New Developments for Scattering Amplitudes

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Motivation

Why are scattering amplitudes interesting?

- Discoveries
  Need both exact measurements and predictions
  LHC delivers more precise measurements
  $\rightarrow$ more precise theoretical predictions are needed
  $\rightarrow$ one important ingredient: scattering amplitudes
- Hidden properties
  Lay in the heart of Quantum Field theories (QFT)
  -Their study improves our understanding of QFTs
  -Capture properties which are hidden in other objects (e.g. Lagrangian)
- Combine several fields of Mathematics
  -Algebraic geometry
  -Grassmanians
  -Number theory
  -Theory of differential equations
  and many more

Introduction

How can we compute a scattering amplitude?

For multi-loop processes with a higher number of legs a direct Feynman Diagram calculation becomes impossible
Reasons:

- Unmangeable number of complicated Feynman integrals
  $\Rightarrow$ Integration-by-parts identities (IBP-Ids) Chetyrkin,Tkachov; Laporta
  $\Rightarrow$ Differential equations Kotikov; Remiddi; Gehrmann, Remiddi; Smirnov; Argeri, Mastrolia; von Manteuffel; Henn; Melnikov; Papadopoulos; Anastasiou, Duhr; Tancredi, Remiddi
- Unphysical gauge freedom which cancels for gauge invariant objects
  $\Rightarrow$ Easier to reconstruct amplitudes from their pole structure which is governed by Analyticity and Unitarity
  $\Rightarrow$ Generalized Unitarity Bern, Dixon, Dunbar, Kosower
  $\Rightarrow$ On-shell techniques Britto, Cachazo, Feng, Witten
  $\Rightarrow$ OPP Integrand-Reduction Ossola, Papadopoulos, Pittau

Introduction

Furthermore through input from algebraic geometry we improved our understanding of QFT

- Integrand-Reduction — Mastrolia, Ossola; Zhang; Mastrolia, Mirabella, Ossola, Peraro
- On-shell formulation of $\mathcal{N}=4$ sYM — Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka

Moreover it revealed hidden properties of QFT

- Color-Kinematic and Gauge/Gravity duality — Bern, Carrasco, Johansson
- Grassmanians — Arkani-Hamed, Cachazo, Cheung, Kaplan; Mason, Skinner
- Dual conformal symmetry — Drummond, Henn, Smirnov, Sokatchev; Bern, Czakon, Dixon, Kosower, Smirnov

Introduction

These developments inspired advancements also on the pheno side
e.g. new concepts like uniform transcendental functions
$\Rightarrow$ Canonical form for differential equations Henn

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$\Rightarrow$ Canonical form for differential equations — Henn

Multipurpose tools for one-loop amplitudes

- BlackHat — Bern, Dixon, Febres-Cordero, Forde, Hoecke, Ita, Kosower, Maitre, Ozeren
- FeynArts/FormCalc/LoopTools — Hahn et al.
- MadLoop — Hirschi, Frederix, Frixione, Garzelli, Maltoni, Pittau
- HelacNLO — Bevilacqua, Czakon, van Hameren, Papadopoulos, Pittau, Worek
- Njets — Badger, Biederman, Uwer, Yundin
- OpenLoops — Cascioli, Maierhoefer, Pozorini
- Recola — Actis, Denner, Hofer, Scharf, Uccirati
- Rocket — Ellis, Giele, Kunszt, Melnikov, Zanderighi
- GoSam — Cullen, Greiner, Heinrich,Mastrolia, Ossola, Reiter, Tramontano

Introduction

These developments inspired advancements also on the pheno side
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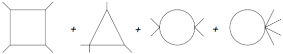
Multipurpose tools for one-loop amplitudes

- BlackHat :: on-shell recurrence + Generalised Unitartiy
- FeynArts/FormCalc/LoopTools :: Feynman Diag.
  + Tensor Red./Integrand Red.
- MadLoop :: tree-level recurrence + Integrand Red.
- HelacNLO :: tree-level recurrence + Integrand Red.
- Njets :: on-shell recurrence + Generalised Unitartiy
- OpenLoops :: recursive tensors + Tensor Red./Integrand Red.
- Recola :: recursive tensors + Tensor Red.
- Rocket :: tree-level recurrence + Generalised Unitartiy
- GoSam :: Feynman Diag. + Tensor Red/Integrand Red.

At two-loops the situation is still less explored

	one-loop	two-loop
graphs	only planar	planar and non-planar
integral basis	known 	determined case by case ?
integrals	known	only for certain cases
IR poles	cancellation between one-loop and tree level	cancellation between two- and one-loop and tree level
appearing functions	logs and dilogs	logs, polylogarithms, generalized polylogs, elliptic functions and more?

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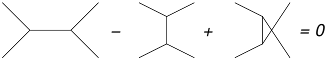
Basic Idea

Bern, Carrasco, Johansson

Split tree-level amplitudes into a kinematical and a color factor

$$\mathcal{A}_n^{tree} = \sum_i \frac{n_i c_i}{D_j}$$

Let the kinematical numerators satisfy the Jacobi identity of the color factors



Proven to be possible for **massless** gauge theories at **tree-level**

Bjerrum-Bohr, Damgaard, Vanhove,
Stieberger,
Feng, Huang, Jia,
Cachazo

Connection to Feynman Diagrams

Mastrolia, Primo, Torres Bobadilla, U.S.

$$\mathcal{A}_n^{tree} = \sum_{i=1}^{N} \frac{c_i \hat{n}_i}{D_i}$$

Feynman numerators do **not** satisfy Jacobi-Identity

$$c_i + c_j + c_k = 0 \quad \Rightarrow \quad \hat{n}_i + \hat{n}_j + \hat{n}_k = \Phi_{[i,j,k]} \neq 0$$

But: generalized gauge freedom

$$\hat{n}_i \to n_i = \hat{n}_i - \Delta_i \quad \text{with} \quad \mathcal{A} - \hat{\mathcal{A}} = \sum_{i=1}^{N} \frac{c_i \Delta_i}{D_i} = 0$$

$\Rightarrow$ Build system of equations

$$\begin{aligned} \Delta_i + \Delta_j + \Delta_k &= \Phi_{[i,j,k]} && \text{M-times} \\ \sum_{j=1}^{N} \frac{\alpha_{i,j} \Delta_j}{D_j} &= 0 && \text{N-M times} \end{aligned}$$

Not all equations are independent

$\Rightarrow$ Additional relations between anomalies $\Phi_{[i,j,k]}$

$\Rightarrow$ only $(n-3)!$ independent color-ordered amplitudes

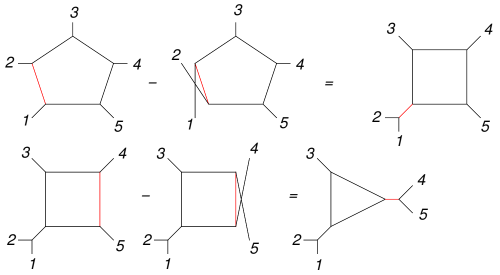
Loop-Level

Bern, Carrasco, Johansson

Loop-amplitudes can be written as

$$\mathcal{A} = \sum_{\text{perms}} \int \left(\prod_{m}^{l} \frac{d^D q_m}{(2\pi)^D} \right) \sum_{\text{graphs}} \frac{n_i c_i}{\prod_j D_j}$$

