Counting surprises in a 1-d effective field theory

Tom Melia, LBNL

work to appear with Brian Henning, Xiaochuan Lu, Hitoshi Murayama

Wilsonian picture of qft

$$S_{eff} = \int \sum_a c_a \mathcal{O}_a$$

Specify the symmetries and particle content of your theory, and then include all possible invariants

Wilsonian picture of qft

$$S_{eff} = \int \sum_a c_a \mathcal{O}_a$$

E.g. SM EFT: SM content & gauge sym.

Important precision programme
at the HL-LHC

Wilsonian picture of qft

$$S_{eff} = \int \sum_a c_a \mathcal{O}_a$$

Even understanding bases at low expansion orders is a difficult problem

Weinberg '79 Dim 5

Buchmuller, Wyler 86; Grzadkowski et al. '10 minimal Dim 6

Lehman, '14 minimal Dim 7

Structure of SM EFT- recent work includes (`13-`15)
e.g. Grojean, Jenkins, Manohar, Trott, Alonso, Elias-Mir,
Espinosa, Masso, Pomarol, Shotwell, Chang, Henning, Lu, Murayama...and more

'Holomorphy' Alonso, Jenkins, Manohar, '14

Information about bases is encoded in Hilbert series - important objects in invariant theory

Lepton flavour invariants

Jenkins, Manohar ’09; Hanany, Jenkins, Manohar, Torri ’11

Emphasizing use in pheno, constructing BSM lagrangians

Lehman, Martin ’15

powerful tool

operators with no derivatives

Information about bases is encoded in Hilbert series - important objects in invariant theory

Well developed in more formal physics literature

Operator spectra and moduli space of SUSY theories

'The Plethystic Program', including contrubtions from:

Hanany, Feng, Benvenuti, He, Dolan, Gray, Hanany, He, Jejjala, Mekareeya, Forchella, Chen, Butti, Vegh, Zaffaroni, Razamat, Rodrugez-Gmez, Zafrir, Seong...and more

All order information

Part I

Introduce the Hilbert series

Demonstrate it's power modulo 'derivative issues'

Part II

(0+1)-D EFT to study 'DIs'

Part III

Conjecture and speculate on all-orders result

Part I

Introduce the Hilbert series

Demonstrate it's power modulo 'derivative issues'

Example: two complex scalar fields charged under U(1)

$$\begin{array}{cccc} \phi_1 & \phi_1^* & \phi_2 & \phi_2^* \\ +1 & -1 & +2 & -2 \end{array} \quad Q:$$

all possible terms invariant under U(1)

Use building blocks:

$$I_1 = \phi_1 \phi_1^* \qquad I_3 = \phi_1^2 \phi_2^*$$

$$I_2 = \phi_2 \phi_2^* \qquad I_4 = (\phi_1^*)^2 \phi_2$$

Relation $I_1^2 \, I_2 = I_3 \, I_4$

treating Is as (complex) numbers less than one

$$\frac{1}{(1-I_1)(1-I_2)(1-I_3)(1-I_4)}$$

generates all possible terms. What if we want all independent terms?

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To account for $I_1^2\, I_2 = I_3\, I_4$ we need

$$\frac{1-I_1^2 I_2}{(1-I_1)(1-I_2)(1-I_3)(1-I_4)} = \frac{1-q_1^2 q_2^2 q_3 q_4}{(1-q_1 q_2)(1-q_3 q_4)(1-q_1^2 q_3)(1-q_2^2 q_4)}$$

using

$$\begin{array}{ll} \phi_1 \to q_1 & \phi_3 \to q_3 \\ \phi_2 \to q_2 & \phi_4 \to q_4 \end{array}$$

$$\begin{array}{ll} I_1 = \phi_1 \phi_1^* & I_3 = \phi_1^2 \phi_2^* \\ I_2 = \phi_2 \phi_2^* & I_4 = (\phi_1^*)^2 \phi_2 \end{array}$$

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$$= \frac{1-q^6}{(1-q^2)(1-q^2)(1-q^3)(1-q^3)}$$

$$q_1, q_2, q_3, q_4 \to q$$

$$\phi_1 \to q_1 \quad \phi_3 \to q_3$$
$$\phi_2 \to q_2 \quad \phi_4 \to q_4$$

$$I_1 = \phi_1 \phi_1^* \qquad I_3 = \phi_1^2 \phi_2^*$$

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$$I_1 = \phi_1 \phi_1^* \qquad I_3 = \phi_1^2 \phi_2^*$$

$$1 + 2q^2 + 2q^3 + 3q^4 + 4q^5 + 6q^6 + 6q^7 + \ldots \qquad I_2 = \phi_2 \phi_2^* \qquad I_4 = (\phi_1^*)^2 \phi_2$$

$$G: \quad U(1)$$

A group G

$$\begin{array}{cccc} & \phi_1 & \phi_1^* & \phi_2 & \phi_2^* \\ Q: & +1 & -1 & +2 & -2 \end{array}$$

