Effective Theories for Point Sources

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Outline

- Motivation
  - The QM of singular potentials
- Formalism
  - Point-particle effective theories
- Atoms
  - Nonrelativistic vs relativistic; spin
- Other applications
  - nonunitary extensions; Hawking radiation; catalysis

QM & singular potentials

- Delta functions & boundary conditions
- Singular potentials

Singular Potentials

- Singular potentials sometimes parameterize small-distance physics

e.g. the delta-function potential can capture finite-size effects of a charge distribution

$$\rho(x) = Ze\left[1 + \frac{1}{6}\,r_p^2\,\nabla^2 + \cdots\right]\delta^3(x)$$

$$V(r) = -\frac{Z\alpha}{r} + \frac{2\pi}{3}\,Z\alpha\,r_p^2\,\delta^3(x)$$

Singular Potentials

- Delta function potentials modify boundary condition near the source

$$-\frac{1}{2m}\nabla^2\psi - g\,\delta^3(x)\,\psi = E\psi$$

$$\lim_{r\to 0}\left[4\pi r^2\,\frac{\partial\psi}{\partial r} + 2mg\,\psi\right] = 0$$

Singular Potentials

- Subtlety: boundary condition makes wave function singular at r=0, so how to make sense of the boundary condition at r=0?

$$\psi(r,\theta,\phi) = R_\ell(r)\, Y_{\ell\,\ell_z}(\theta,\phi)$$

$$R_\ell(r) = C_1\, r^\ell + \boxed{C_2\, r^{-\ell-1}}$$

nonzero when g nonzero

$$\lim_{r\to 0}\left[4\pi r^2\,\frac{\partial\psi}{\partial r} + 2mg\,\psi\right] = 0$$

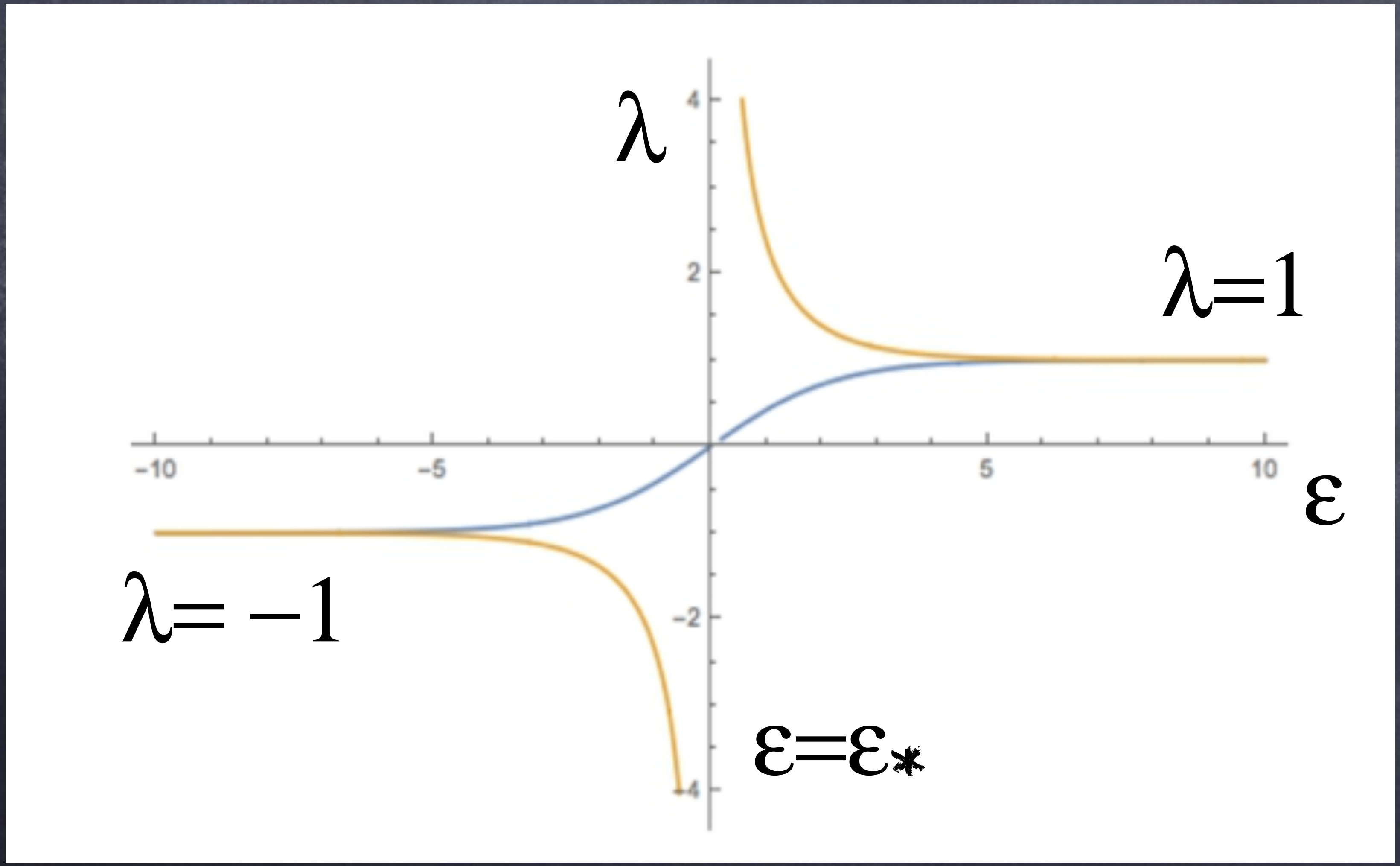
Singular Potentials

- Regulate by evaluating at nonzero but small radius $r = \varepsilon$
- Renormalize g: give $g = g(\varepsilon)$ an implicit ε dependence to cancel explicit ε dependence in physical quantities (ie from C_2/C_1)

$$\lambda = 1 + \frac{2mg}{2\pi\epsilon} \qquad \frac{\partial\lambda}{\partial \ln\epsilon} = \frac{1}{2}\left(1 - \lambda^2\right)$$

Singular Potentials

- Physical quantities depend only on RG invariant combinations, like ε_*



$$\lambda = 1 + \frac{2mg}{2\pi\epsilon}$$

Singular Potentials

- Similar issues arise for other singular potentials, like $V(r) = -h/r^2$

$$-\frac{1}{2m}\nabla^2\psi - \frac{h}{r^2}\psi = E\psi$$

$$R_\ell(r) = C_1 r^s + C_2 r^{-s-1}$$

$$s(s+1) = \ell(\ell+1) - 2mh$$

For l = 0 both solutions are singular at r=0

Singular Potentials

- Which solution should be kept? (ie what boundary condition should be used at r=0?) How to deal with divergence at r=0?

