Some Recent Results on Renormalization-Group Flows of Quantum Field Theories

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Outline

- Renormalization-group flow from UV to IR in asymptotically free gauge theory; types of IR behavior; role of an exact or approximate IR fixed point

- Higher-loop calculations of UV to IR evolution, including IR zero of β and anomalous dimension γ_m of fermion bilinear; comparison with γ_m measurements from lattice

- Higher-loop calculation of structural properties of β

- Study of scheme dependence

- RG evolution in IR-free theories: U(1), $\lambda|\vec{\phi}|^4$ and Yukawa theories; question of possible UV zero in beta functions

- Conclusions

RG Flow from UV to IR; Types of IR Behavior and Role of IR Fixed Point

First, consider an asymptotically free, vectorial gauge theory with gauge group G and N_f massless fermions in representation R of G.

Asymptotic freedom $\Rightarrow$ theory is weakly coupled, properties are perturbatively calculable for large Euclidean momentum scale μ in deep ultraviolet (UV).

The question of how this theory flows from large μ in the UV to small μ in the infrared (IR) is of fundamental field-theoretic interest.

For some fermion contents, the (perturbatively calculated) beta function of the theory may have an exact or approximate IR fixed point (zero of β).

Denote running gauge coupling at scale μ as $g = g(\mu)$, and let $\alpha(\mu) = g(\mu)^2/(4\pi)$ and $a(\mu) = g(\mu)^2/(16\pi^2) = \alpha(\mu)/(4\pi)$.

The dependence of $\alpha(\mu)$ on μ is described by the renormalization group β function

$$\beta_\alpha \equiv \frac{d\alpha}{dt} = -2\alpha \sum_{\ell=1}^{\infty} b_\ell \, a^\ell = -2\alpha \sum_{\ell=1}^{\infty} \bar{b}_\ell \, \alpha^\ell$$

where $dt = d \ln \mu$, $\ell =$ loop order of the coeff. b_ℓ, and $\bar{b}_\ell = b_\ell/(4\pi)^\ell$.

Coefficients b_1 and b_2 in β are independent of regularization/renormalization scheme, while b_ℓ for $\ell \geq 3$ are scheme-dependent.

Asymptotic freedom means $b_1 > 0$, so $\beta < 0$ for small $\alpha(\mu)$, in neighborhood of UV fixed point (UVFP) at $\alpha = 0$.

As the scale μ decreases from large values, $\alpha(\mu)$ increases. Denote α_{cr} as minimum value for formation of bilinear fermion condensates and resultant spontaneous chiral symmetry breaking (SχSB).

Two generic possibilities for β and resultant UV to IR flow:

- β has no IR zero, so as μ decreases, $\alpha(\mu)$ increases, eventually beyond the perturbatively calculable region. This is the case for QCD.
- β has a IR zero, α_{IR}, so as μ decreases, $\alpha \to \alpha_{IR}$. In this class of theories, there are two further generic possibilities: $\alpha_{IR} < \alpha_{cr}$ or $\alpha_{IR} > \alpha_{cr}$.

If $\alpha_{IR} < \alpha_{cr}$, the zero of β at α_{IR} is an exact IR fixed point (IRFP) of the renorm. group (RG) as $\mu \to 0$ and $\alpha \to \alpha_{IR}$, $\beta \to \beta(\alpha_{IR}) = 0$, and the theory becomes exactly scale-invariant with nontrivial anomalous dimensions (Caswell, Banks-Zaks).

If β has no IR zero, or an IR zero at $\alpha_{IR} > \alpha_{cr}$, then as μ decreases through a scale Λ, $\alpha(\mu)$ exceeds α_{cr} and SχSB occurs, so fermions gain dynamical masses $\sim \Lambda$.

If SχSB occurs, then in low-energy effective field theory applicable for $\mu < \Lambda$, one integrates these fermions out, and β fn. becomes that of a pure gauge theory, with no IR zero. Hence, if β has a zero at $\alpha_{IR} > \alpha_{cr}$, this is only an approx. IRFP of RG.