A set of variables which transform under the action of G

$$I_1 = \phi_1 \phi_1^* \qquad I_3 = \phi_1^2 \phi_2^*$$

$$I_2 = \phi_2 \phi_2^* \qquad I_4 = (\phi_1^*)^2 \phi_2$$

Generators

$$I_1^2 I_2 - I_3 I_4 = 0$$

Relation between generators

$$\frac{1 - q_1^2 q_2^2 q_3 q_4}{(1 - q_1 q_2)(1 - q_3 q_4)(1 - q_1^2 q_3)(1 - q_2^2 q_4)}$$

Hilbert series

Two algebra concepts

A *commutative ring* is like a field but without a division operation

If R is a commutative ring, a subset $I \subset R$ is an *ideal* if

$$0 \in I$$
$$\text{If } a, b \in I, \text{ then } a + b \in I$$
$$\text{If } a \in I, \text{ and } b \in R \text{ then } b \cdot a \in I$$

Four Point Algebra re-phrasing

1 $k[\phi_1, \phi_1^*, \phi_2, \phi_2^*]$ all polynomials in fields: *polynomial ring*

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3 $R_I = k[I_1, I_2, I_3, I_4]$ $\qquad I_1^2 I_2 - I_3 I_4 = 0$

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define the ideal:

$$\langle I_1^2 I_2 - I_3 I_4 \rangle = \{ h \cdot (I_1^2 I_2 - I_3 I_4) \ \text{ for all } h \in R_I \}$$

(all possible polynomial consequences of relation)

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quotient ring: $R_I / \langle I_1^2 I_2 - I_3 I_4 \rangle$

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quotient ring: $R_I / \langle I_1^2 I_2 - I_3 I_4 \rangle$

4 Hilbert series: $HS(R_I / \langle I_1^2 I_2 - I_3 I_4 \rangle)$

$$G: \quad U(1)$$

A group G

$$\begin{array}{cccc} & \phi_1 & \phi_1^* & \phi_2 & \phi_2^* \\ Q: & +1 & -1 & +2 & -2 \end{array}$$

A set of variables which transform under the action of G

$$I_1 = \phi_1 \phi_1^* \qquad I_3 = \phi_1^2 \phi_2^*$$

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Hilbert series

$G: \quad U(1)$

A group G

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$Q:$

A set of variables which transform under the action of G

ideally...

$$I_1 = \phi_1 \phi_1^* \qquad I_3 = \phi_1^2 \phi_2^*$$

$$I_2 = \phi_2 \phi_2^* \qquad I_4 = (\phi_1^*)^2 \phi_2$$

Generators

$$I_3^2 I_2 - I_1 I_4 = 0$$

Relation between generators

$$\frac{1 - q_1^2 q_2^2 q_3 q_4}{(1 - q_1 q_2)(1 - q_3 q_4)(1 - q_1^2 q_3)(1 - q_2^2 q_4)}$$

Hilbert series

$$G: \quad U(1)$$

$$\begin{array}{cccc} \phi_1 & \phi_1^* & \phi_2 & \phi_2^* \\ +1 & -1 & +2 & -2 \end{array}$$

$Q:$

A group G

A set of variables which transform under the action of G

Molien formula: straight to the Hilbert series

$$\int d\mu_G \prod_i \frac{1}{\det(1 - q_i U_i(G))}$$

Haar measure

Variables

Representation in which variable i transforms under G

$G: \quad U(1)$

A group G

$\phi_1 \quad \phi_1^* \quad \phi_2 \quad \phi_2^*$

counts

$q_1 \quad q_2 \quad q_3 \quad q_4$

A set of variables which transform under the action of G

$$\int \frac{d\theta}{2\pi} \frac{1}{(1-q_1 e^{i\theta})(1-q_2 e^{-i\theta})(1-q_3 e^{2i\theta})(1-q_4 e^{-2i\theta})}$$

Haar measure

Repr.

change variables $\quad z = e^{i\theta}$

$G: \quad U(1)$

A group G

$\phi_1 \quad \phi_1^* \quad \phi_2 \quad \phi_2^*$

counts

$q_1 \quad q_2 \quad q_3 \quad q_4$

A set of variables which transform under the action of G

$$\int \frac{d\theta}{2\pi} \frac{1}{(1-q_1 e^{i\theta})(1-q_2 e^{-i\theta})(1-q_3 e^{2i\theta})(1-q_4 e^{-2i\theta})}$$

Haar measure

Repr.