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For l = 0 both solutions are singular at r=0

Needn't be useful to think perturbatively in s ~ h.

Singular Potentials

- Which solution should be kept? (ie what boundary condition should be used at r=0?) How to deal with divergence at r=0?

$$-\frac{1}{2m}\nabla^2\psi - \frac{h}{r^2}\psi - \boxed{g\,\delta^3(x)\psi} = E\psi$$

Claim: Both questions answered by a compulsory delta-function potential.

Singular Potentials

- Must again renormalize g: but now g=0 is no longer a fixed point, so delta function cannot vanish for all scales!

$$\lambda = 1 + \frac{2mg}{2\pi\epsilon} \qquad \frac{\partial\lambda}{\partial \ln\epsilon} = \frac{1}{2}\Big(\zeta^2 - \lambda^2\Big)$$

$$\zeta = \sqrt{1 - 8mh}$$

Singular Potentials

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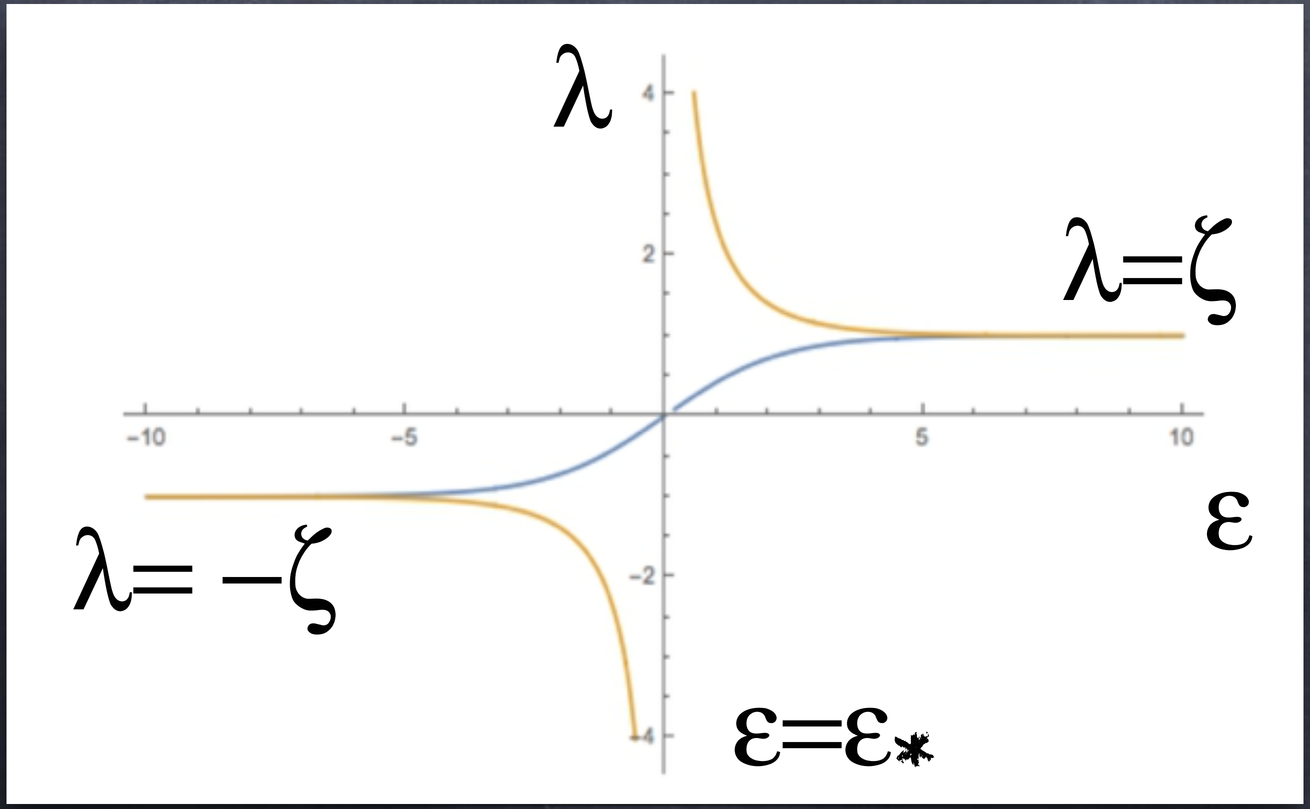


$$\lambda = 1 + \frac{2mg}{2\pi\epsilon}$$

$$\zeta = \sqrt{1 - 8mh}$$

Singular Potentials

- Presence of delta function clarifies many physical properties
  - eg: ambiguities in boundary conditions are associated with choice for localised action
    - need not be self-adjoint
  - eg: bound state sometimes exists even for repulsive potentials: $V=+h/r^2$
    - bound state supported by delta-function