If α_{IR} is only slightly greater than α_{cr}, then, as $\alpha(\mu)$ approaches α_{IR}, since $\beta = d\alpha/dt \to 0$, $\alpha(\mu)$ varies very slowly as a function of the scale μ, i.e., there is approximately scale-invariant (= dilatation-invariant, walking) behavior.

SχSB at Λ also breaks the approx. dilatation symmetry, leads to a resultant approx. NGB, the dilaton (Yamawaki et al.; Bardeen et al.., 1986). This is not massless, since β is nonzero at $\alpha = \alpha_{cr}$ where SχSB occurs.

Denote the n-loop β fn. as $\beta_{n\ell}$ and the IR zero of $\beta_{n\ell}$ as $\alpha_{IR,n\ell}$.

At the $n = 2$ loop level,

$$\alpha_{IR,2\ell} = -\frac{4\pi b_1}{b_2}$$

which is physical for $b_2 < 0$. 1-loop coefficient b_1 is

$$b_1 = \frac{1}{3}(11C_A - 4N_f T_f)$$

where $C_A \equiv C_2(G)$ is quadratic Casimir invariant, $T_f \equiv T(R)$ is trace invariant. Focus here on $G = \mathrm{SU}(N_c)$.

Asymp. freedom requires $N_f < N_{f,b1z}$, where

$$N_{f,b1z} = \frac{11C_A}{4T_f}$$

e.g., for $R =$ fundamental rep., $N_f < (11/2)N_c$.

2-loop coeff. b_2 is (with $C_f \equiv C_2(R)$) (Caswell, Jones)

$$b_2 = \frac{1}{3}\left[34C_A^2 - 4(5C_A + 3C_f)N_f\,T_f\right]$$

For small N_f, $b_2 > 0$; b_2 decreases as fn. of N_f and vanishes with sign reversal at $N_f = N_{f,b2z}$, where

$$N_{f,b2z} = \frac{34C_A^2}{4T_f(5C_A + 3C_f)}$$

For arbitrary G and R, $N_{f,b2z} < N_{f,b1z}$, so there is always an interval in N_f for which β has an IR zero, namely

$$I: \quad N_{f,b2z} < N_f < N_{f,b1z}$$

- for SU(2), I: $\quad 5.55 < N_f < 11$
- for SU(3), I: $\quad 8.05 < N_f < 16.5$
- As $N_c \to \infty$, I: $\quad 2.62 N_c < N_f < 5.5 N_c$.

(expressions evaluated for $N_f \in \mathbb{R}$, but it is understood that physical values of N_f are nonnegative integers.)

As N_f decreases from the upper to lower end of interval I, α_{IR} increases. Denote

$$N_f = N_{f,cr} \quad \text{at} \quad \alpha_{IR} = \alpha_{cr}$$

Value of $N_{f,cr}$ is of fundamental importance, since it separates the (zero-temp.) chirally symmetric and broken IR phases.

Intensive current lattice studies of SU(N_c) gauge theories with N_f copies of fermions in various representations R; progress toward determining $N_{f,cr}$ for various N_c and R.

Higher-Loop Corrections to UV $\rightarrow$ IR Evolution of Gauge Theories

Because of this strong-coupling physics, one should calculate the IR zero in β, α_{IR}, and resultant value of γ_m evaluated at α_{IR} to higher-loop order: done for general R in Ryttov and Shrock, PRD83, 056011 (2011) [arXiv:1011.4542] and Pica and Sannino, PRD83, 035013 (2011) [arXiv:1011.5917].

Although coeffs. in β at $\ell \geq 3$ loop order are scheme-dependent, results give a measure of accuracy of the 2-loop calc. of the IR zero of β, and similarly with γ_m evaluated at this IR zero.

The value of higher-loop calculations has been amply shown in comparison of QCD predictions with experimental data, e.g., in $\overline{\mathrm{MS}}$ scheme due to Bardeen et al.