$$\frac{1}{2\pi} \int \frac{dz}{z} \frac{1}{(1-q_1 z)(1-\frac{q_2}{z})(1-q_3 z^2)(1-\frac{q_4}{z^2})}$$

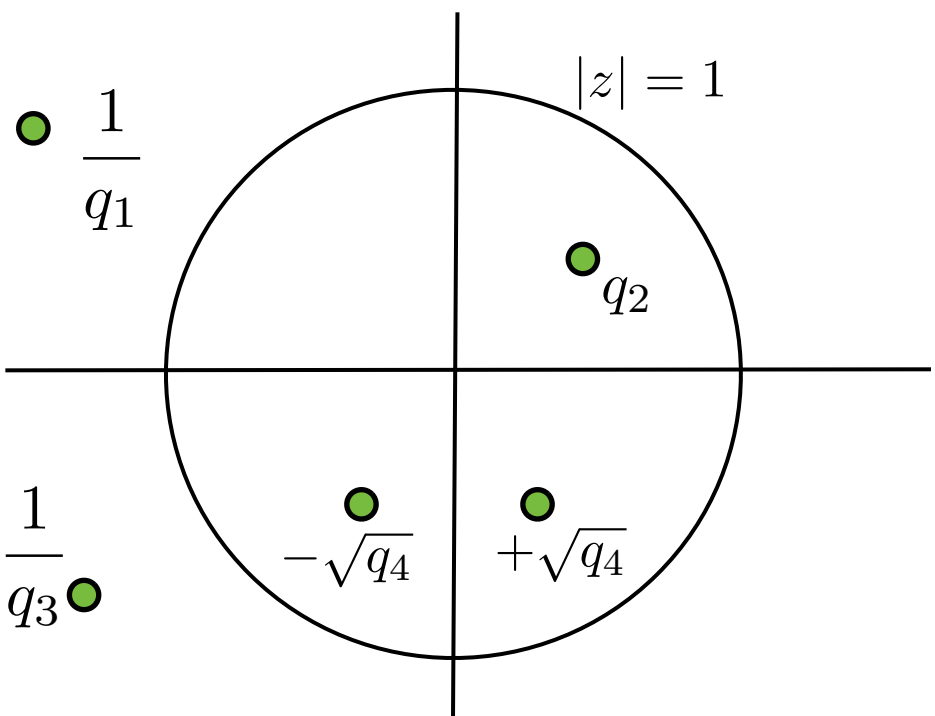
$G: \quad U(1)$

A group G

$$\begin{array}{cccc} \phi_1 & \phi_1^* & \phi_2 & \phi_2^* \\ q_1 & q_2 & q_3 & q_4 \end{array}$$

A set of variables which transform under the action of G

$$\frac{1}{2\pi}\int_{|z|=1} dz \frac{z^2}{q_1 q_3 (z - \frac{1}{q_1})(z - q_2)(z^2 - \frac{1}{q_3})(z^2 - q_4)}$$



Cauchy theorem sum of three residues

$$\frac{1 - q_1^2 q_2^2 q_3 q_4}{(1 - q_1 q_2)(1 - q_3 q_4)(1 - q_1^2 q_4)(1 - q_2^2 q_3)}$$

Molien formula

$$\int d\mu_G \prod_i \frac{1}{\det(1 - q_i U_i(G))}$$

see Lehman & Martin for a very nice review and examples

However, our monomials sit in a Lagrangian to be used in an action

$$S_{eff} = \int \sum_a c_a \mathcal{O}_a$$

‘Derivative issues’

1. Integration by parts relations

$$\mathcal{O}_a \simeq \mathcal{O}_b \quad \text{if} \quad \mathcal{O}_b = \mathcal{O}_a + \partial_\mu \mathcal{O}_c$$

2. Using classical eqs. of motion leaves S-matrix elements (physical observables) invariant

$$\mathcal{O}_a \simeq \mathcal{O}_b \quad \text{if} \quad \mathcal{O}_b = \mathcal{O}_a + \mathcal{O}_c \frac{\delta S_{free}}{\delta \psi_i}$$

Part II

(0+1)-D EFT to study ‘DIs'