Conjecture: Numerators satisfy BCJ equations
e.g. at one-loop we have



Double Copy Procedure

Amplitudes in SUperGRAvity can be obtained by replacing the color factor with a copy of the kinematical factor

$$\mathcal{A}^{\text{sYM}} = \int \left(\prod_m^l \frac{d^4 q_m}{(2\pi)^4} \right) \sum_{\text{graphs}} \frac{n_i c_i}{\prod_j D_j}$$

$$\Rightarrow \mathcal{A}^{\text{SUGRA}} = \int \left(\prod_m^l \frac{d^D q_m}{(2\pi)^D} \right) \sum_{\text{graphs}} \frac{n_i \tilde{n}_i}{\prod_j D_j}$$

Is known to work for several theories

Gauge numerator n	Gauge numerator $\tilde{n}$	Gravity
$\mathcal{N}$=4 sYM	$\mathcal{N}$=4 sYM	$\mathcal{N}$=8 SUGRA
$\mathcal{N}$=4 sYM	$\mathcal{N}$=0 sYM	$\mathcal{N}$=4 SUGRA
YM	YM	Einstein Gravity + dilaton

Calculation of $\mathcal{N}$=8 SUGRA loop amplitudes becomes feasible
$\Rightarrow$ Investigate UV behavior of $\mathcal{N}$=8 SUGRA through direct computation

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MultiLoop Integrand Reduction

A general multi-loop amplitude can be written as

Mastrolia, Ossola
Zhang
Mastrolia, Mirabella, Ossola, Peraro

$$\mathcal{A} = \int \prod_i^l \frac{d^D q_i}{(2\pi)^D} \frac{N(q_1, .., q_l)}{D_1 D_2 .. D_n}$$

The propagators define a ring of multiples called **Ideal**

$$\langle D_1, .., D_n \rangle = \left\{ \sum_k p_k D_k \, : \, p_k \in P[q] \right\}$$

The Buchbinder algorithm constructs a **Gröbner basis** from an ideal

$$\langle D_1, .., D_n \rangle = \langle g_1, .., g_m \rangle$$

Perform a **multivariate polynomial division** of the integrand by an ideal

$$N(q_1, .., q_l) = \sum_k N_{1..k-1k+1..n} D_k + \Delta_{1..n}$$

Recursion Relation

This gives us an recursion relation

Integrand Recursion Relation

$N(q_1, .., q_l) = \sum_k N_{1..k-1k+1..n} D_k + \Delta_{1..n}$

Used to arrive at

$$\begin{aligned} N(q_1, .., q_l) = \Delta_{12..n} \\ &+\Delta_{23..n} D_1 + .. + \Delta_{12..n-1} D_n + \ldots \\ &+\Delta_1 D_2 D_3..D_n + .. + \Delta_n D_1 D_2..D_{n-1} \end{aligned}$$

Divide-and-Conquer approach

- Generate the integrand
- Perform divison algorithm directly

Fit-on-the-cut appraoch

- Obtain parametric form of the residue via the division of a generic integrand
- Determine the coefficients by sampling

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Combine integrand reduction and the color-kinematic duality by following this general algorithm

Algorithm

- Find all non-equivalent graphs with cubic vertices
- Perform a parametric integrand reduction on each graph
- Determine residues from unitarity cuts
- Fix leftover freedom (If any) by demanding the color-kinematic duality

Note that:

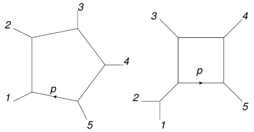
- Integrand reduction is usually performed on sets of denomintors not graphs
- Therefore some graphs are not independent in the sense of integrand reduction
- That's why there is room for leftover freedom which can be used by the color-kinematic duality

$\Rightarrow$ Integrand reduction is a well suited tool to investigate the loop conjecture

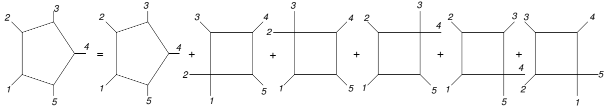
Basic Setup at One-Loop

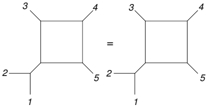
For a one-loop five-point amplitude in $\mathcal{N} = 4$ SYM we find two basic graphs:

Carrasco, Johansson



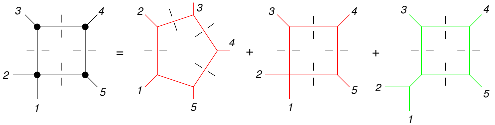
Integrand reduction of the pentagon and the box numerator



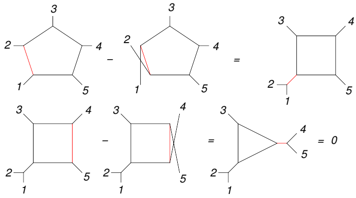


The Quadruple-Cut and Color-Kinematic Duality

The quadruple-cut gets contributions from the fivefold-cut and the fourfold-cut residue

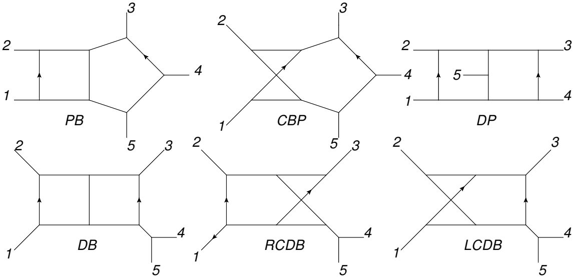


Disentangle with the BCJ equations



Basic Setup at Two-Loops

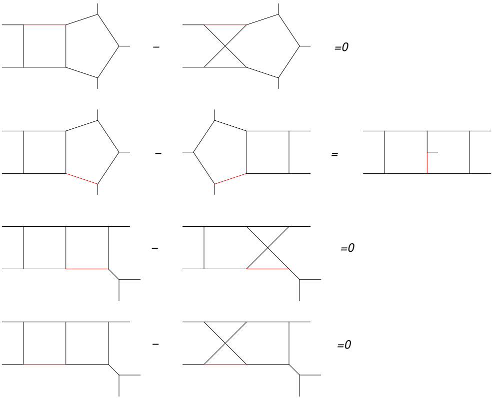
At two-loops we find six graphs



Carrasco, Johansson
Mastrolia, Mirabella, Ossola, Peraro

The Duality between different Graphs

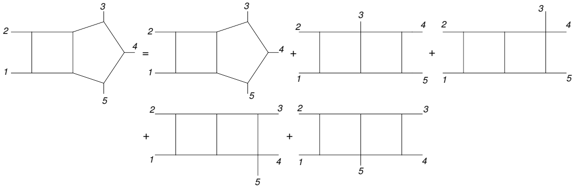
BCJ equations connect different graphs

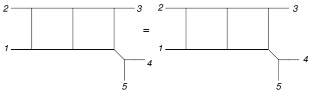


Focus on integrand reduction of the **pentabox** and of the **doublebox**

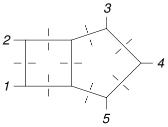
Integrand Reduction and Eightfold-Cut

Integrand reduction of the pentabox and the doublebox





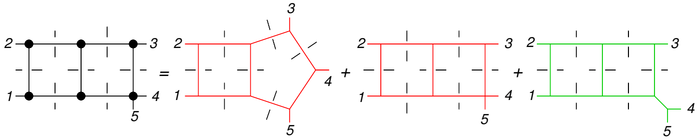
Fit the eightfold-cut residue from unitarity cuts