Values of $\bar{b}_\ell = b_\ell/(4\pi)^\ell$ for N_c, where interval I is $8.05 < N_f < 16.5$:

N_c	N_f	$\bar{b}_1$	$\bar{b}_2$	$\bar{b}_3$	$\bar{b}_4$
3	0	0.875	0.646	0.720	1.173
3	1	0.822	0.566	0.582	0.910
3	2	0.769	0.485	0.450	0.681
3	3	0.716	0.405	0.324	0.485
3	4	0.663	0.325	0.205	0.322
3	5	0.610	0.245	0.091	0.194
3	6	0.557	0.165	-0.016	0.099
3	7	0.504	0.084	-0.118	0.039
3	8	0.451	0.004	-0.213	0.015
3	9	0.398	-0.076	-0.303	0.025
3	10	0.345	-0.156	-0.386	0.072
3	11	0.292	-0.236	-0.463	0.154
3	12	0.239	-0.317	-0.534	0.273
3	13	0.186	-0.397	-0.599	0.429
3	14	0.133	-0.477	-0.658	0.622
3	15	0.080	-0.557	-0.711	0.852
3	16	0.0265	-0.637	-0.758	1.121

3-loop coefficient in β function (in $\overline{\mathrm{MS}}$ scheme, Vermaseren et al.):

$$b_3 = \frac{2857}{54} C_A^3 + T_f N_f \left[2C_f^2 - \frac{205}{9} C_A C_f - \frac{1415}{27} C_A^2 \right] + (T_f N_f)^2 \left[\frac{44}{9} C_f + \frac{158}{27} C_A \right]$$

Here, $b_3 < 0$ for $N_f \in I$. Since $\beta_{3\ell} = -[\alpha^2/(2\pi)](b_1 + b_2 a + b_3 a^2)$, $\beta_{3\ell} = 0$ away from $\alpha = 0$ at two values:

$$\alpha = \frac{2\pi}{b_3} \Big(-b_2 \pm \sqrt{b_2^2 - 4b_1 b_3}\ \Big)$$

Since $b_2 < 0$ and $b_3 < 0$, can rewrite as

$$\alpha = \frac{2\pi}{|b_3|} \Big(-|b_2| \mp \sqrt{b_2^2 + 4b_1 |b_3|}\ \Big)$$

Soln. with $-$ sqrt is negative, hence unphysical; soln. with $+$ sqrt is $\alpha_{IR,3\ell}$.

We showed that with $b_3 < 0$ the value of the IR zero decreases when calculated at the 3-loop level, i.e.,

$$\alpha_{IR,3\ell} < \alpha_{IR,2\ell}$$

This can be seen as follows:

$$\begin{aligned}
\alpha_{IR,2\ell} - \alpha_{IR,3\ell} &= \frac{4\pi b_1}{|b_2|} - \frac{2\pi}{|b_3|}\Big(-|b_2| + \sqrt{b_2^2 + 4b_1|b_3|}\ \Big) \\
&= \frac{2\pi}{|b_2 b_3|}\Big[2b_1|b_3| + b_2^2 - |b_2|\sqrt{b_2^2 + 4b_1|b_3|}\ \Big]
\end{aligned}$$

The expression in square brackets is positive if and only if

$$(2b_1|b_3| + b_2^2)^2 - b_2^2(b_2^2 + 4b_1|b_3|) > 0$$

This difference is equal to the positive-definite quantity $4b_1^2 b_3^2$, which proves the inequality.

In RS, Phys. Rev. D 87, 105005 (2013) [arXiv:1301.3209] we generalized this:

If a scheme had $b_3 > 0$ in I, then, since $b_2 \to 0$ at lower end of I, $b_2^2 - 4b_1 b_3 < 0$, so this scheme would not have a physical $\alpha_{IR,3\ell}$ in this region.

Since the existence of the IR zero in β at 2-loop level is scheme-independent, one may require that a scheme should maintain this property to higher-loop order, and hence that $b_3 < 0$ for $N_f \in I$.