(0+1)-dimensional QFT (=quantum mechanics)
N real scalar fields

$$\phi_1 \dots \phi_N$$

$$S_{eff} = \int_{\mathcal{M}} \sum_a c_a \mathcal{O}_A$$

polynomials in $\{\phi_i\}$ and their derivatives: $\phi, \partial\phi, \partial^2\phi \dots$

space-time manifold: circle or real line

$$S_{free} + S_{int}$$

$$S_{free} = \int_{\mathcal{M}} \sum_{i=1}^{N} \frac{1}{2}(\partial\phi_i)^2 - \frac{1}{2}m_i^2\phi_i^2$$

$$(\partial^2 + m_i^2)\phi_i = 0$$

We are interested in the function

$$H(t, \{u_i\}) = \sum_{k, r_1, .., r_N = 0}^{\infty} c_{k\, r_1 .. r_N} u_1^{r_1} \ldots u_N^{r_N} t^k$$

number of independent terms of the form

$$\phi_1^{r_1} \ldots \phi_N^{r_N} \partial^k$$

$\{u_i\}$ count $\{\phi_i\}$

t counts ∂

the k derivatives are understood to act in some unspecified way

We are interested in the function

$$H(t, \{u_i\}) = \sum_{k, r_1, .., r_N = 0}^{\infty} c_{k\, r_1 .. r_N} u_1^{r_1} \ldots u_N^{r_N} t^k$$

Cases:

(A) no IBP & no EoM easy

(B) IBP & no EoM easy

(C) no IBP & EoM easy

(D) IBP & EoM surprisingly hard for N>1!

We are interested in the function

$$H(t, \{u_i\}) = \sum_{k, r_1, .., r_N = 0}^{\infty} c_{k\, r_1 .. r_N} u_1^{r_1} \dots u_N^{r_N} t^k$$

One flavour, N=1

(A) no IBP & no EoM

$$H^{free}(t, u_1) = \frac{1}{(1-u_1)(1-tu_1)(1-t^2u_1)(1-t^3u_1)\dots}$$

$$c_{k\, r_1} = P_{r_1}(k)$$ partition of k into r1 parts

We are interested in the function

$$H(t, \{u_i\}) = \sum_{k, r_1, .., r_N = 0}^{\infty} c_{k\, r_1 .. r_N} u_1^{r_1} \dots u_N^{r_N} t^k$$

One flavour, N=1

(B) IBP & no EoM

$$\underbrace{\phi_1^{r_1} \partial^k}_{P_{r_1}(k)} \qquad \partial(\underbrace{\phi_1^{r_1} \partial^{k-1}}_{P_{r_1}(k-1)})$$

We are interested in the function

$$H(t,\{u_i\}) = \sum_{k,r_1,..,r_N=0}^{\infty} c_{k\,r_1..r_N} u_1^{r_1} \dots u_N^{r_N} t^k$$

One flavour, N=1

(B) IBP & no EoM

$$\underbrace{\phi_1^{r_1}\partial^k}_{P_{r_1}(k)} \qquad \partial(\underbrace{\phi_1^{r_1}\partial^{k-1}}_{P_{r_1}(k-1)})$$

$$c^{IBP}_{k\,r_1} = c^{free}_{k\,r_1} - c^{free}_{k-1\,r_1}$$

We are interested in the function

$$H(t,\{u_i\}) = \sum_{k,r_1,..,r_N=0}^{\infty} c_{k\,r_1..r_N} u_1^{r_1} \dots u_N^{r_N} t^k$$

One flavour, N=1

(B) IBP & no EoM

$$H^{IBP}(t,u_1) = \sum_{k,r_1=0}^{\infty} (c^{free}_{k\,r_1} - c^{free}_{k-1\,r_1}) u_1^{r_1} t^k$$

$$H^{IBP}(t,u_1) = H^{free}(t,u_1) - tH^{free}(t,u_1)$$

$$H^{IPB}(t,u_1) = \frac{1-t}{(1-u_1)(1-tu_1)(1-t^2u_1)(1-t^3u_1)\dots}$$

We are interested in the function

$$H(t, \{u_i\}) = \sum_{k, r_1, .., r_N = 0}^{\infty} c_{k\, r_1 .. r_N} u_1^{r_1} \dots u_N^{r_N} t^k$$

One flavour, N=1

(C) no IBP & EoM

$$H^{EoM}(t, u_1) = \frac{1}{(1 - u_1)(1 - t u_1)}$$

We are interested in the function

$$H(t, \{u_i\}) = \sum_{k, r_1, .., r_N = 0}^{\infty} c_{k\, r_1 .. r_N} u_1^{r_1} \dots u_N^{r_N} t^k$$

One flavour, N=1

(D) IBP & EoM

$$H^{IBP+EoM}(t, u_1) = \frac{1}{(1 - u_1)}$$

We are interested in the function

$$H(t,\{u_i\}) = \sum_{k,r_1,..,r_N=0}^{\infty} c_{k\,r_1..r_N} u_1^{r_1} \dots u_N^{r_N} t^k$$

	One flavour	N flavours
free	$\frac{1}{(1-u_1)(1-tu_1)(1-t^2u_1)\dots}$	$\frac{1}{\prod_{i=1}^{N}(1-u_i)(1-tu_i)(1-t^2u_i)\dots}$
IBP	$\frac{1-t}{(1-u_1)(1-tu_1)(1-t^2u_1)\dots}$	$\frac{1-t}{\prod_{i=1}^{N}(1-u_i)(1-tu_i)(1-t^2u_i)\dots}$
EoM	$\frac{1}{(1-u_1)(1-tu_1)}$	$\frac{1}{\prod_{i=1}^{N}(1-u_i)(1-tu_i)}$
IBP+ EoM	$\frac{1}{(1-u_1)}$	not trivial