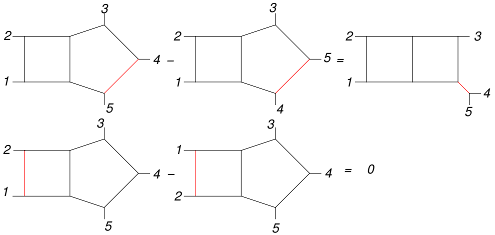
$$= c_0 + c_1 q \cdot p_1 + c_2 k \cdot p_4 + c_3 k \cdot p_3$$

The Sevenfold-Cuts and Color-Kinematic Duality

Sevenfold-cut gets contributions from the eightfold-cut and the sevenfold-cut residues
e.g. leaving D_7 uncut



Disentangle with the BCJ equations

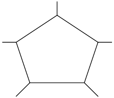


The UV poles of the One-Loop Amplitude

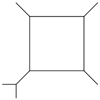
The full one-loop amplitude is

$$\mathcal{A}^{\text{1-loop}} = ig^5 \sum_{\text{all perm}} \frac{1}{10}\beta_{12345}c^P \text{Int}^P + \frac{1}{4}\frac{\gamma_{12}}{s_{12}}c^B\text{Int}^B$$

$$\text{Int}^P = \int \frac{d^Dq}{(2\pi)^D}\frac{1}{D_1 D_2 D_3 D_4 D_5}$$



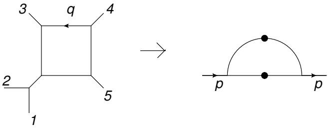
$$\text{Int}^B = \int \frac{d^Dq}{(2\pi)^D}\frac{1}{D_2 D_3 D_4 D_5}$$



The leading UV divergence comes from the box integral since it has one less propagator

Small-Momentum-Injection

Reduce the graph to a two-point function by keeping a small momentum to flow through the diagram



This integral can be solved with the well known formula

$$\int \frac{d^D q}{(2\pi)^D} \frac{1}{D_1^2 D_2^2} = i \frac{(p^2)^{D/2-4}}{(4\pi)^{D/2}} G(2,2)$$

$$= i \frac{(p^2)^{D/2-4}}{(4\pi)^{D/2}} \frac{\Gamma(-D/2+4)\Gamma(D/2-2)\Gamma(D/2-2)}{\Gamma(D-4)}$$

which diverges at D=8.

In $D = 8 - 2\epsilon$ we find

$$\int \frac{d^D q}{(2\pi)^D} \frac{1}{D_1^2 D_2^2} \overset{D=8-2\epsilon}{\longrightarrow} \frac{i}{6\epsilon(4\pi)^4}$$

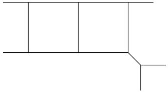
The UV Pole of the Two-Loop Amplitude

The full two-loop amplitude is

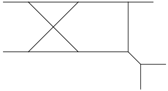
$$\mathcal{A}^{2-loop} = -g^7 \sum_{\text{all perm}} \Big(\frac{1}{2} c^{BP} \text{Int}^{BP} + \frac{1}{4} c^{CBP} \text{Int}^{CBP} + \frac{1}{4} c^{DP} \text{Int}^{DP} + \frac{1}{2} c^{DB} \text{Int}^{DB} + \frac{1}{4} c^{LCDB} \text{Int}^{LCDB} + \frac{1}{4} c^{RCDB} \text{Int}^{RCDB} \Big)$$

At two-loops the doubleboxes diverge first

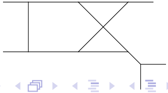
$$\text{Int}^{DB} = \int \frac{d^D k}{(2\pi)^D} \frac{d^D q}{(2\pi)^D} \frac{N^{DB}}{s_{45} D_1 D_2 D_3 D_4 D_5 D_7 D_8}$$



$$\text{Int}^{LCDB} = \int \frac{d^D k}{(2\pi)^D} \frac{d^D q}{(2\pi)^D} \frac{N^{LCDB}}{s_{45} D_1 D_2 D_3 D_4 D_5 D_7 D_8}$$

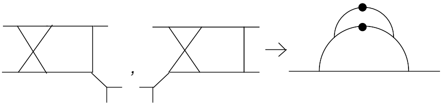


$$\text{Int}^{RCDB} = \int \frac{d^D k}{(2\pi)^D} \frac{d^D q}{(2\pi)^D} \frac{N^{RCDB}}{s_{45} D_1 D_2 D_3 D_4 D_5 D_7 D_8}$$

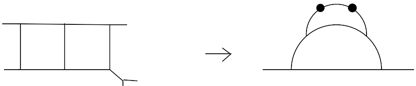


Small Momentum Injection

Use small momentum injection to compute the integral



$$\xrightarrow{D=7-2\epsilon} -\frac{\pi}{30\pi^7\epsilon}$$



$$\xrightarrow{D=7-2\epsilon} -\frac{\pi}{20\pi^7\epsilon}$$

Used small-momentum-injection formula recursively

Short Summary

Color-Kinematic duality

- Connection between Feynman diagrams and color-dual numerators at tree level
- Combined for the first time MultiLoop Integrand Reduction, Unitarity and Color-Kinematic Duality
- Presented a systematical way to obtain an amplitude in the color-dual form
- Extracted the leading UV pole of these amplitudes
  1-loop diverges at D=8
  2-loop diverges at D=7
- Through the double copy procedure amplitudes in YM theory are connected to gravity amplitudes
- Can be used to study UV behavior of gravity

Concerning the Integrand Reduction:

- Symmetries can constrain the form of the residues
- Each scalar product stands for a potential master integral
  $\rightarrow$ Symmetry reduces number of potential master integrals

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Feynman integrals are functions of

- Mandelstam variables
- Internal and external masses
- Spacetime dimensions

Facts

- Not all Feynman integrals are independent
- IBP-ids connect different Feynman integrals
- We can find an integral basis called master integrals

$\Downarrow$

Exploit this by

- Taking derivatives of the master integrals in respect to the kinematic invariants
- Reduce result back to master integrals
- Solve the obtained first order differential equation analytically

First order differential equation

$$\partial_x \vec{f}(x,\epsilon) = A(x,\epsilon)\vec{f}(x,\epsilon)$$

where ϵ is the dimensional regularization parameter $D = 4 - 2\epsilon$.