So the inequality $\alpha_{IR,3\ell} < \alpha_{IR,2\ell}$ holds in all such schemes, not just in $\overline{\mathrm{MS}}$. Analysis of zeros of β function up to 4-loop order in a general scheme: RS, PRD 87, 105005 (2013) [arXiv:1301.3209].

With $\overline{\mathrm{MS}}$, from 3- to 4-loop level, slight increase: $\alpha_{IR,4\ell} \gtrsim \alpha_{IR,3\ell}$; small change, so overall, $\alpha_{IR,4\ell} < \alpha_{IR,2\ell}$.

Our result of smaller fractional change in value of IR zero of β at higher-loop order agrees with expectation that calc. to higher loop order should give more stable result.

Numerical values of $\alpha_{IR,n\ell}$ at the $n = 2,\ 3,\ 4$ loop level for SU(2), SU(3) and fermions in fund. rep. (3-loop and 4-loop calc. in $\overline{\rm MS}$ scheme):

N_c	N_f	$\alpha_{IR,2\ell}$	$\alpha_{IR,3\ell}$	$\alpha_{IR,4\ell}$
2	6	11.42	1.645	2.395
2	7	2.83	1.05	1.21
2	8	1.26	0.688	0.760
2	9	0.595	0.418	0.444
2	10	0.231	0.196	0.200
3	10	2.21	0.764	0.815
3	11	1.23	0.578	0.626
3	12	0.754	0.435	0.470
3	13	0.468	0.317	0.337
3	14	0.278	0.215	0.224
3	15	0.143	0.123	0.126
3	16	0.0416	0.0397	0.0398

(Perturbative calc. not applicable if $\alpha_{IR,n\ell}$ too large.) We have performed the corresponding higher-loop calculations for SU(N_c) gauge theories with N_f fermions in the adjoint, symmetric and antisymmetric rank-2 tensor representations.

It is of interest to calculate the anomalous dimension $\gamma_m \equiv \gamma$ for the fermion bilinear, with series expansion

$$\gamma = \sum_{\ell=1}^{\infty} c_\ell a^\ell = \sum_{\ell=1}^{\infty} \bar{c}_\ell \alpha^\ell$$

where $\bar{c}_\ell = c_\ell/(4\pi)^\ell$ is the ℓ-loop coefficient.

The 1-loop coeff. c_1 is scheme-independent, the c_ℓ with $\ell \geq 2$ are scheme-dependent and have been calculated up to 4-loop level in $\overline{\mathrm{MS}}$ scheme (Vermaseren, Larin, and van Ritbergen): $c_1 = 6C_f$,

$$c_2 = 2C_f\Big[\frac{3}{2}C_f + \frac{97}{6}C_A - \frac{10}{3}T_f N_f\Big]$$

etc. for c_3, c_4.

Denote γ calculated to n-loop ($n\ell$) level as $\gamma_{n\ell}$ and, evaluated at the n-loop value of the IR zero of β, as

$$\gamma_{IR,n\ell} \equiv \gamma_{n\ell}(\alpha = \alpha_{IR,n\ell})$$

In the IR chirally symmetric phase, an all-order calculation of γ evaluated at an all-order calculation of α_{IR} would be an exact property of the theory.

In the chirally broken phase, just as the IR zero of β is only an approx. IRFP, so also, the γ is only approx., describing the running of $\bar{\psi}\psi$ and the dynamically generated running fermion mass near the zero of β having large-momentum (large k) behavior

$$\Sigma(k) \sim \Lambda \left(\frac{\Lambda}{k} \right)^{2-\gamma}$$

where γ is bounded above as $\gamma < 2$. Schwinger-Dyson estimates suggest γ could be ~ 1 in walking regime with SχSB (Yamawaki et al., Appelquist..,Holdom). The upper bound $\gamma < 2$ also holds for the chirally symmetric conformal IR phase; from a unitarity argument, $\dim(\bar{\psi}\psi) = 3 - \gamma_m > 1$, so $\gamma < 2$.