Formalism

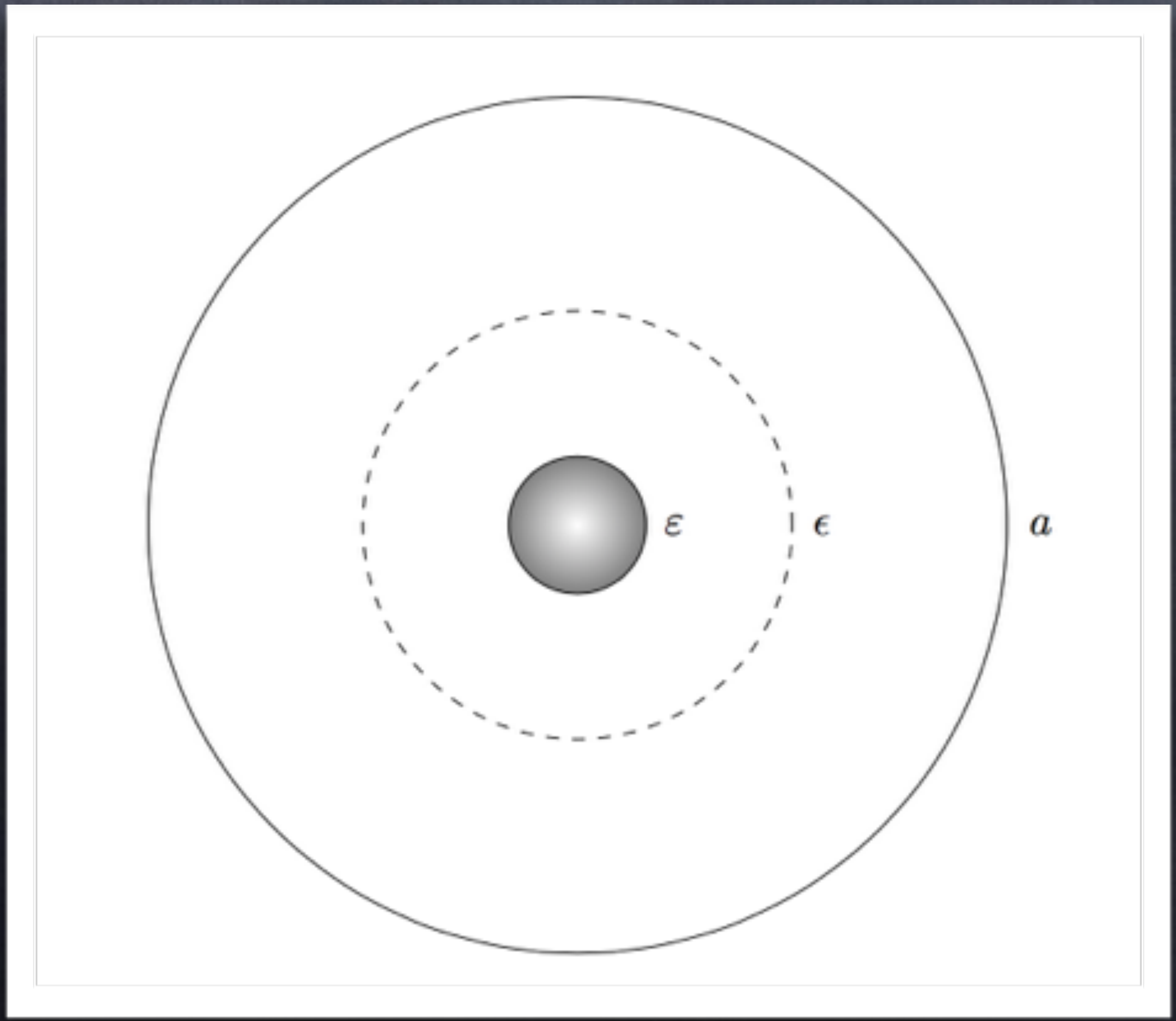
- Near/far matching
- Boundary action
- PP EFT

Formalism

- Can systematise the delta-function reasoning to more general point-source action:
  - Relate near-source boundary condition to effective point-particle action describing low-energy interactions of the source and its environment.
  - Resembles relation between branes and environment (Goldberger & Wise, Goldberger & Rothstein)

Formalism

- Simple statements require hierarchy of scales:



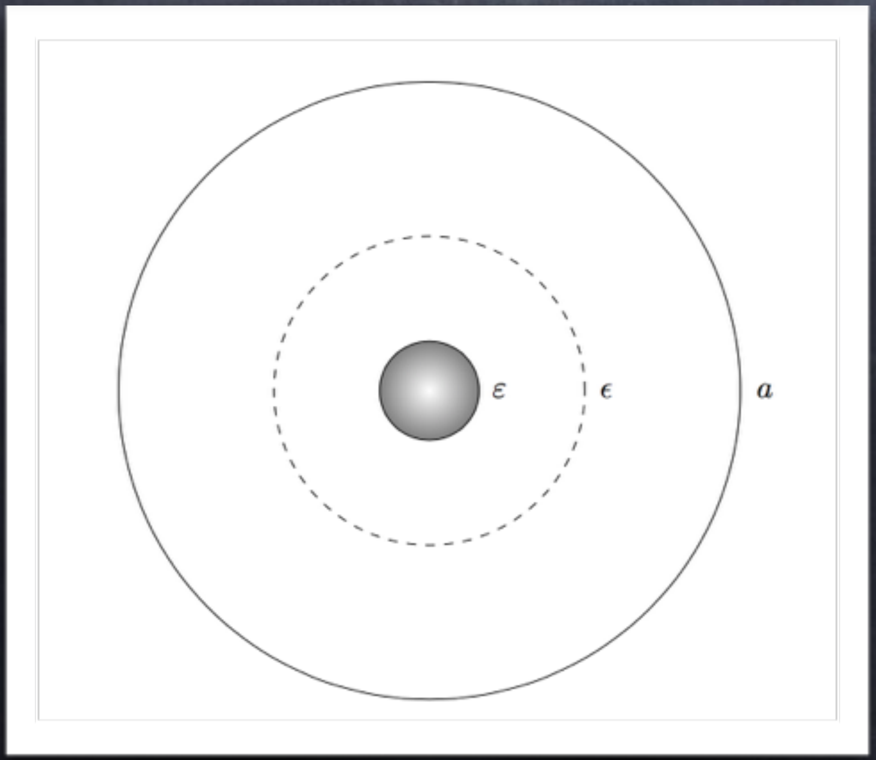
$$\varepsilon \ll a$$

so can choose

$$\varepsilon \ll \epsilon \ll a$$

Near-far matching

- Eg: electrostatic multipole expansion
  - Wish to describe physics at scale a using minimal input from microscopic scale ε



$$-\nabla^2 V = \rho$$

Near-far matching

- Eg: electrostatic multipole expansion
  - Fix integration constants in far-field solution by matching to near-field solution

$$V(r) = \int \mathrm{d}x \, \frac{\rho(x)}{|r - x|} = \frac{Q}{r} + \frac{\hat{r} \cdot D}{r^2} + \cdots$$

far-field

$$V_\ell(r) \simeq C_1 \, r^\ell + \frac{C_2}{r^{\ell+1}}$$

near-field

Near-far matching

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$$V(r) = \int \mathrm{d}x \, \frac{\rho(x)}{|r-x|} = \frac{Q}{r} + \frac{\hat{r}\cdot D}{r^2} + \cdots$$

$$V_\ell(r) \simeq C_1 \, r^\ell + \frac{C_2}{r^{\ell+1}}$$

agreement fixes C_2 but not C_1 for each l

Point-source action

- Alternative formulation: Describe higher multipoles using a point-source action

'bulk' action

$$L = \frac{1}{2}\Big[E^2 - B^2\Big] + L_p$$

multipole moments

$$L_p = \Big[Q + D \cdot \nabla + \cdots\Big] V(x) \delta^3(x)$$

$$E = -\nabla V$$

Point-source action

- Get boundary condition by integrating field equations over gaussian pillbox of radius intermediate between ε and a
  - Get higher multipoles by integrating weighted by powers of x

$$-\int_\epsilon \mathrm{d}^3x\, \nabla^2 V = -\oint \mathrm{d}^2\Omega\, \epsilon^2 n \cdot \nabla V = Q$$