Properties of $A(x,\epsilon)$

- Block triangular
- Rational in x and ϵ
- Satisfies integrability condition (for at least two invariants x and y)

$$\partial_y A(x,\epsilon) - \partial_x A(y,\epsilon) + [A(x,\epsilon), A(y,\epsilon)] = 0$$

Bottom-up Approach

- Solve each line in $A(x,\epsilon)$ bottom up
- Previously solved integrals will appear as inhomogenous parts of the next DEQ

Matrix Approach Henn

- Conjecture: We can find a basis such that

$$\partial_x \vec{g}(x,\epsilon) = \epsilon\tilde{A}(x)\vec{g}(x,\epsilon)$$

- Makes integration simple
- But finding the basis is difficult

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Change of Basis

$$\begin{aligned}\vec{g}(x,\epsilon) &= B(x,\epsilon)\vec{f}(x,\epsilon)\\ \hat{A}(x,\epsilon) &= B^{-1}(x,\epsilon)A(x,\epsilon)B(x,\epsilon) - B^{-1}(x,\epsilon)\partial_x B(x,\epsilon)\end{aligned}$$

Reformulate the Problem

Find a change of basis which brings us to the canonical form
$\Rightarrow$ Solving a DEQ for $B(x,\epsilon)$

$$\partial_x B(x,\epsilon) = A(x,\epsilon)B(x,\epsilon) - \epsilon B(x,\epsilon)\hat{A}(x)$$

Can be as hard as initial problem

Assumption — Ageri, Di Vita, Mastrolia, Mirabella, Schlenk, Tancredi, **U.S.**

Linear DEQ

$$A(x,\epsilon) = A_0(x) + \epsilon A_1(x)$$

We have to solve

$$\partial_x B(x) = A_0(x)B(x)$$

Can be done with Magnus Theorem

Magnus theorem

Starting from a first order differential equation

$$\partial_x \vec{f}(x) = A(x)\vec{f}(x)$$

The solution is given by the Magnus exponential

$$\vec{f}(x) = e^{\Omega[A](x,x_0)}\vec{f}(x_0) \equiv e^{\Omega[A](x)}\vec{f}(x_0) \qquad \Omega[A](x) = \sum_{n=1}^{\infty} \Omega_n[A](x)$$

$$\Omega_1[A](x) = \int_{x_0}^{x} d\tau_1 A(\tau_1)$$

$$\Omega_2[A](x) = \frac{1}{2}\int_{x_0}^{x} d\tau_1 \int_{x_0}^{\tau_1} d\tau_2 [A(\tau_1), A(\tau_2)] \;, \ldots$$

Connected to the Dyson Series

$$\begin{aligned}
\vec{f}(x) &= \left(1 + \sum_{n=1}^{\infty} P_n(x)\right)\vec{f}(x_0) \;, \; P_n(x) = \int_{x_0}^{x} d\tau_1 \ldots \int_{x_0}^{\tau_{n-1}} d\tau_n \; A(\tau_1)A(\tau_2)\ldots A(\tau_n) \\
P_1(x) &= \Omega_1(x) \\
P_2(x) &= \Omega_2(x) + \frac{1}{2}\Omega_1^2(x) \;, \ldots.
\end{aligned}$$

Finding the canonical form with Magnus

From a linear DEQ

$$A(x,\epsilon) = A_0(x) + \epsilon A_1(x)$$

Magnus Theorem provides a basis change to the canonical form

$$B(x) = e^{\Omega[A_0](x)} \qquad \vec{g}(x,\epsilon) = B(x)\vec{f}(x,\epsilon)$$

Reducing DEQ — Lee

DEQ is rational in x

Mosers algorithm reduces all poles to simple poles (if possible)

$$A(x,\epsilon) = \sum_{n=1}^{k} \frac{\mathcal{S}_n(\epsilon)}{x - x_n} \qquad \text{with } x_n \text{ being constant}$$

Referred to as Fuchsian form

If Eigenvalues of $\mathcal{S}_n$ are linear in ϵ with integer coefficients
We can shift them to multiples of ϵ

A similarity transformation $\epsilon\tilde{\mathcal{S}} = T^{-1}(\epsilon)\mathcal{S}(\epsilon)T(\epsilon)$ gives us the canonical form

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Solving DEQ in the matrix approach

The canonical form

$$\partial_x g(x,\epsilon) = \epsilon \tilde{A}(x) g(x,\epsilon)$$

implies

$$\vec{g}(x,\epsilon) = \left(1 + \sum_{n=1}^{\infty} \epsilon^n P_n(x)\right) \vec{g}(x_0,\epsilon) \quad P_n(x) = \int_{x_0}^{x} d\tau_1 \ldots \int_{x_0}^{\tau_{n-1}} d\tau_n \, \tilde{A}(\tau_1)\tilde{A}(\tau_2)\ldots\tilde{A}(\tau_n)$$

- $\tilde{A}(x)$ determines the types of functions which will appear
- If $\tilde{A}(x)$ is Fuchsian

$$\tilde{A}(x) = \sum_{n=1}^{k} \frac{\mathcal{S}_n}{x - x_n} \qquad \text{with } x_n \text{ being constant}$$

then $g(x,\epsilon)$ can be written in terms of Goncharov Polylogarithms

Goncharov Polylogarithm

Goncharov

Basic definition

$$
\begin{aligned}
G(a_1; x) &= \int_0^x \frac{dt}{t - a_1} \\
G(a_1, \ldots, a_n; x) &= \int_0^x \frac{dt}{t - a_1} G(a_2, \ldots, a_n ; t) \\
G(\vec{0}_n; x) &= \frac{1}{n!} \log^n(x)
\end{aligned}
$$

n is the weight of the Polylog

Connections to other functions:

- Logarithm: $G(a_1, \ldots, a_1; x) = \frac{1}{n!} \log^n(1 - \frac{x}{a_1})$
- classical Polylog $G(\vec{0}_n, 1; x) = -\mathrm{Li}_n(x)$
- Harmonic Polylogs (HPLs) Remiddi, Vermaseren
  $G(\vec{a}; x) = (-1)^p H(\vec{a}, x)$ with $a_i \in \{-1, 0, 1\}$
- Two-dimensional harmonic polylogarithms Gehrmann, Remiddi
  $G(\vec{a}; x) = (-1)^p H(\vec{a}, x)$ with $a_i \in \{0, 1, -y, 1 - y\}$

Properties

Invariant under rescaling

for $k \in \mathbb{C}$

$$G(k\vec{a};\ k\,x) = G(\vec{a}; x)$$

Shuffle relations

$$G(\vec{a};\ x)G(\vec{b};\ x) = \sum_{\vec{c}\in\vec{a}\sqcup\!\sqcup\vec{b}} G(\vec{c};\ x)$$

where $\vec{a} \sqcup\!\sqcup \vec{b}$ means taking every permutation of the elements in $\vec{a}$ and $\vec{b}$ such that the individual ordering of $\vec{a}$ and $\vec{b}$ is kept

e.g. $G(a_1;\ x)G(b_1, b_2;\ x) = G(a_1, b_1, b_2;\ x) + G(b_1, a_1, b_2;\ x) + G(b_1, b_2, a_1;\ x)$