Illustrative numerical values of $\gamma_{IR,n\ell}$ for SU(2) and SU(3) at the $n = 2,\ 3, 4$ loop level and fermions in the fundamental representation with $N_f \in I$:

N_c	N_f	$\gamma_{IR,2\ell}$	$\gamma_{IR,3\ell}$	$\gamma_{IR,4\ell}$
2	7	(2.67)	0.457	0.0325
2	8	0.752	0.272	0.204
2	9	0.275	0.161	0.157
2	10	0.0910	0.0738	0.0748
3	10	(4.19)	0.647	0.156
3	11	1.61	0.439	0.250
3	12	0.773	0.312	0.253
3	13	0.404	0.220	0.210
3	14	0.212	0.146	0.147
3	15	0.0997	0.0826	0.0836
3	16	0.0272	0.0258	0.0259

Plots of γ as function of N_f for SU(2), SU(3):

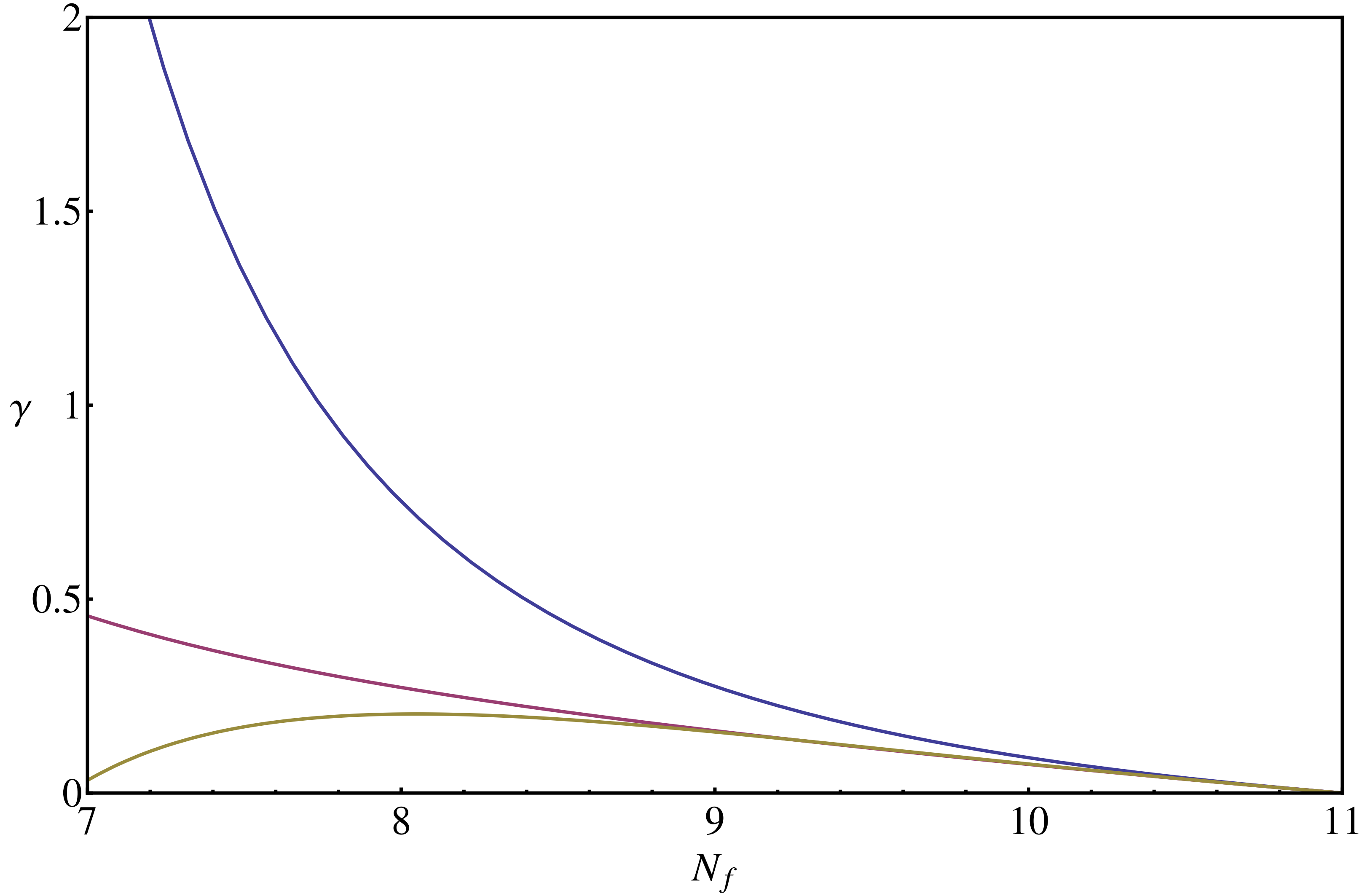

Figure 1: $\boldsymbol{n}$-loop anomalous dimension $\boldsymbol{\gamma_{IR,n\ell}}$ at $\boldsymbol{\alpha_{IR,n\ell}}$ for SU(2) with $\boldsymbol{N_f}$ fermions in fund. rep: (i) blue: $\boldsymbol{\gamma_{IR,2\ell}}$; (ii) red: $\boldsymbol{\gamma_{IR,3\ell}}$; (iii) brown: $\boldsymbol{\gamma_{IR,4\ell}}$.

