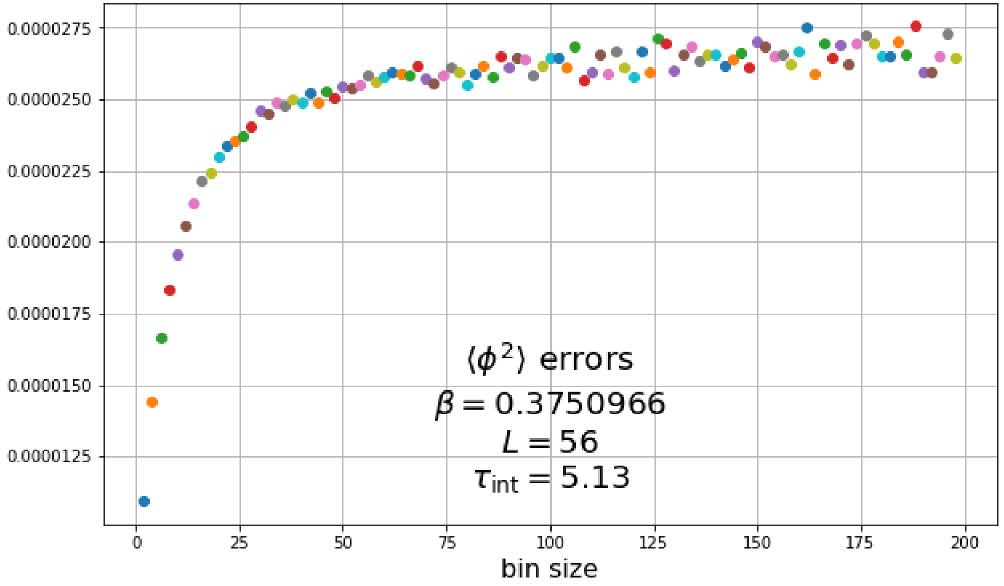
Measurements binned by 10

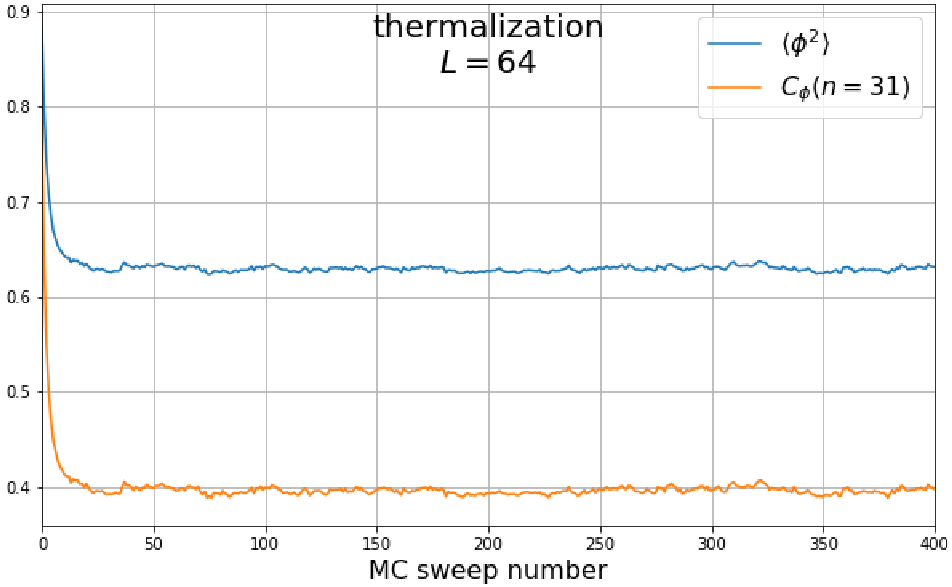
About 200k indep. samples per volume

Volumes: *L = 24, 36, 48, 56, 64, 72*

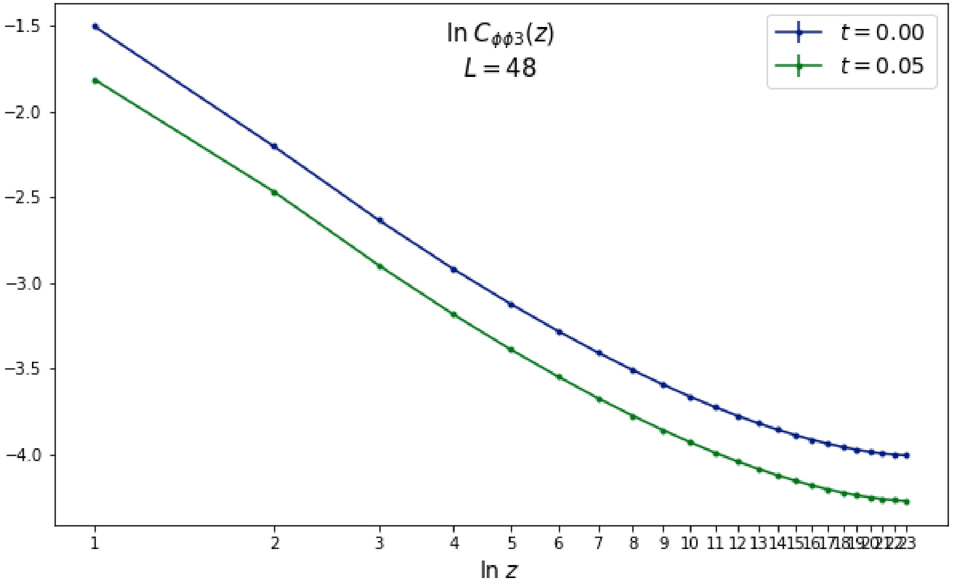
GF integrated with 4$^{\text{th}}$ order Runge-Kutta