Many more known/unknown relations
$\Rightarrow$ Led to development of Symbol and Coproduct

Goncharov, Spradlin, Vergu, Volovich; Duhr

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Recap: Solution of the canonical DEQ

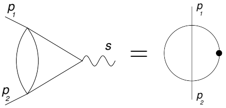
$$\vec{g}(x,\epsilon) = \left(1 + \sum_{n=1}^{\infty} \epsilon^n P_n(x)\right) \vec{g}(x_0,\epsilon)$$

We can obtain $\vec{g}(x_0,\epsilon)$ by

Known limits

- Taking the limit $x \to x_0$ to a known function
- Fix the boundary constant by matching the solution to the known function

$\lim_{s \to 0}$



Pseudo-thresholds

- Solution has physical and unphysical divergences
- Physical divergences correspond to particle thresholds
- Unphysical divergences are absent in the final result
- Demanding their absence gives relations between the boundary constants of the master integrals
- Leftover constants must be provided (usually elementary integrals)

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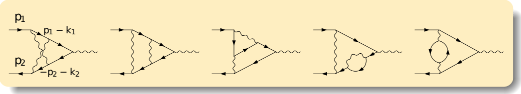
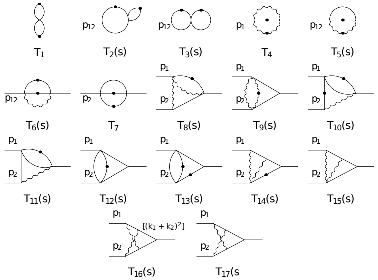
massive QED vertex at two-loop

Bonciani, Mastrolia, Remiddi

17 MI's for all relevant topologies



$$-\frac{s}{m^2} = \frac{(1-x)^2}{x}$$



The f's obey an ϵ-linear DE [ADVMMSST '14]

$$\begin{array}{lllll} f_1 = \epsilon^2 T_1 & f_2 = \epsilon^2 T_2 & f_3 = \epsilon^2 T_3 & f_4 = \epsilon^2 T_4 & f_5 = \epsilon^2 T_5 \\ f_6 = \epsilon^2 T_6 & f_7 = \epsilon^2 T_7 & f_8 = \epsilon^3 T_8 & f_9 = \epsilon^3 T_9 & f_{10} = \epsilon^2 T_{10} \\ f_{11} = \epsilon^3 T_{11} & f_{12} = \epsilon^3 T_{12} & f_{13} = \epsilon^2 T_{13} & f_{14} = \epsilon^3 T_{14} & f_{15} = \epsilon^4 T_{15} \\ f_{16} = \epsilon^4 T_{16} & f_{17} = \epsilon^4 T_{17} & & & \end{array}$$

After getting rid of A_0, the g's obey a canonical DE

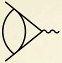
$$\partial_x g(\epsilon, x) = \epsilon\, \hat{A}_1(x)\, g(\epsilon, x) \qquad \hat{A}_1(x) = \frac{M_1}{x} + \frac{M_2}{1+x} + \frac{M_3}{1-x}$$

$$M_1 = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&2&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
-1&0&0&0&5&-6&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&2&-2&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
-1&0&0&-4&0&-2&0&-2&0&0&0&0&0&0&0&0&0\\
0&0&0&-2&0&0&0&0&2&0&0&0&0&0&0&0&0\\
-\frac{1}{2}&0&0&0&1&-2&-3&0&0&3&3&0&0&0&0&0&0\\
0&0&0&0&1&-1&2&0&0&-2&-2&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&-1&-1&0&0&0&0\\
0&-1&0&0&0&0&-3&0&0&0&0&3&3&0&0&0&0\\
0&-1&0&0&1&-\frac{1}{2}&0&2&2&0&0&0&0&2&2&0&0\\
0&0&0&0&0&\frac{1}{2}&0&-\frac{1}{2}&0&0&0&0&0&-1&-1&0&0\\
-\frac{1}{2}&0&0&-2&-1&0&-2&1&0&2&0&-2&0&0&-2&-2&2\\
0&0&0&0&-1&\frac{1}{2}&0&3&-2&0&-6&-2&0&0&-4&-4&4
\end{pmatrix}$$

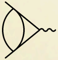
$$M_2 = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&-2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&-4&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&-6&3&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-4&0&0&0&0&0&0&0&0\\
0&0&0&0&-1&\frac{1}{2}&0&0&0&-4&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0\\
0&0&0&0&0&0&-6&0&0&0&0&0&-2&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&-4&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-4
\end{pmatrix}$$

$$M_3 = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&2&-2&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&-2&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&-2&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-6&0&0&2&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&-2&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-12&0&0&0&0&0&4&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&2&0&0&0&-4&0&0&-4&-2&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&4
\end{pmatrix}$$

Bonciani, Remiddi, **P.M.** (2013)



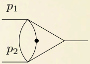
$$
\begin{aligned}
M_{-2} &= \frac{1}{2}\,,\\
M_{-1} &= \frac{5}{2} - \left[1 - \frac{2}{(1-x)}\right] H(0,x)\,,\\
M_0 &= \frac{19}{2} + \zeta(2) + \left[1 - \frac{2}{(1-x)}\right][\zeta(2) - 5H(0,x) + 2H(-1,0,x)]\\
&\quad + \frac{2}{(1-x)} H(0,0,x) + \left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right][\zeta(2)H(0,x)\\
&\quad + H(0,0,0,x)]\,.
\end{aligned}
$$



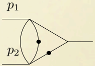
$(p_2 \cdot k_1)$

$$
\begin{aligned}
\frac{N_{-2}}{a} &= \frac{1}{8} + \frac{1}{16}\left[x + \frac{1}{x}\right],\\
\frac{N_{-1}}{a} &= \frac{9}{32}\left[2 + x + \frac{1}{x}\right] - \frac{1}{8}\left[4 + x - \frac{1}{x}\right] H(0,x) + \frac{1}{(1-x)} H(0,x)\,,\\
\frac{N_0}{a} &= \frac{63}{32} + \frac{\zeta(2)}{2} + \frac{63}{64}\left[\left(1 + \frac{16}{63}\zeta(2)\right) x + \frac{1}{x}\right] - \frac{\zeta(2)}{(1-x)} - \frac{1}{16}\left[32 + 9x\right.\\
&\quad \left. - \frac{9}{x}\right] H(0,x) + \frac{(16+\zeta(2))}{4(1-x)} H(0,x) - \frac{\zeta(2)}{4(1+x)} H(0,x) - \frac{1}{4}\left[2 - \frac{1}{x}\right.\\
&\quad \left. - \frac{4}{(1-x)}\right] H(0,0,x) + \frac{1}{4}\left[4 + x - \frac{1}{x} - \frac{8}{(1-x)}\right] H(-1,0,x)\\
&\quad + \frac{1}{4}\left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right] H(0,0,0,x)\,.
\end{aligned}
$$