We are interested in the function

$$H(t,\{u_i\}) = \sum_{k,r_1,..,r_N=0}^{\infty} c_{k\,r_1..r_N}\, u_1^{r_1}\dots u_N^{r_N}\, t^k$$

	One flavour	N flavours
free	$\frac{}{(1}$	$\frac{}{- t^2 u_i)\dots}$
IBP	$\frac{1-t}{(1-u_1)(1-tu_1)(1-t^2u_1)\dots}$	$\frac{1-t}{\prod_{i=1}^N(1-u_i)(1-tu_i)(1-t^2u_i)\dots}$
EoM	$\frac{1}{(1-u_1)(1-tu_1)}$	$\frac{1}{\prod_{i=1}^N(1-u_i)(1-tu_i)}$
IBP+ EoM	$\frac{1}{(1-u_1)}$	not trivial

Interlude: we just intuited the formulas. Prove them in a more rigorous fashion, and gain some physical and mathematical insight.

We are interested in the function

$$H(t, \{u_i\}) = \sum_{k, r_1, .., r_N = 0}^{\infty} c_{k\, r_1 .. r_N} u_1^{r_1} \ldots u_N^{r_N} t^k$$

fix the powers of {r_i} and define

$$H_{r_1 .. r_N}(t) = \sum_{k=0}^{\infty} c_{k\, r_1 .. r_N} t^k$$

so that

$$H(t, \{u_i\}) = \sum_{r_1, .., r_N = 0}^{\infty} u_1^{r_1} \ldots u_N^{r_N} H_{r_1 .. r_N}(t)$$

Back to the one flavour case

aim: $$H_r(t) = \sum_{k=0}^{\infty} c_{k,r} t^k$$

Move to momentum space

$$\phi(\theta) = \sum_{p=-\infty}^{\infty} c_p \, e^{i\,p\,\theta}$$

derivatives pull down powers of p

$$\partial^k \phi(\theta) = \sum_{p=-\infty}^{\infty} c_p \, p^k \, e^{i\,p\,\theta}$$

We find

$$\int_0^{2\pi} d\theta \phi^r \partial^k \sim \sum_{p_1,..p_r=-\infty}^{\infty} w(p_1,\ldots,p_r) c_{p_1} \ldots c_{p_r} \int_0^{2\pi} d\theta e^{i\theta(p_1+\ldots+p_r)}$$

One-to-one correspondence between term and degree k polynomials

$$\mathbb{R}[p_1,\ldots,p_r]^{S_r}$$

Power sum sym. polynomials:

$$P_n = \sum_{i=1}^{r} p_i^n$$

$$\mathbb{R}[p_1,\ldots,p_r]^{S_r} = \mathbb{R}[P_1,\ldots,P_r]$$

We find

$$\int_0^{2\pi} d\theta \phi^r \partial^k \sim \sum_{p_1,..p_r=-\infty}^{\infty} w(p_1,\ldots,p_r) c_{p_1} \ldots c_{p_r} \int_0^{2\pi} d\theta e^{i\theta(p_1+\ldots+p_r)}$$

(A) no IBP & no EoM

As its freely generated the Hilbert series is easy:

$$H_r(t) = HS(\mathbb{R}[P_1,\ldots,P_r]) = \frac{1}{(1-t)(1-t^2)\ldots(1-t^r)}$$

$$H(t,u) = \sum_{r=0}^{\infty} u^r H_r(t) = \frac{1}{(1-u)(1-ut)(1-ut^2)\ldots}$$

We find

$$\int_0^{2\pi} d\theta \phi^r \partial^k \sim \sum_{p_1,..p_r=-\infty}^{\infty} w(p_1,\ldots,p_r) c_{p_1} \ldots c_{p_r} \int_0^{2\pi} d\theta e^{i\theta(p_1+\ldots+p_r)}$$

(B) IBP & no EoM

Momentum conservation $\int_0^{2\pi} d\theta e^{i\theta(p_1+\ldots+p_r)} \sim \delta_{P_1,0}$

$$p_1 + \ldots + p_r = P_1 = 0$$

$$w_a \simeq w_b \qquad \text{for} \qquad w_a = w_b + w_c P_1$$

ideal $\langle P_1 \rangle$

We find

$$\int_0^{2\pi} d\theta \phi^r \partial^k \sim \sum_{p_1,..p_r=-\infty}^{\infty} w(p_1,\ldots,p_r) c_{p_1} \ldots c_{p_r} \int_0^{2\pi} d\theta e^{i\theta(p_1+\ldots+p_r)}$$