Point-source action

- Boundary condition relates normal derivatives (for bosonic fields) to coeffs of PP action
    - Get higher multipoles by integrating weighted by powers of x

$$\left[r^2 \partial_r V_\ell - \ell V\right]_\epsilon = Q_\ell$$

Point-source action

- Boundary condition relates normal derivatives (for bosonic fields) to coeffs of PP action
  - Get higher multipoles by integrating weighted by powers of x

$$\Big[r^2\partial_r V_\ell - \ell V\Big]_\epsilon = Q_\ell$$

- Noteworthy: $V_\ell \propto r^\ell$ drops out of b.c. and $Q_\ell(\epsilon)$ keeps position of surface irrelevant

Boundary action

- Can regard boundary condition as resulting from variation of fields on boundary, weighted by an appropriate boundary action

$$\mathcal{I} = \oint \mathrm{d}^2\Omega \, L_{\mathrm{bdy}}(V_\ell)$$

RG is statement that position of boundary is not unique

Boundary action need not be local

but is for s-wave $\quad L_p = 4\pi\epsilon^2 \, L_{\mathrm{bdy}}$

Formalism

- All of this generalises to case where L_p is a more complicated function of fields
  - eg for Schrodinger field lowest-dimension interaction corresponds to a delta-function potential

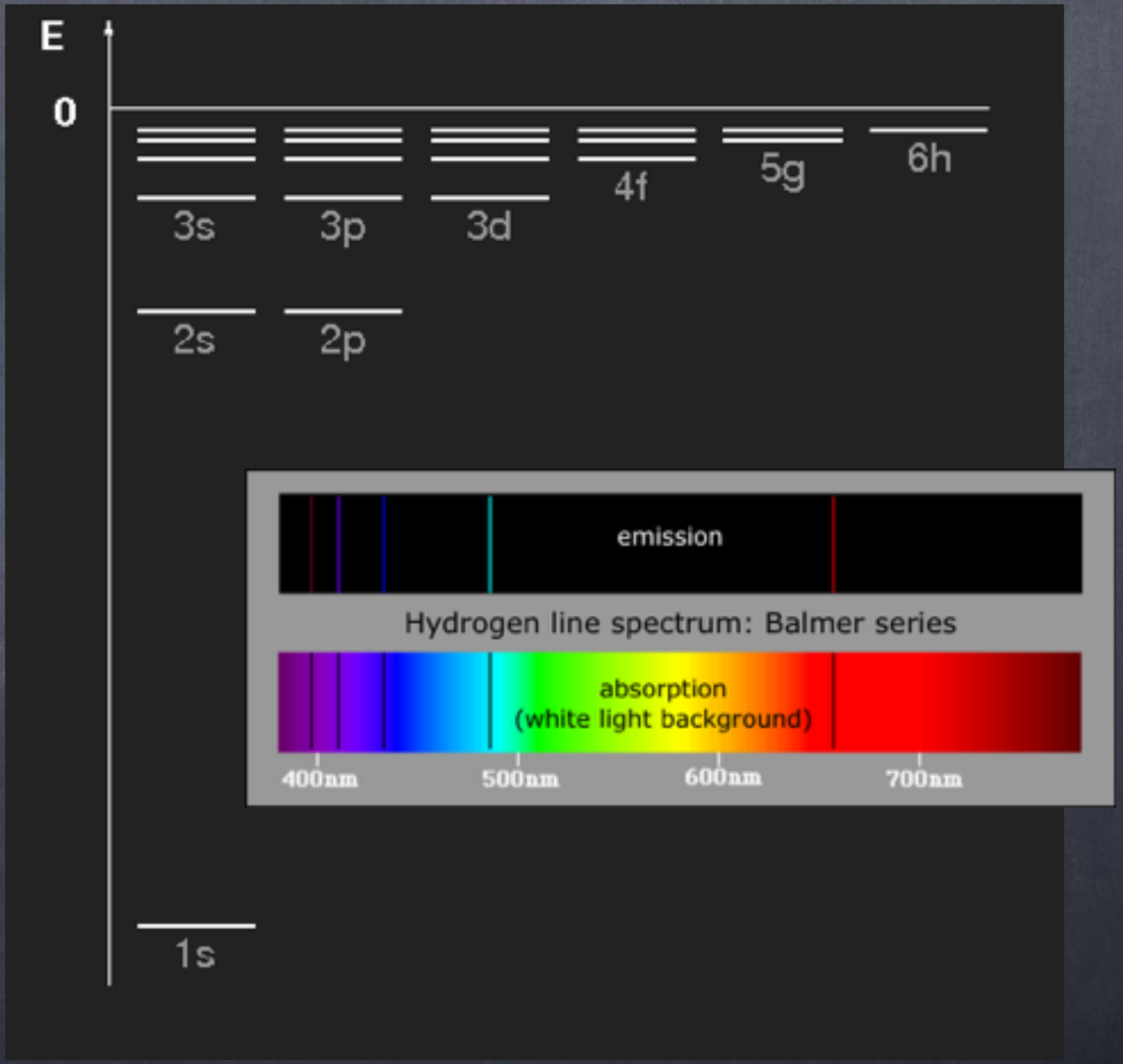
$$L_p = -g\,\psi^*\psi\,\delta^3(x)$$

$$4\pi\epsilon^2\Big[\partial_r\psi\Big]_\epsilon = 2mg\,\psi$$

explains why Robin conditions so common

Atoms

- Schrodinger
- Klein-Gordon
- Dirac



NR Atoms

- Natural language for describing finite-source size effects for atoms
    - Work in limit of infinite nuclear mass but finite nuclear size
    - Track (for simplicity) only spherically symmetric effective interactions

$$L_p = \left[-M + Ze \left(1 + \frac{1}{6} r_p^2 \nabla^2 \right) V - g\, \psi^* \psi + \cdots \right] \delta^3(x)$$

NR Atoms

- Maxwell's equations imply scalar potential:

$$V(r) = \frac{Z\alpha}{r} + \frac{2\pi}{3}\, Z\alpha\, r_p^2\, \delta^3(x) \qquad\qquad r_p^2 = \langle r^2 \rangle_N$$