AdVMMSST (2014)



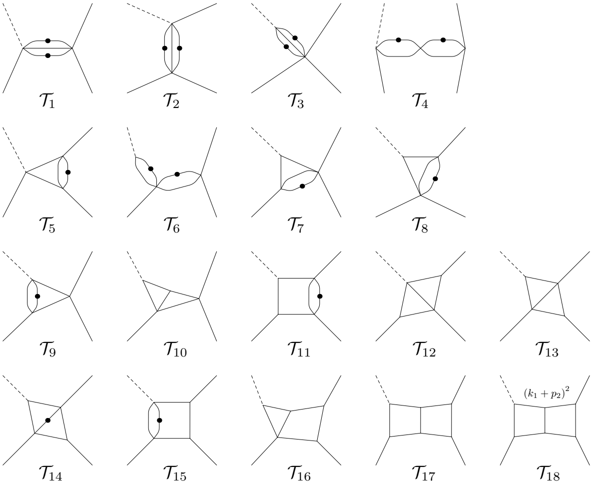
$$
\begin{aligned}
g_{12}^{(0)} &= 0\,,\\
g_{12}^{(1)} &= 0\,,\\
g_{12}^{(2)} &= 0\,,\\
g_{12}^{(3)} &= -\,\mathrm{H}(0,0,0;x) - \zeta_2 \mathrm{H}(0;x)\,,\\
g_{12}^{(4)} &= -2\,\mathrm{H}(-1,0,0,0;x) + 2\,\mathrm{H}(0,-1,0,0;x) + 2\,\mathrm{H}(0,0,-1,0;x)\\
&\quad - 3\,\mathrm{H}(0,0,0,0;x) - 4\,\mathrm{H}(0,1,0,0;x) + \zeta_2(-2\,\mathrm{H}(-1,0;x)\\
&\quad + 6\,\mathrm{H}(0,-1;x) - \mathrm{H}(0,0;x)) + 2\,\zeta_3\,\mathrm{H}(0;x) + \frac{\zeta_4}{4}\,,
\end{aligned}
$$



$$
\begin{aligned}
g_{13}^{(0)} &= 0\,,\\
g_{13}^{(1)} &= 0\,,\\
g_{13}^{(2)} &= \mathrm{H}(0,0;x) + \frac{3\,\zeta_2}{2}\,,\\
g_{13}^{(3)} &= -2\,\mathrm{H}(-1,0,0;x) - 2\,\mathrm{H}(0,-1,0;x) + 4\,\mathrm{H}(0,0,0;x) + 4\,\mathrm{H}(1,0,0;x)\\
&\quad + \zeta_2(-6\,\mathrm{H}(-1;x) + 2\,\mathrm{H}(0;x) - 3\log 2) - \frac{\zeta_3}{4}\,,\\
g_{13}^{(4)} &= 4\,\mathrm{H}(-1,-1,0,0;x) + 4\,\mathrm{H}(-1,0,-1,0;x) - 8\,\mathrm{H}(-1,0,0,0;x)\\
&\quad - 8\,\mathrm{H}(-1,1,0,0;x) + 4\,\mathrm{H}(0,-1,-1,0;x) - 8\,\mathrm{H}(0,-1,0,0;x)\\
&\quad - 8\,\mathrm{H}(0,0,-1,0;x) + 10\,\mathrm{H}(0,0,0,0;x) + 12\,\mathrm{H}(0,1,0,0;x)\\
&\quad - 8\,\mathrm{H}(1,-1,0,0;x) - 8\,\mathrm{H}(1,0,-1,0;x) + 16\,\mathrm{H}(1,0,0,0;x)\\
&\quad + 16\,\mathrm{H}(1,1,0,0;x) + 12\,\mathrm{Li}_4\frac{1}{2} + \frac{\log^4 2}{2} + 2\,\zeta_2\,(12\log 2\,\mathrm{H}(-1;x)\\
&\quad + 12\log 2\,\mathrm{H}(1;x) + 6\,\mathrm{H}(-1,-1;x) - 2\,\mathrm{H}(-1,0;x) - 8\,\mathrm{H}(0,-1;x)\\
&\quad + \mathrm{H}(0,0;x) - 12\,\mathrm{H}(1,-1;x) + 4\,\mathrm{H}(1,0;x) + 3\log^2 2)\\
&\quad - 2\,\zeta_3(5\,\mathrm{H}(-1;x) + 4\,\mathrm{H}(0;x) + 11\,\mathrm{H}(1;x)) - \frac{47\,\zeta_4}{4}\,,
\end{aligned}
$$

Higgs+Jet at two-loop

Gehrmann, Remiddi; Di Vita, Mastrolia, Yundin, **U.S.**



Higgs+Jet at two-loop

Gehrmann, Remiddi; Di Vita, Mastrolia, Yundin, **U.S.**

- Dimensionless variables

$$x = \frac{s}{m_h^2} \qquad y = \frac{t}{m_h^2}$$

- ϵ-linear basis

$$\partial_x \vec{f}(x,y,\epsilon) = (A_{1,0}(x,y) + \epsilon A_{1,1}(x,y))\,\vec{f}(x,y,\epsilon)$$
$$\partial_y \vec{f}(x,y,\epsilon) = (A_{2,0}(x,y) + \epsilon A_{2,1}(x,y))\,\vec{f}(x,y,\epsilon)$$

- Canonical form with Magnus

$$\partial_x \vec{g}(x,y,\epsilon) = \epsilon \hat{A}_1(x,y)\vec{g}(x,y,\epsilon)$$
$$\partial_y \vec{g}(x,y,\epsilon) = \epsilon \hat{A}_2(x,y)\vec{g}(x,y,\epsilon)$$

- Log-form

$$\hat{A}(x,y) = M_1\log(x) + M_2\log(1-x) + M_3\log(y) + M_4\log(1-y)$$
$$+M_5\log(\frac{x+y}{x}) + M_6\log(\frac{1-x-y}{1-x})$$

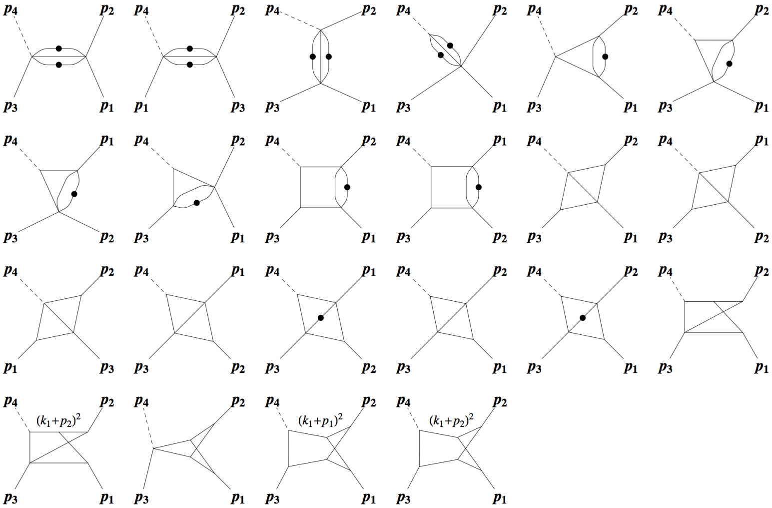
$$\partial_x \hat{A}(x,y) = \hat{A}_1(x,y) \qquad \partial_y \hat{A}(x,y) = \hat{A}_2(x,y)$$