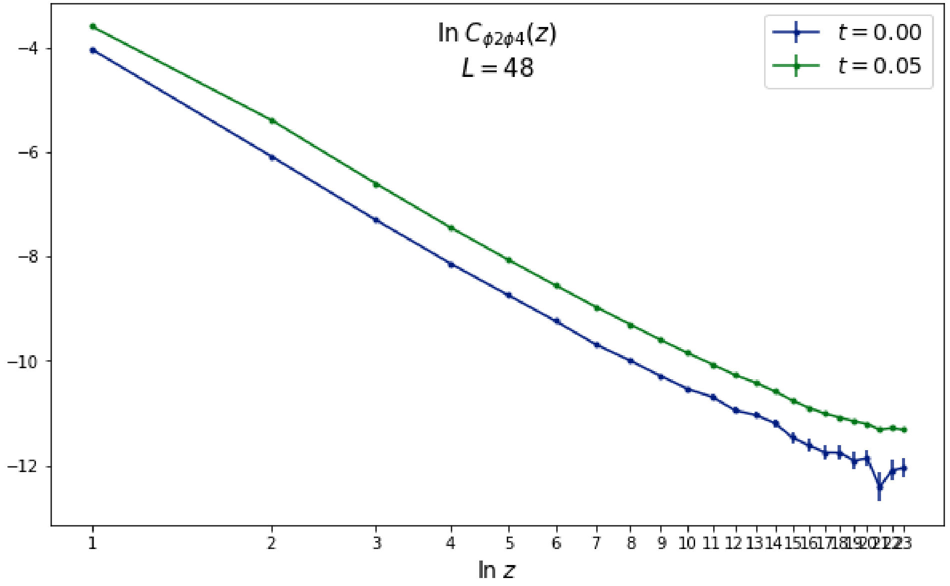
Similar story in 2d, but 1/5 the statistics (so far)





Log-Log Plots





Correlators at large distances are noisy, but GF smooths out much of the noise; noise is much larger in the even operator subspace