(B) IBP & no EoM

$$H_r(t) = HS(\mathbb{R}[P_1,\ldots,P_r]/\langle P_1\rangle) = HS(\mathbb{R}[P_2,\ldots,P_r]) = \frac{1}{(1-t^2)\ldots(1-t^r)}$$

$$H(t,u) = \sum_{r=0}^{\infty} u^r H_r(t) = \frac{1-t}{(1-u)(1-ut)(1-ut^2)\ldots}$$

We find

$$\int_0^{2\pi} d\theta \phi^r \partial^k \sim \sum_{p_1,..p_r=-\infty}^{\infty} w(p_1,\ldots,p_r) c_{p_1}\ldots c_{p_r} \int_0^{2\pi} d\theta e^{i\theta(p_1+\ldots+p_r)}$$

(C) no IBP & EoM

$$\partial^k \phi \sim 0 \qquad k \geq 2$$

$$\Longrightarrow \; P_k = 0, \quad k \geq 2 \qquad\qquad \Longrightarrow \; P_1^{r+1} = 0$$

ideal $\langle P_1^{r+1}, P_2, \ldots, P_r \rangle$

We find

$$\int_0^{2\pi} d\theta \phi^r \partial^k \sim \sum_{p_1,..p_r=-\infty}^{\infty} w(p_1,\ldots,p_r) c_{p_1}\ldots c_{p_r} \int_0^{2\pi} d\theta e^{i\theta(p_1+\ldots+p_r)}$$

(C) no IBP & EoM

$$H_r(t) = HS(\mathbb{R}[P_1,\ldots,P_r]/\langle P_1^{r+1}, P_2,\ldots,P_r\rangle)$$
$$= HS(\mathbb{R}[P_1]/\langle P_1^{r+1}\rangle) = \frac{1-t^{r+1}}{1-t}$$

$$H(t,u) = \sum_{r=0}^{\infty} u^r H_r(t) = \frac{1}{(1-u)(1-ut)}$$

We find

$$\int_0^{2\pi} d\theta \phi^r \partial^k \sim \sum_{p_1,..p_r=-\infty}^{\infty} w(p_1,\ldots,p_r) c_{p_1} \ldots c_{p_r} \int_0^{2\pi} d\theta e^{i\theta(p_1+\ldots+p_r)}$$

(D) IBP & EoM

ideals $\langle P_1 \rangle$ $\langle P_1^{r+1}, P_2, \ldots, P_r \rangle$

$$H(t,u) = \sum_{r=0}^{\infty} u^r \cdot 1 = \frac{1}{(1-u)}$$

Generalise for N flavours

$$\int_0^{2\pi} d\theta \phi_1^{r_1} \dots \phi_N^{r_N} \partial^k \sim$$

$$\sum_{\{p^{(1)}\},..\{p^{(N)}\}=-\infty}^{\infty} w(\{p^{(1)}\},\dots,\{p^{(N)}\}) \times (c_1^{(1)} \dots c_{r_1}^{(1)}) \dots (c_1^{(N)} \dots c_{r_N}^{(N)}) \int_0^{2\pi} d\theta \, exp\left(i\theta \sum_{i=1}^{N} \sum_{j=1}^{r_i} p_j^{(i)}\right)$$

	One flavour		N flavours
Ring	$\mathbb{R}[P_1, P_2, \dots, P_r]$	$\longrightarrow$	$\mathbb{R}[\{P_1^{(i)}\}, \{P_2^{(i)}\}, \dots, \{P_{r_i}^{(i)}\}]$
IBP	$\langle P_1 \rangle$	$\longrightarrow$	$\left\langle \sum_{i=1}^{N} P_1^{(i)} \right\rangle$
EoM	$\langle P_1^{r+1}, P_2, \dots, P_r \rangle$	$\longrightarrow$	$\left\langle (P_1^{(1)})^{r_1+1}, \dots, (P_1^{(N)})^{r_N+1}, \{P_2^{(i)}\},, \{P_3^{(i)}\}, \dots, \{P_{r_i}^{(i)}\} \right\rangle$

$$\mathbb{R}[P_1^{(1)}, \dots, P_1^{(N)}] / \left\langle \sum_{i=1}^{N} P_1^{(i)}, (P_1^{(1)})^{r_1+1}, \dots, (P_1^{(N)})^{r_N+1} \right\rangle$$

Generalise for N flavours

number of independent terms $\phi_1^{r_1} \dots \phi_N^{r_N} \partial^k$

$$\mathbb{R}[P_1^{(1)}, \dots, P_1^{(N)}] / \left\langle \sum_{i=1}^{N} P_1^{(i)}, (P_1^{(1)})^{r_1+1}, \dots, (P_1^{(N)})^{r_N+1} \right\rangle$$