- Alphabet

$$\{x, 1-x, y, 1-y, x+y, 1-x-y\}$$

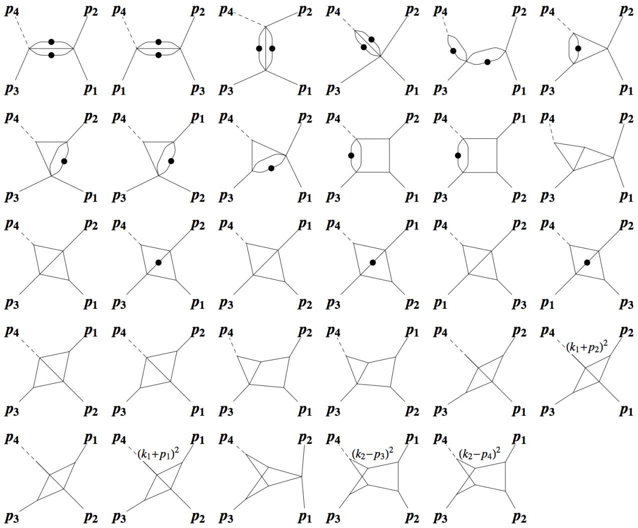
Higgs+Jet at two-loop

Gehrmann, Remiddi; Di Vita, Mastrolia, Yundin, **U.S.**



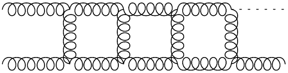
Higgs+Jet at two-loop

Gehrmann, Remiddi; Di Vita, Mastrolia, Yundin, **U.S.**

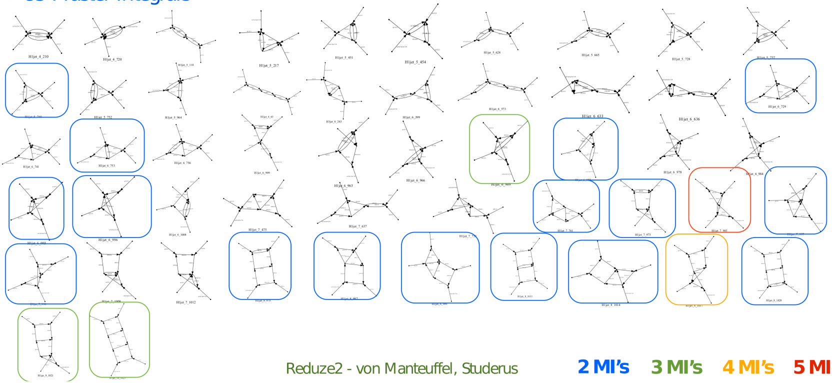


Higgs+Jet at three-loop

Di Vita, Mastrolia, Yundin, **U.S.**



- 85 Master Integrals



Reduze2 - von Manteuffel, Studerus

2 MI's **3 MI's** **4 MI's** **5 MI's**

taken from Pierpaolo's slides at Amplitudes 2014

Higgs+Jet at three-loop

Di Vita, Mastrolia, Yundin, **U.S.**

- ϵ-linear basis

$$\partial_x \vec{f}(x,y,\epsilon) = \left(A_{1,0}(x,y) + \epsilon A_{1,1}(x,y)\right) \vec{f}(x,y,\epsilon)$$
$$\partial_y \vec{f}(x,y,\epsilon) = \left(A_{2,0}(x,y) + \epsilon A_{2,1}(x,y)\right) \vec{f}(x,y,\epsilon)$$

- Canonical form with Magnus

$$\partial_x \vec{g}(x,y,\epsilon) = \epsilon \hat{A}_1(x,y) \vec{g}(x,y,\epsilon)$$
$$\partial_y \vec{g}(x,y,\epsilon) = \epsilon \hat{A}_2(x,y) \vec{g}(x,y,\epsilon)$$

- Log-form

$$\hat{A}(x,y) = M_1 \log(x) + M_2 \log(1-x) + M_3 \log(y) + M_4 \log(1-y)$$
$$+ M_5 \log\left(\frac{x+y}{x}\right) + M_6 \log\left(\frac{1-x-y}{1-x}\right)$$
$$\partial_x \hat{A}(x,y) = \hat{A}_1(x,y) \qquad \partial_y \hat{A}(x,y) = \hat{A}_2(x,y)$$

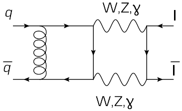
- Alphabet

$$\{x, 1-x, y, 1-y, x+y, 1-x-y\}$$

same as in the two-loop case!

Two-loop Correction to Drell-Yan

Bonciani, Di Vita, Mastrolia, U.S.



Three different classes of processes with zero,one and two equal massive internal legs
Master integrals for the latter two are still unknown

- Dimensionless variables

$$x = \frac{s}{m^2} \qquad\qquad y = \frac{t}{m^2}$$

- ϵ-linear basis

$$\partial_x \vec{f}(x,y,\epsilon) = \left(A_{1,0}(x,y) + \epsilon A_{1,1}(x,y)\right)\vec{f}(x,y,\epsilon)$$
$$\partial_y \vec{f}(x,y,\epsilon) = \left(A_{2,0}(x,y) + \epsilon A_{2,1}(x,y)\right)\vec{f}(x,y,\epsilon)$$

- Canonical form with Magnus

$$\partial_x \vec{g}(x,y,\epsilon) = \epsilon \hat{A}_1(x,y)\vec{g}(x,y,\epsilon)$$
$$\partial_y \vec{g}(x,y,\epsilon) = \epsilon \hat{A}_2(x,y)\vec{g}(x,y,\epsilon)$$

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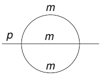
Two-loop equal-mass sunrise



DEQ is known to evaluate to elliptic functions Laporta, Remiddi (2004)

Easy to find a DEQ $\partial_s \vec{f} = A(s,\epsilon)\vec{f}$ which is Fuchsian and linear in $\epsilon = \frac{(D-4)}{2}$

$$A(s,\epsilon) = \frac{1}{s}\begin{pmatrix} 0 & 0 & 0 & 0 \\ -\epsilon & -1+\epsilon & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}-\epsilon & -1+\epsilon \end{pmatrix} + \frac{1}{s-m^2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ \epsilon & -2\epsilon & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{\epsilon}{2} & \frac{3}{4}+\frac{3}{2}\epsilon & -1+2\epsilon \end{pmatrix} + \frac{1}{s-9m^2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2\epsilon & -1-2\epsilon & 4 \\ 0 & \frac{\epsilon}{2} & \frac{1}{4}+\frac{1}{2}\epsilon & 1 \end{pmatrix}$$