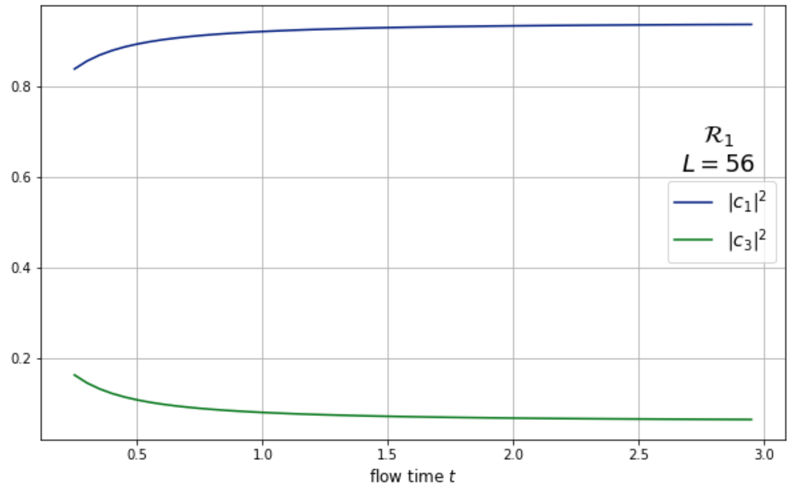
Log-Log plots indicate clear power law behavior, with modifications at large distance due to finite volume and subleading operators

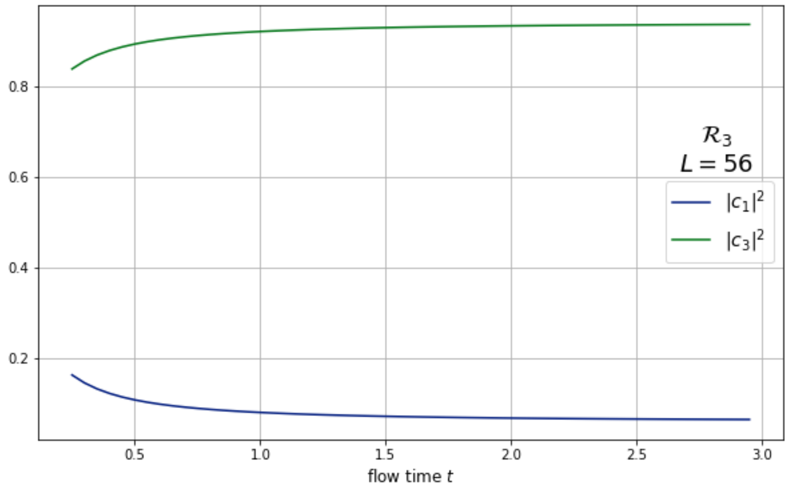
Diagonalization Coefficients

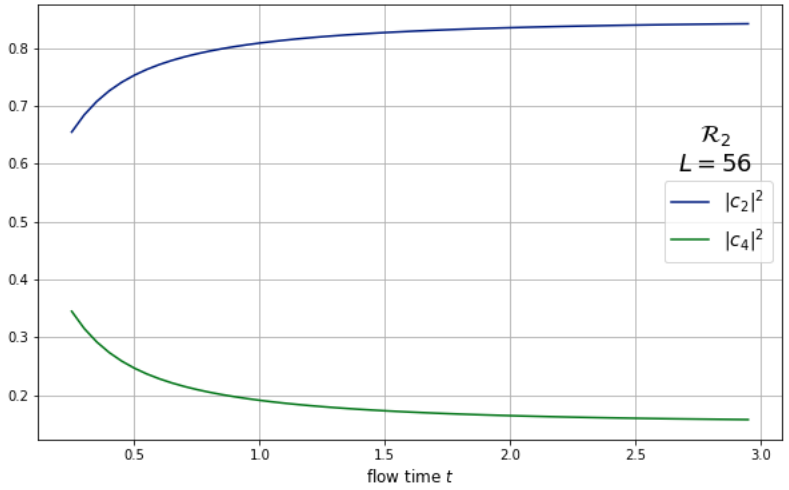
- Coefficients of monomial operators in scaling operators