Generalise for N flavours

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tmelia — M2-binary — 80×24

```
dhcp-192-190-216-153:~ tmelia$
dhcp-192-190-216-153:~ tmelia$
dhcp-192-190-216-153:~ tmelia$ M2
Macaulay2, version 1.7
with packages: ConwayPolynomials, Elimination, IntegralClosure, LLLBases,
               PrimaryDecomposition, ReesAlgebra, TangentCone

i1 : r1=QQ[x,y,z];

i2 : i1=ideal(x+y+z,x^5,y^6,z^7);

o2 : Ideal of r1

i3 : hilbertSeries( r1 / i1 )

                5    8     9     10
     1 - T - T  + T  + 2T  - 2T
o3 = ---------------------------
                      3
                (1 - T)

o3 : Expression of class Divide
```

Generalise for N flavours

number of independent terms $\phi_1^{r_1} \dots \phi_N^{r_N} \partial^k$

$$\mathbb{R}[P_1^{(1)}, \dots, P_1^{(N)}] / \left\langle \sum_{i=1}^{N} P_1^{(i)}, (P_1^{(1)})^{r_1+1}, \dots, (P_1^{(N)})^{r_N+1} \right\rangle$$

tmelia — M2-binary — 80×24

```
dhcp-192-190-216-153:~ tmelia$
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                5    8     9     10
     1 - T - T  + T  + 2T  - 2T
o3 = ---------------------------
                     3
                (1 - T)

o3 : Expression of class Divide
```

```
                 2     3     4     5     6     7
     1 + 2T + 3T  + 4T  + 5T  + 5T  + 4T  + 2T
o4 = -----------------------------------------
                          1
```

We are interested in the function

$$H(t,\{u_i\}) = \sum_{k,r_1,..,r_N=0}^{\infty} c_{k\,r_1..r_N}\, u_1^{r_1}\ldots u_N^{r_N}\, t^k$$

	One flavour	N flavours
	end interlude	
free	$\frac{1}{(1-u_1)(1-tu_1)(1-t^2u_1)\ldots}$	$\frac{1}{\prod_{i=1}^{N}(1-u_i)(1-tu_i)(1-t^2u_i)\ldots}$
IBP	$\frac{1-t}{(1-u_1)(1-tu_1)(1-t^2u_1)\ldots}$	$\frac{1-t}{\prod_{i=1}^{N}(1-u_i)(1-tu_i)(1-t^2u_i)\ldots}$
EoM	$\frac{1}{(1-u_1)(1-tu_1)}$	$\frac{1}{\prod_{i=1}^{N}(1-u_i)(1-tu_i)}$
IBP+ EoM	$\frac{1}{(1-u_1)}$	not trivial

We are interested in the function

$$H(t,\{u_i\}) = \sum_{k,r_1,..,r_N=0}^{\infty} c_{k\,r_1..r_N} u_1^{r_1} \dots u_N^{r_N} t^k$$

	One flavour	N flavours
free	$\frac{1}{(1-u_1)(1-tu_1)(1-t^2u_1)\dots}$	$\frac{1}{\prod_{i=1}^{N}(1-u_i)(1-tu_i)(1-t^2u_i)\dots}$
IBP	$\frac{1-t}{(1-u_1)(1-tu_1)(1-t^2u_1)\dots}$	$\frac{1-t}{\prod_{i=1}^{N}(1-u_i)(1-tu_i)(1-t^2u_i)\dots}$
EoM	$\frac{1}{(1-u_1)(1-tu_1)}$	$\frac{1}{\prod_{i=1}^{N}(1-u_i)(1-tu_i)}$
IBP+ EoM	$\frac{1}{(1-u_1)}$	not trivial

Part III

Conjecture and speculate on all-orders result

N=2 is easy too

$$H_{r_1\, r_2}(t) = HS(\mathbb{R}[x,y]/\langle x+y, x^{r_1+1}, y^{r_2+1}\rangle)$$

$$= \frac{(1-t^{\mathrm{Min}(r_1,r_2)+1})}{(1-t)}$$

$$H(t,u_1,u_2) = \sum_{r_1\geq r_2} u_1^{r_1}u_2^{r_2}\frac{(1-t^{r_2+1})}{(1-t)} + \sum_{r_2\geq r_1} u_1^{r_1}u_2^{r_2}\frac{(1-t^{r_1+1})}{(1-t)} - \sum_{r_1=r_2=r} u_1^{r}u_2^{r}\frac{(1-t^{r+1})}{(1-t)}$$

$$= \frac{1}{(1-u_1)(1-u_1u_2)(1-tu_1u_2)} + \frac{1}{(1-u_2)(1-u_1u_2)(1-tu_1u_2)} - \frac{1}{(1-u_1u_2)(1-tu_1u_2)}$$

$$= \frac{1}{(1-u_1)(1-u_2)(1-tu_1u_2)}$$

N=3 is very difficult to do this way

but pattern recognising...