- Schrodinger equation has effective potential:

$$U = -\,\frac{Z\alpha}{r} + g_{\text{eff}}\, \delta^3(x)$$

$$g_{\text{eff}} = g + \frac{2\pi}{3}\, Z\alpha r_p^2$$

NR Atoms

- Naive energy shift:

$$\delta E_n = g_{\text{eff}} |\psi(0)|^2 = \frac{g_{\text{eff}}}{\pi} \left(\frac{Z\alpha m}{n} \right)^3$$

$$g_{\text{eff}} = g + \frac{2\pi}{3} Z\alpha\, r_p^2 = \frac{2\pi}{3} Z\alpha\, r_{\text{eff}}^2$$

$$r_p^2 = \langle r^2 \rangle_N$$

g parameterizes how different charge distributions do not contribute precisely proportional to their rms charge radius

NR Atoms

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$$r_p^2 = \langle r^2 \rangle_N$$

Higher derivative bilinear terms capture how ψ'/ψ varies as a function of source size kR

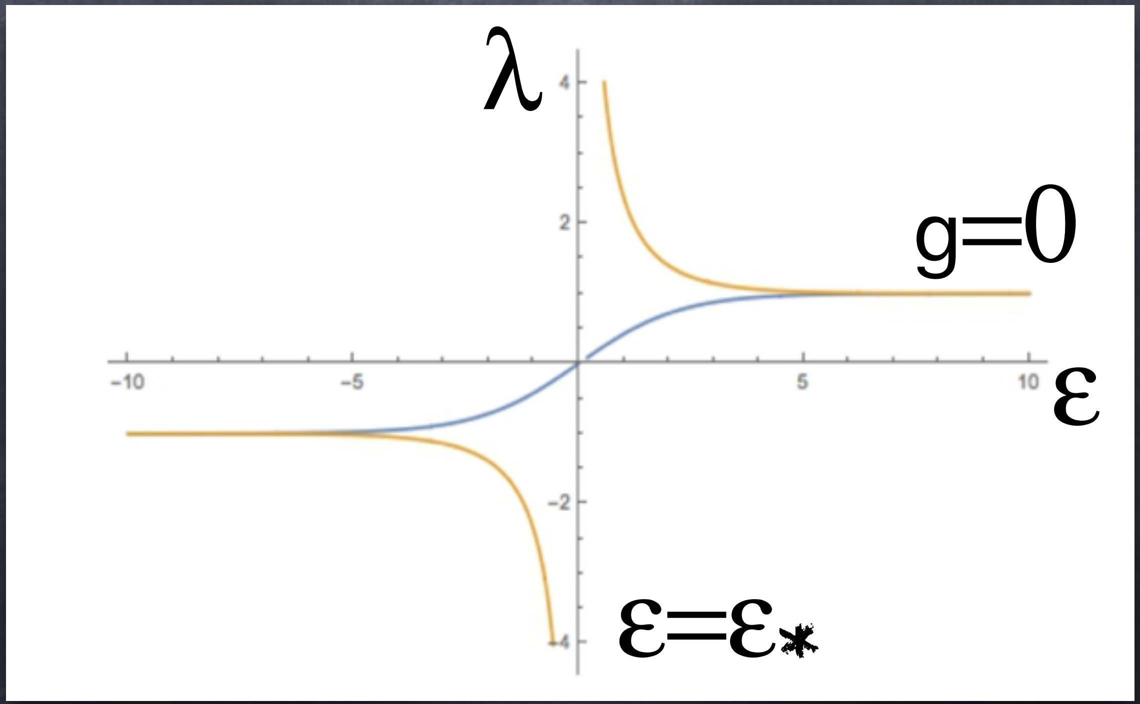
$$(\psi^* \nabla^2 \psi)\delta^3(x)$$

NR Atoms

- Naive because: nonzero g requires ψ must diverge at r=0, so $|\psi(0)|^2$ ill-defined for small enough ε
- RG can resum all orders in the dimensionless ratio: mg/ε to give

$$\delta E_n = \frac{2\pi\epsilon_\star}{\pi m}\left(\frac{Z\alpha m}{n}\right)^3$$

$$\frac{\epsilon_*}{\epsilon} = \left[\frac{\lambda-\zeta}{\lambda+\zeta}\right]^{1/\zeta}$$



NR Atoms

- This is overkill for ordinary atoms, since $g \sim Z\alpha\,\varepsilon^2$ implies $mg/\varepsilon \sim m\varepsilon\, Z\alpha \sim \varepsilon/a_B$ is small so naive answer usually dominates.
- An exception is pionic atoms, for which g is large since it captures local effects of the short-ranged nucleus-pion strong interaction

$$\delta E_n = \frac{2\pi\epsilon_\star}{\pi m}\left(\frac{Z\alpha m}{n}\right)^3 \quad \text{where} \quad a_s = \frac{\epsilon_\star}{2}$$

Reln btwn dE and scatt length: Deser formula

KG Atoms

- The Schrodinger treatment also fails if matching to point source occurs at $r < (Z\alpha)^2 a_B$ since then relativistic effects cannot be ignored.
  - eg for spinless particles Klein-Gordon equation:

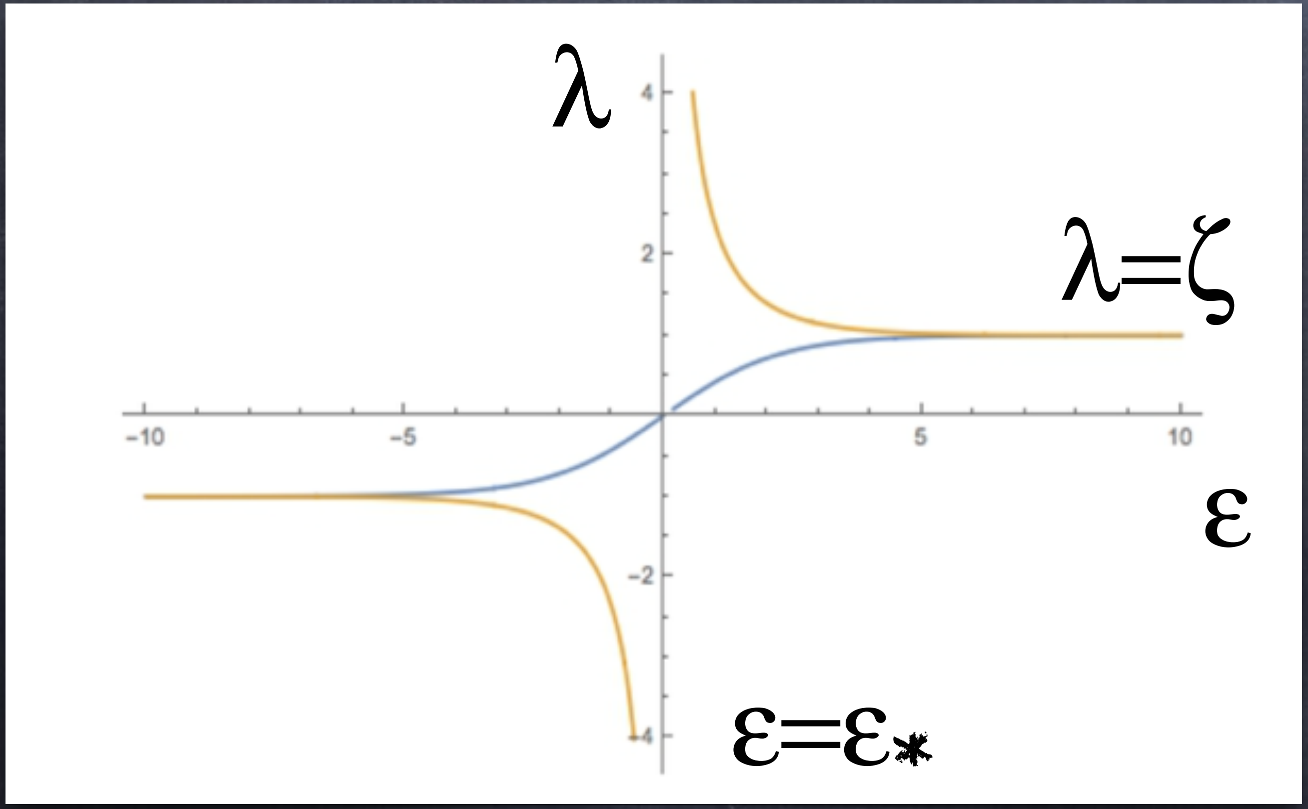
$$-(\partial_t + Z\alpha/r)^2\phi + \nabla^2\phi - m^2\phi = 0$$

Radial eq dominated by $1/r^2$ potential at small r:

$$V_{\text{eff}}(r) = -\frac{2EZ\alpha}{r} - \frac{(Z\alpha)^2}{r^2}$$

KG Atoms

- If $\varepsilon < (Z\alpha)^2 a_B$ then running of g cannot be ignored since $(Z\alpha)^2/r^2$ term drives g away from zero



$$\delta E_n = \frac{2\pi\epsilon_\star}{\pi m}\left(\frac{Z\alpha m}{n}\right)^3$$

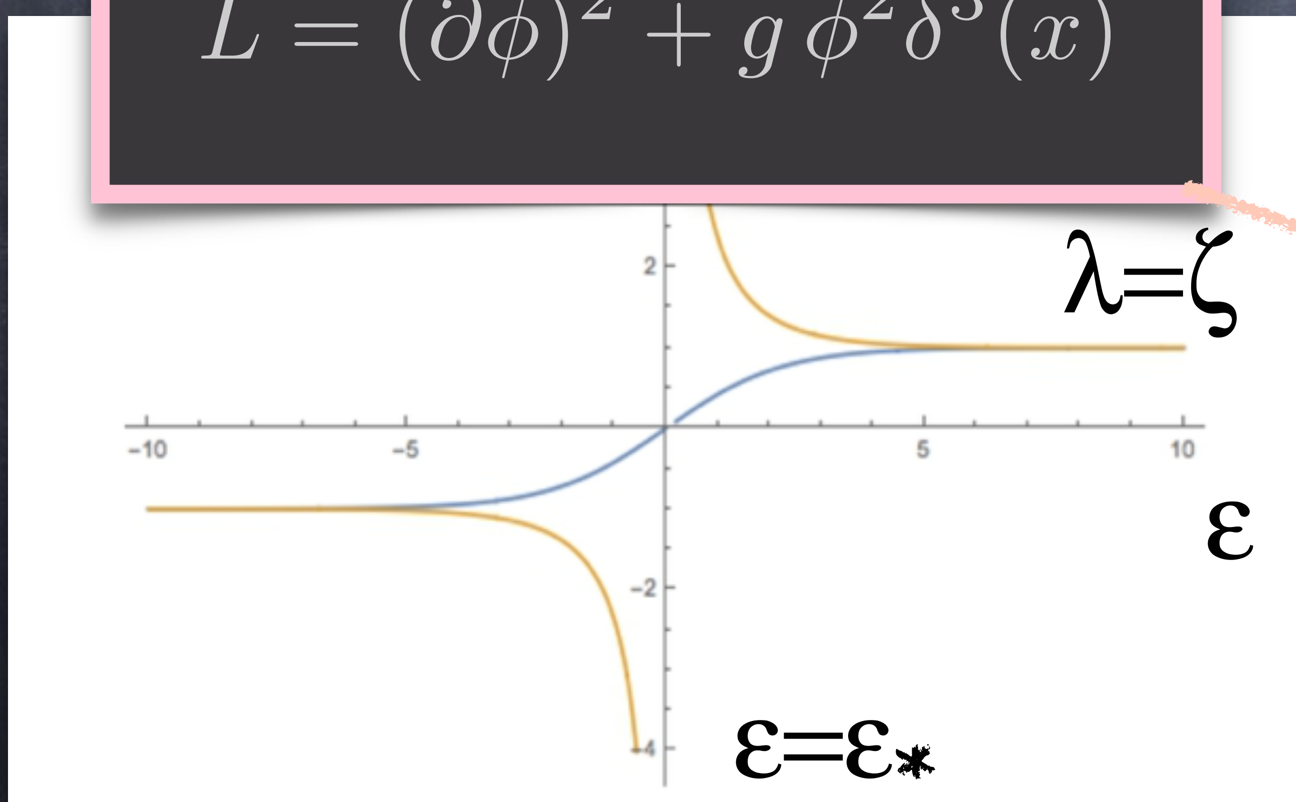
$$\frac{2\pi\epsilon_*}{m} \simeq \frac{g + 2\pi\epsilon(Z\alpha)^2/m}{1 + mg/(2\pi\epsilon)}$$

KG Atoms

- If $\varepsilon < (Z\alpha)^2 a_B$ then running of g cannot be … term drives g away