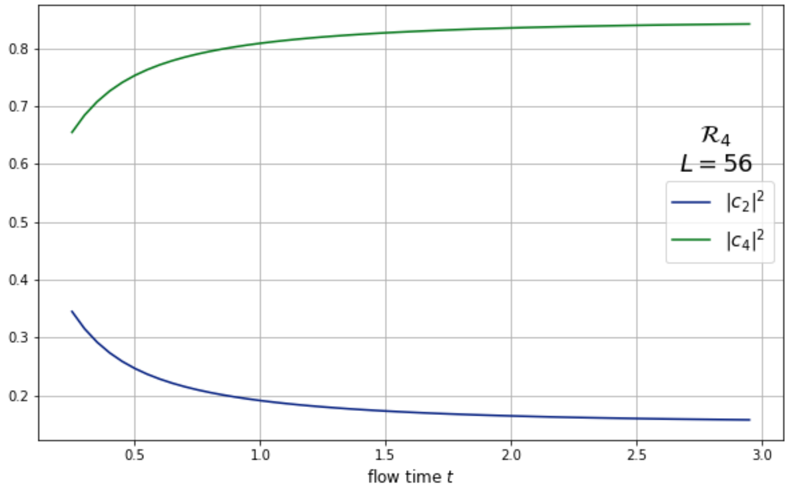
$$\mathcal{R}_1(\phi) = c_1\phi + c_3\phi^3$$