$$H(t, u_1, u_2, u_3) = \frac{1 - tu_1u_2u_3}{(1-u_1)(1-u_2)(1-u_3)(1-tu_1u_2)(1-tu_1u_3)(1-tu_2u_3)}$$

Now make a guess for N=4

$$H(t, u_1, u_2, u_3, u_4) = \frac{\prod_{i<j<k}(1 - t u_i u_j u_k)}{\prod_i (1 - u_i) \prod_{i<j}(1 - t u_i u_j)\,(1 - t u_1 u_2 u_3 u_4)}$$

Now make a guess for N=4

$$H(t,u_1,u_2,u_3,u_4) = \frac{\prod_{i<j<k}(1-tu_iu_ju_k)}{\prod_i(1-u_i)\prod_{i<j}(1-tu_iu_j)\,(1-tu_1u_2u_3u_4)}$$

Need correction

$$-\frac{t^2u_1u_2u_3u_4}{\prod_{i<j}(1-tu_iu_j)(1-tu_1u_2u_3u_4)}$$

Now make a guess for N=4

$$H(t,u_1,u_2,u_3,u_4) = \frac{\prod_{i<j<k}(1-tu_iu_ju_k)}{\prod_i(1-u_i)\prod_{i<j}(1-tu_iu_j)\,(1-tu_1u_2u_3u_4)}$$

$$-\frac{t^2u_1u_2u_3u_4}{\prod_{i<j}(1-tu_iu_j)(1-tu_1u_2u_3u_4)}$$

Combining, numerator factors out: $(1-tu_1u_2u_3u_4)$

$$H(t,u_1,u_2,u_3,u_4) = \frac{1-t(s3-s4)-t^2(s4-s1s4)-t^3s4^2}{\prod_i(1-u_i)\prod_{i<j}(1-tu_iu_j)}$$

What matching conditions?

$$H(t, u_1) = \frac{1}{(1 - u_1)}$$

$$H(t, u_1, u_2) = \frac{1}{(1 - u_1)(1 - u_2)(1 - tu_1u_2)}$$

$$H(t, u_1, u_2, u_3) = \frac{1 - tu_1u_2u_3}{(1 - u_1)(1 - u_2)(1 - u_3)(1 - tu_1u_2)(1 - tu_1u_3)(1 - tu_2u_3)}$$

$$H(t, u_1, u_2, u_3, u_4) = \frac{1 - t(s3 - s4) - t^2(s4 - s1s4) - t^3 s4^2}{\prod_i (1 - u_i) \prod_{i<j} (1 - tu_iu_j)}$$

Look at the residue as $t \to \dfrac{1}{u_1 u_2}$

$$H(t, u_1, u_2, u_3, u_4) = \frac{1 - t(s3 - s4) - t^2(s4 - s1s4) - t^3 s4^2}{\prod_i (1 - u_i) \prod_{i<j} (1 - t u_i u_j)}$$

$$\to \frac{-1}{(1 - 1/u_1)(1 - 1/u_2)(1 - u_3/u_1)(1 - u_3/u_2)(1 - u_4/u_1)(1 - u_4/u_2)}$$

This requires some *very* non-trivial factorization of the numerator!

Conjecture

$$H_N(t, \{u_i\}) = \frac{\mathrm{Num}_N(t, \{u_i\})}{\prod_i (1-u_i) \prod_{i<j} (1 - t u_i u_j)}$$

$$\mathrm{Num}_N(t, \{u_i\}) = 1 - \sum_{k=1}^{k_{max}} C_k(\{u_i\}) t^k, \quad k_{max} = (N-1)(N-2)/2$$

Further, the residue as $t \to 1/u_i u_j$

$$\frac{-1}{(1-\frac{1}{u_i})(1-\frac{1}{u_j}) \prod_{k \neq i,j} (1-\frac{u_k}{u_i})(1-\frac{u_k}{u_j})}$$

Summary

Reviewed the Hilbert series, Molien formula for no derivative invariants

Investigated including derivatives using a (0+1)-D EFT of N real scalars

In momentum space understood Hilbert series at fixed orders

$$\mathbb{R}[P_1^{(1)}, \ldots, P_1^{(N)}] / \left\langle \sum_{i=1}^{N} P_1^{(i)}, (P_1^{(1)})^{r_1+1}, \ldots, (P_1^{(N)})^{r_N+1} \right\rangle$$

Presented a conjecture for the full Hilbert series of the theory