For KG field shd expect g linear in ε

$$L = (\partial\phi)^2 + g\,\phi^2\delta^3(x)$$



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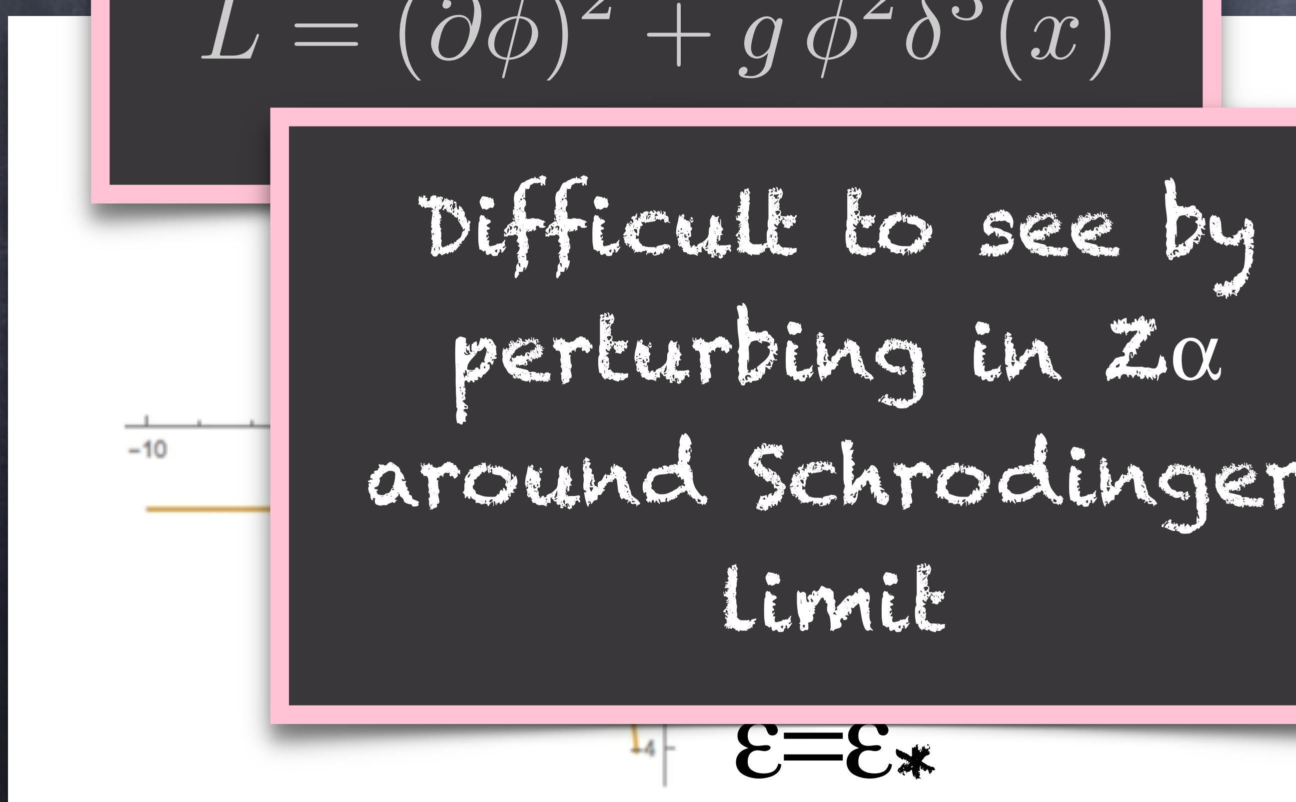


Difficult to see by perturbing in $Z\alpha$ around Schrodinger limit

$$\delta E_n = \frac{2\pi\epsilon_\star}{\pi m}\left(\frac{Z\alpha m}{n}\right)^3$$

$$\frac{2\pi\epsilon_*}{m} \simeq \frac{g + 2\pi\epsilon(Z\alpha)^2/m}{1 + mg/(2\pi\epsilon)}$$

Dirac Atoms

- For spinning (Dirac) electrons:
  - Solutions again vary like r^s with $s = (Z\alpha)^2/2$
  - relativistic effects also cannot be ignored when $\varepsilon < (Z\alpha)^2 a_B$
  - Corrections differ when $m\varepsilon < Z\alpha$
  - ...but not in a way that is measurable (yet)

Dirac Atoms

- Restrict to parity preserving case
  - two contact interactions rather than one

$$L = -\overline{\psi}(\gamma \cdot \partial + m)\psi - \overline{\psi}\Big(c_s + c_v \gamma^0\Big)\psi\, \delta^3(x)$$

  - source physics only enters through boundary condition for f/g at r = ε

$$\psi_{L\pm} = f_\pm(r) U_1(\theta, \varphi) + i g_\pm(r) U_2(\theta, \varphi)$$

Dirac Atoms

- Boundary condition relates f/g to c_s and c

$$L = -\overline{\psi}(\gamma \cdot \partial + m)\psi - \overline{\psi}\left(c_s + c_v \gamma^0\right)\psi\, \delta^3(x)$$

$$\oint \mathrm{d}^2\Omega\, \epsilon^2 (n \cdot \gamma)\psi_\epsilon = (c_s + c_v \gamma^0)\psi_\epsilon$$

$$4\pi\epsilon^2 \left(\frac{g_+}{f_+}\right)_\epsilon = c_s + c_v$$

$$4\pi\epsilon^2 \left(\frac{f_-}{g_-}\right)_\epsilon = c_s - c_v$$

Dirac Atoms

- Leading deviations to parity-odd states arise at NLO in momentum expansion. For charge distribution of radius R

$$\frac{g_+}{f_+} = Z\alpha\Big[f_0 + f_2(kR)^2 + \cdots\Big]$$

$$k^2 = \left(\omega + \frac{Z\alpha}{R}\right)^2 - m^2$$

$$(kR)^2 \simeq 2mRZ\alpha \quad \text{if } mR \gg Z\alpha$$

$$(kR)^2 \simeq (Z\alpha)^2 \quad \text{if } mR \ll Z\alpha$$

$$\delta E = 2Z\alpha\, R^2\Big[1 + 2\left(f_0 + f_2(kR)^2\right)\Big]\left(\frac{Z\alpha m}{n}\right)^3$$