$f^T = \Big($  ,  ,  ,  $\Big)$

But: Magnus Series does not converge

Eigenvalues are linear but transformation is not possible

$\Rightarrow \epsilon^0$-part is related to elliptic functions

Conclusion

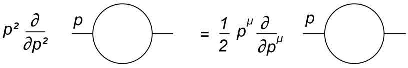
- Combined Integrand-Reduction, Unitarity method and Color-Kinematic duality
- Symmetries constrain the number scalar products appearing in the residue
  $\Rightarrow$ Reduces the number of possible master integrals
- Extracted the leading ultra-violet divergence for the amplitude
  Connected to leading UV divergence of $\mathcal{N} = 8$ SUGRA
- Provided an alternative derivation of the BCJ conform numerators which only depends on geometric constraints of the amplitude (see thesis)

- Magnus Series finds a canonical basis for a DEQ which is linear in ϵ
- Has been applied to a variety of known and unknown processes
  -QED Vertex at two-loop
  -$2 \to 2$ non-planar massless box
  -Higgs+jet at two-loop
  -Ladder topology for Higgs+jet at three-loop
  -Mixed NNLO corrections to Drell-Yan

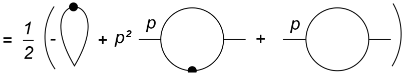
Back up slides

The one-loop massless bubble

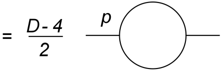
- Construct differential operator



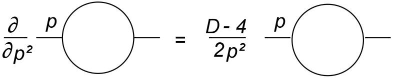
- Apply derivative



- Reduce back to master integrals with IBP-ids

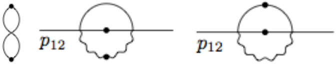


- Gives us the differential equation



Example: massive QED Sunrise

- Three master integrals



- Dimensionless variable

$$-\frac{s}{m^2} = \frac{(1-x)^2}{x}$$

- Linear and Fuchsian DEQ

$$\partial_x \vec{f}(x) = A(x,\epsilon)\vec{f}(x)$$

$$A(x,\epsilon) = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{\epsilon}{2(1-x)} - \frac{\epsilon}{2(1+x)} & -\frac{1+6\epsilon}{1+x} + \frac{1+3\epsilon}{x} + \frac{1}{1-x} & -\frac{1-2\epsilon}{2(1-x)} - \frac{\epsilon}{x} - \frac{1-6\epsilon}{2(1+x)} \\ 0 & \frac{2\epsilon}{x} + \frac{4\epsilon}{1-x} & \frac{1}{x} + \frac{2}{1-x} \end{pmatrix}$$

Let's apply Magnus and Lees algorithm

QED Sunrise with Magnus

Split ϵ^0-part into diagonal and off-diagonal part and transform with diagonal part first

$$A_0(x) = D_0(x) + N_0(x), \qquad B_1 = e^{\int dx\, D_0(x)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{x}{1-x^2} & 0 \\ 0 & 0 & \frac{x}{(1-x)^2} \end{pmatrix}$$

Transformed DEQ

$$\hat{A}(x) = B_1^{-1} A(x) B_1 - B_1^{-1} \partial_x B_1 = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{\epsilon}{x} & \frac{3\epsilon}{x} - \frac{6\epsilon}{1+x} & -\frac{1-2\epsilon}{(x-1)^2} - \frac{\epsilon}{x} \\ 0 & \frac{2\epsilon}{x} & 0 \end{pmatrix}$$

QED Sunrise with Magnus

Transform with off-diagonal part

$$\hat{A}_0(x) = \hat{N}_0(x), \qquad B_2 = e^{\int dx\, \hat{N}_0(x)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{1-x} \\ 0 & 0 & 1 \end{pmatrix}$$

Transformed DEQ II

$$\tilde{A}(x) = B_2^{-1}\hat{A}(x)B_2 - B_2^{-1}\partial_x B_2$$

$$\tilde{A}(x) = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{\epsilon}{x} & \frac{2\epsilon}{1-x} + \frac{5\epsilon}{x} - \frac{6\epsilon}{1+x} & -\frac{2\epsilon}{1-x} - \frac{6\epsilon}{x} + \frac{3\epsilon}{1+x} \\ 0 & \frac{2\epsilon}{x} & -\frac{2\epsilon}{1-x} - \frac{2\epsilon}{x} \end{pmatrix}$$

Masters for canonical basis are given by

$$\vec{g} = B_2^{-1}B_1^{-1}\vec{f} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{(1-x)(x+1)}{x} & \frac{1-x}{x} \\ 0 & 0 & \frac{(x-1)^2}{x} \end{pmatrix} \vec{f}$$

QED Sunrise with Lee

Focus only on 2x2 system of the Sunrise, since the rest of the system is already ϵ-factorized
System is Fuchsian
$\Rightarrow$ Shift all Eigenvalues to multiples of ϵ

$$x \to 1 : \{-1-2\epsilon, -2+2\epsilon\} \quad x \to 0 : \{1+\epsilon, 1+2\epsilon\}$$
$$x \to -1 : \{0, -1-6\epsilon\} \quad x \to \infty : \{1+\epsilon, 1+2\epsilon\}$$

A balance transformation will shift one eigenvalue by $+1$ the other one by -1

$$\mathcal{B}(\mathbb{P}, x_1, x_2|x) = \bar{\mathbb{P}} + c\frac{x-x_2}{x-x_1}\mathbb{P}, \quad \mathbb{P} = u(\lambda_1)v^{\dagger}(\lambda_2), \quad \bar{\mathbb{P}} = 1-\mathbb{P}$$

After four transformation we brought all Eigenvalues in the right form

$$x \to 1 : \{-2\epsilon, 2\epsilon\} \quad x \to 0 : \{\epsilon, 2\epsilon\}$$
$$x \to -1 : \{0, -6\epsilon\} \quad x \to \infty : \{\epsilon, 2\epsilon\}$$

QED Sunrise with Lee

The last step is to find a similarity transformation which brings us to the canonical form

For each pole matrix we solve

$$\frac{A(x,\epsilon)}{\epsilon}T(\epsilon,\mu) = T(\epsilon,\mu)\frac{A(x,\mu)}{\mu}$$

Finally we arrive at

$$\partial_x \vec{g} = \epsilon \begin{pmatrix} \frac{5}{x} - \frac{6}{1+x} + \frac{2}{1-x} & \frac{6}{x} - \frac{3}{1+x} + \frac{2}{1-x} \\ -\frac{2}{x} & -\frac{2}{1-x} - \frac{2}{x} \end{pmatrix} \vec{g}$$

Combining all transformation we find for our masters

$$\vec{g} = \begin{pmatrix} \frac{(1-x)(1+x)}{x} & \frac{1-x}{x} \\ 0 & -\frac{(x-1)^2}{x} \end{pmatrix} \vec{f} \quad \text{Reminder:} \quad B_{\text{Magnus}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{(1-x)(x+1)}{x} & \frac{1-x}{x} \\ 0 & 0 & \frac{(x-1)^2}{x} \end{pmatrix}$$