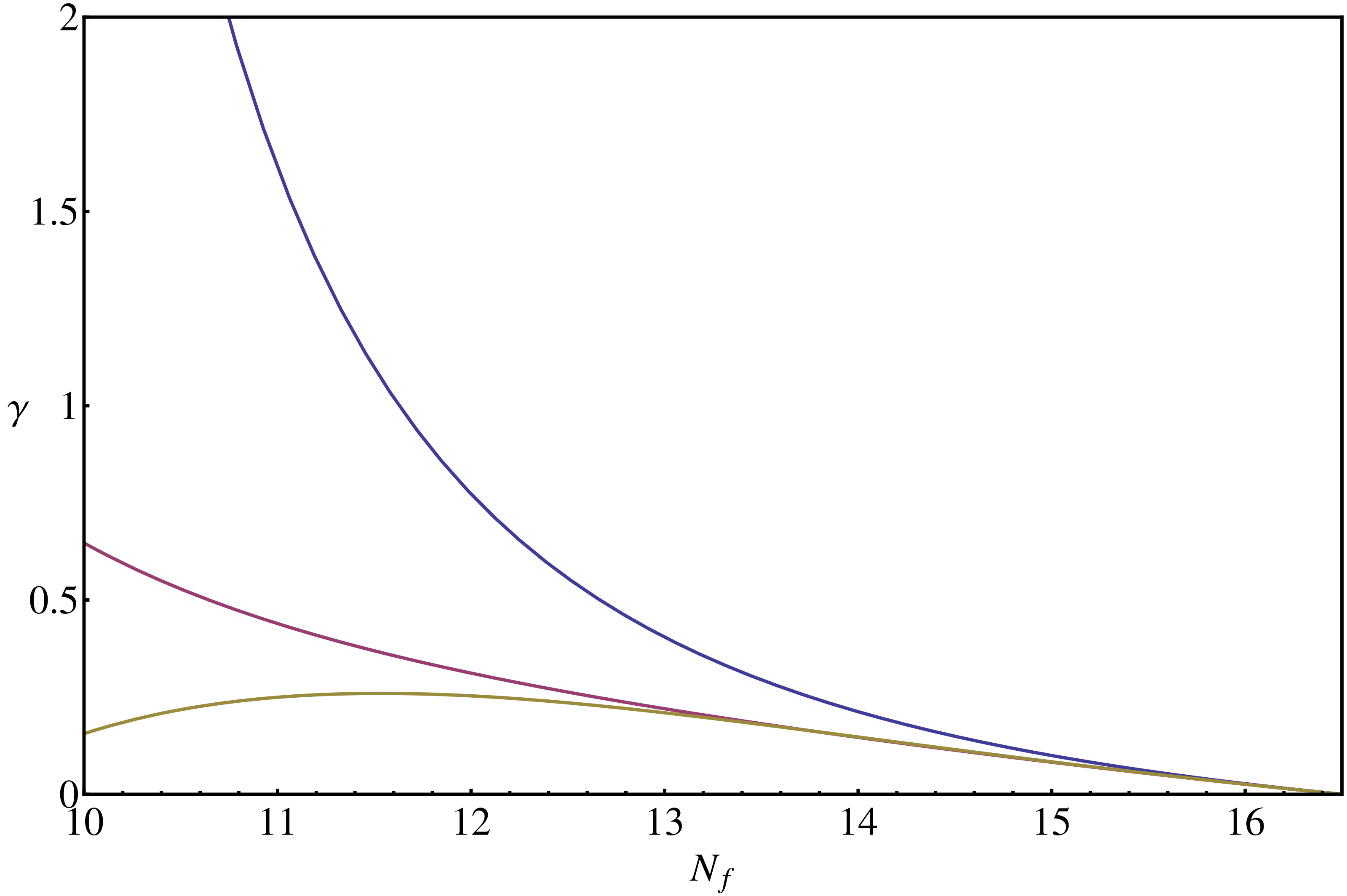

Figure 2: $\boldsymbol{n}$-loop anomalous dimension $\boldsymbol{\gamma_{IR,n\ell}}$ at $\boldsymbol{\alpha_{IR,n\ell}}$ for SU(3) with $\boldsymbol{N_f}$ fermions in fund. rep: (i) blue: $\boldsymbol{\gamma_{IR,2\ell}}$; (ii) red: $\boldsymbol{\gamma_{IR,3\ell}}$; (iii) brown: $\boldsymbol{\gamma_{IR,4\ell}}$.

A necessary condition for a perturbative calculation to be reliable is that higher-order contributions do not modify the result too much. We find that the 3-loop and 4-loop results are closer to each other for a larger range of N_f than the 2-loop and 3-loop results.

So our higher-loop calcs. of α_{IR} and γ allow us to probe the theory reliably down to smaller values of N_f and thus stronger couplings, closer to $N_{f,cr}$. Of course, perturbative calculations are not applicable when α is too large.

Comparisons with Lattice Measurements

We compare these calculations with lattice measurements here.

N.B.: for some theories with given gauge group and fermion content, there is not yet a consensus as to whether the theory is chirally symmetric or chirally broken in the IR.