$$\mathcal{R}_2(\phi) = c_2\phi^2 + c_4\phi^4$$

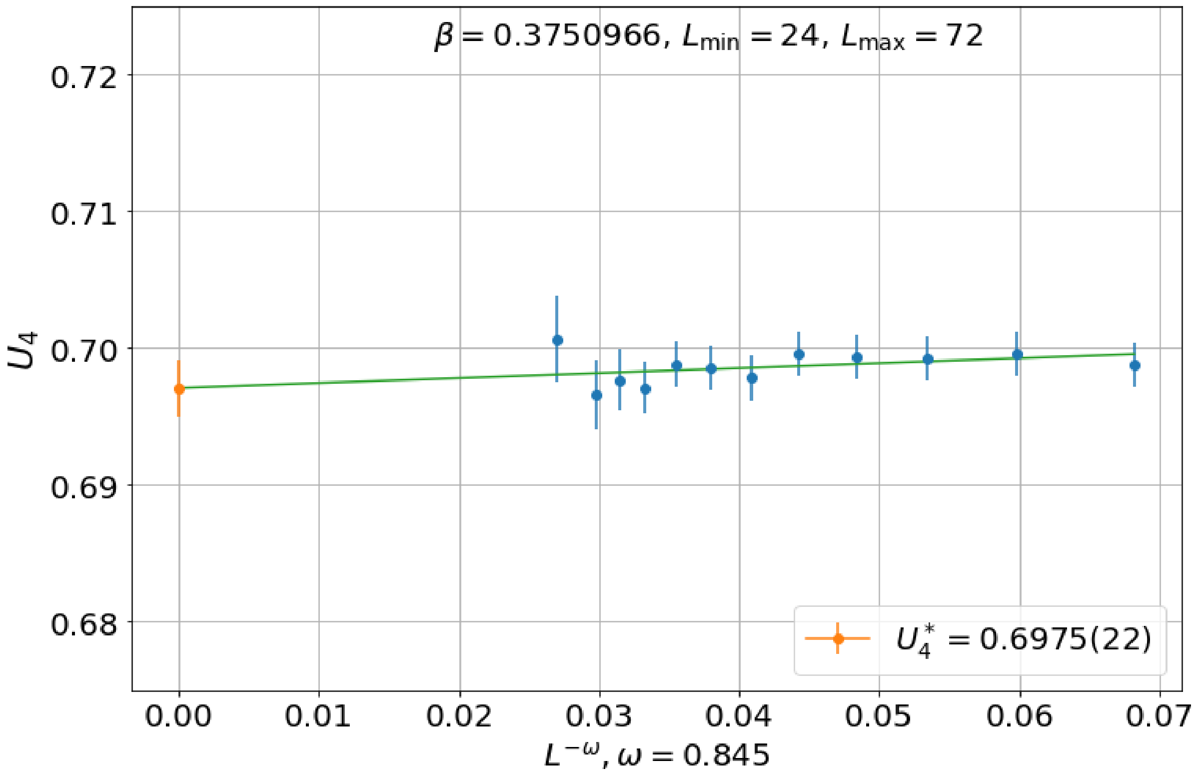






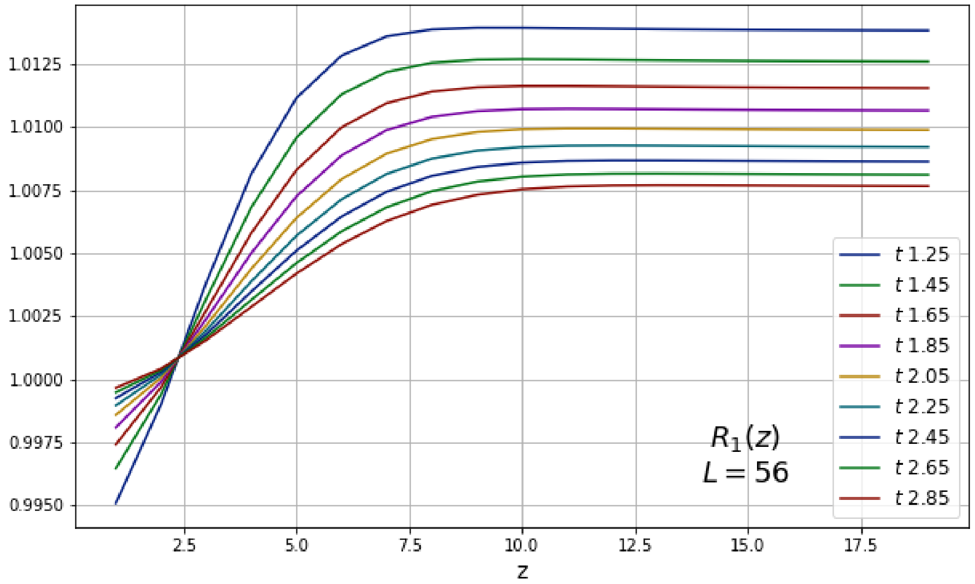


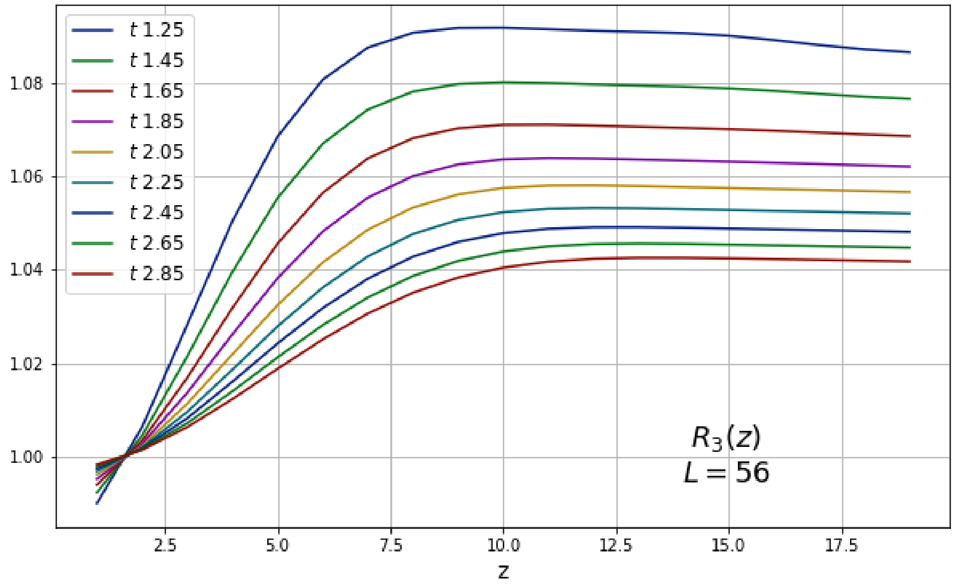
Binder Extrapolation in 3d



Ratios

Ratios of the leading two scaling operator correlators in the odd subspace





Leading Dimensions

We measured mixed correlations in the odd and even bases

$$\{\phi, \phi^3\}, \quad \{\phi^2, \phi^4\}$$

Infinite volume extrapolations were performed as before

The precisely-known values to compare against:

$$2\Delta_1 = 1.03580(40)$$

$$2\Delta_2 = 2.8234(15)$$

(Right) Extrapolation for leading dimensions in each subspace, using all ratio distances past *z = 10*

The Δ_1 exponent deviates from the input by several standard deviations

Δ_2 is consistent with the Hasenbusch value