Other applications

- Non-hermitian extensions
- Hawking radiation
- Catalysis

Other applications

- Linking ambiguities to an action keeps choices physical

sometimes self-adjointness is not really the extension you seek

eg: trapped polarizable atoms attracted to a charged wire

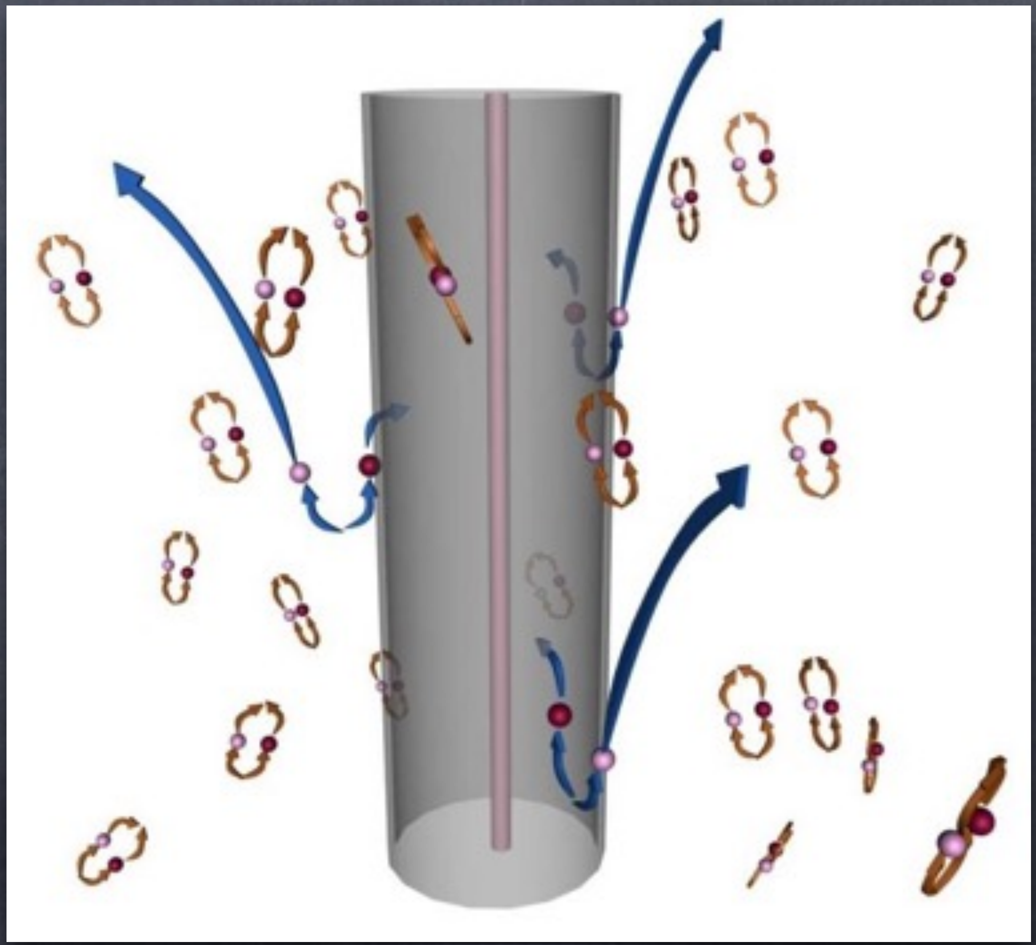
$$V = -\frac{h}{r^2} + (g + i\eta)\,\delta^3(r)$$

Other applications

- Near-horizon radial equation in black-hole spacetime same as Schrodinger eq with potential

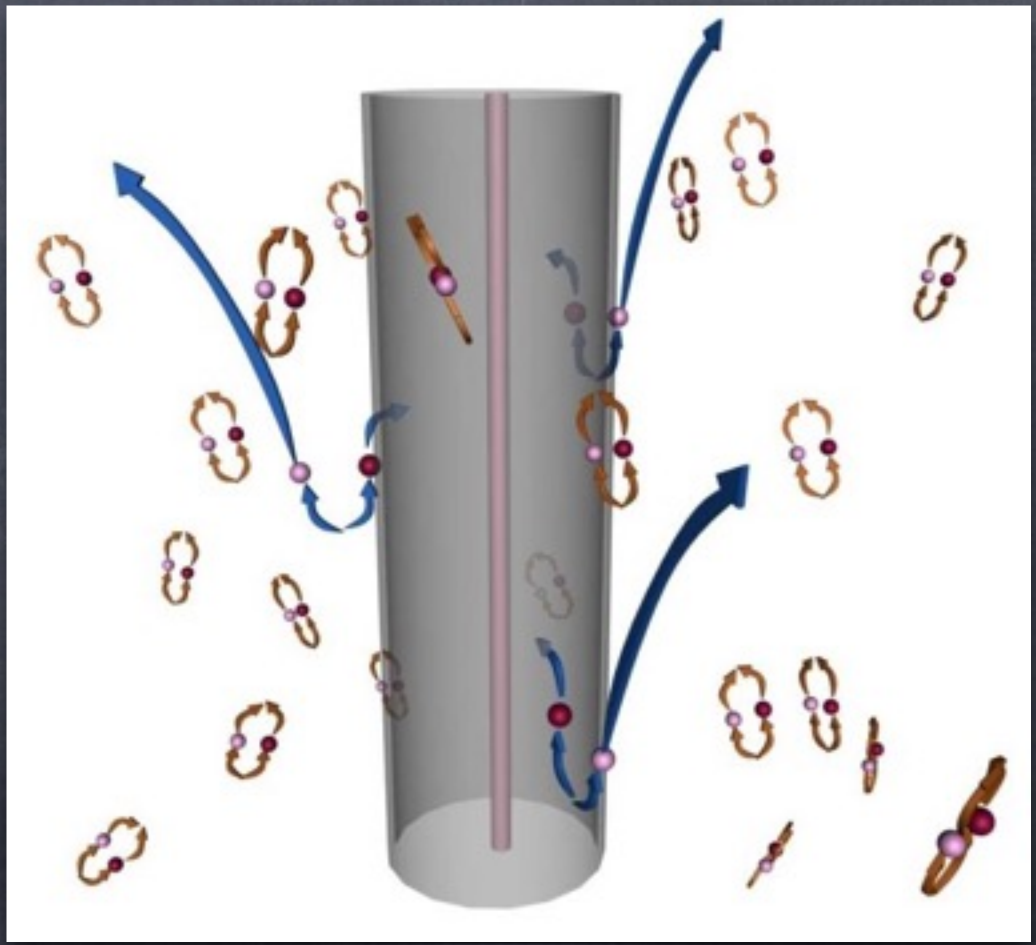
  $V_{eff} \sim 1/(r-r_s)^2$

  Implied delta function related to Wilczek-Robinson anomaly term



Other applications

- Scattering from small objects (eg monopoles) can be much larger than their geometric size (monopole catalysis)



related to RG scale ε_* being very different from geometrical size

Summary

- Singular potentials underline the need to think carefully about boundary condition near the source
  - Near source bc given by source PPEFT
  - Linking ambiguities to source properties makes choices physical
- Parameterize finite source size effects in atoms
  - potential surprises for small r (but none measurable yet)