One of the most heavily studied cases on the lattice is for the gauge group SU(3) with $N_f = 12$ fermions in the fundamental representation (with extrapolations to the continuum limit and to massless fermions):

For this theory, Appelquist et al. (LSD); Deuzeman, Lombardo, and Pallante; Hasenfratz et al.; DeGrand et al.; Aoki et al. (LatKMI) find that the IR behavior is chirally symmetric (Jin and Mawhinney, and Kuti et al. found it is chirally broken).

For this SU(3) theory with $N_f = 12$, our calculations give

$$\gamma_{IR,2\ell} = 0.77, \qquad \gamma_{IR,3\ell} = 0.31, \qquad \gamma_{IR,4\ell} = 0.25$$

some lattice results (N.B. - error estimates do not include all systematic uncertainties):

$\gamma = 0.414 \pm 0.016$ (Appelquist et al. (LSD Collab.), PRD 84, 054501 (2011).

$\gamma \sim 0.35$ (DeGrand, PRD 84, 116901 (2011).

$0.2 \lesssim \gamma \lesssim 0.4$ (Kuti et al. (method-dep.) arXiv:1205.1878, arXiv:1211.3548, 1211.6164, PTP, finding SχSB).

$\gamma \simeq 0.4$ (Y. Aoki et al. (LatKMI Collab.) PRD 86, 054506 (2012) [arXiv:1207.3060]);

$\gamma = 0.27(3)$ (Hasenfratz et al., arXiv:1207.7162; $\gamma \simeq 0.25$; Hasenfratz et al., arXiv:1310.1124).

$\gamma = 0.235(46)$ (Lombardo, Miura, Nunes, Pallante (LMNP), arXiv:1410.0298).

So 2-loop value is larger than, and the 3-loop and 4-loop values closer to, lattice data.

Thus, our higher-loop calculations of γ yield better agreement with these lattice measurements than two-loop calculations.

SU(N_c) with fermions in fund. rep. and other N_f values:

SU(3) with $N_f = 10$: Appelquist et al., arXiv:1204.6000 get $\gamma_{IR} \sim 1$

SU(3) with $N_f = 8$, possibly in chirally broken phase; under study by several groups:

Aoki et al. (LatKMI), PRD 87, 094511 (2013) [arXiv:1302.6859]; and Y. Aoki et al. (LatKMI), PRD 89, 111502 (2014) [arXiv:1403.5000] get $\gamma_{IR} \sim 1$.

Appelquist et al. (LSD) PRD 90, 114502 (2014) [arXiv:1405.4752], also get $\gamma_{IR} \sim 1$.

Lattice results are consistent with $\gamma_{IR} \sim 1$ in quasi-scale invariant (walking) regime of chirally broken phase. For these theories, the coupling is probably too strong for perturbative methods to be accurate.

We have done calculations for higher-dimension fermion reps. R and some of these have been studied on the lattice.

Further Higher-Loop Structural Properties of β

In addition to $\alpha_{IR,n\ell}$, further interesting structural properties of the n-loop beta fn. $\beta_{n\ell}$ include, e.g., the derivative $\beta'_{IR,n\ell} \equiv \frac{d\beta_{n\ell}}{d\alpha}$ evaluated at $\alpha_{IR,n\ell}$.

In quasi-scale-invariant case where $\alpha_{IR} \gtrsim \alpha_{cr}$, dilaton mass relevant in dynamical EWSB models depends on how small β is for $\alpha \simeq \alpha_{cr}$ near to α_{IR}. To estimate β at α_{cr}, use Taylor series expansion:

$$\beta_{n\ell}(\alpha_{cr}) = \beta'_{IR,n\ell}(\alpha_{IR,n\ell}) \times (\alpha_{cr} - \alpha_{IR,n\ell}) + O\Big((\alpha_{cr} - \alpha_{IR,n\ell})^2\Big)$$

We calculated $\beta'_{IR,n\ell}$ analytically and numerically in RS, Phys. Rev. D87, 105005 (2013) [arXiv:1301.3209].

We prove a general inequality: for a given gauge group G, fermion rep. R, and $N_f \in I$ (in a scheme with $b_3 < 0$, which thus preserves the existence of the 2-loop IR zero in β at 3-loop level), $\beta'_{IR,n\ell}$ decreases from $n = 2$ loop to $n = 3$ loop level. We calculate the $n = 4$ loop $\beta'_{IR,4\ell}$ and find a further decrease, so

$$\beta'_{IR,4\ell} < \beta'_{IR,3\ell} < \beta'_{IR,2\ell}$$

These results show that when calculated to higher-loop order, the derivative $\beta'_{IR,n\ell}$ decreases, consistent, via the Taylor series expansion, with a decrease in the value of β itself for $\alpha \simeq \alpha_{IR,n\ell}$.

Some numerical values:

N_c	N_f	$\beta'_{IR,2\ell}$	$\beta'_{IR,3\ell}$	$\beta'_{IR,4\ell}$
2	7	1.20	0.728	0.677
2	8	0.400	0.318	0.300
2	9	0.126	0.115	0.110
2	10	0.0245	0.0239	0.0235
3	10	1.52	0.872	0.853
3	11	0.720	0.517	0.498
3	12	0.360	0.2955	0.282
3	13	0.174	0.156	0.149
3	14	0.0737	0.0699	0.0678
3	15	0.0227	0.0223	0.0220
3	16	0.00221	0.00220	0.00220

Illustrative figure for SU(3) with $N_f = 12$ fermions:

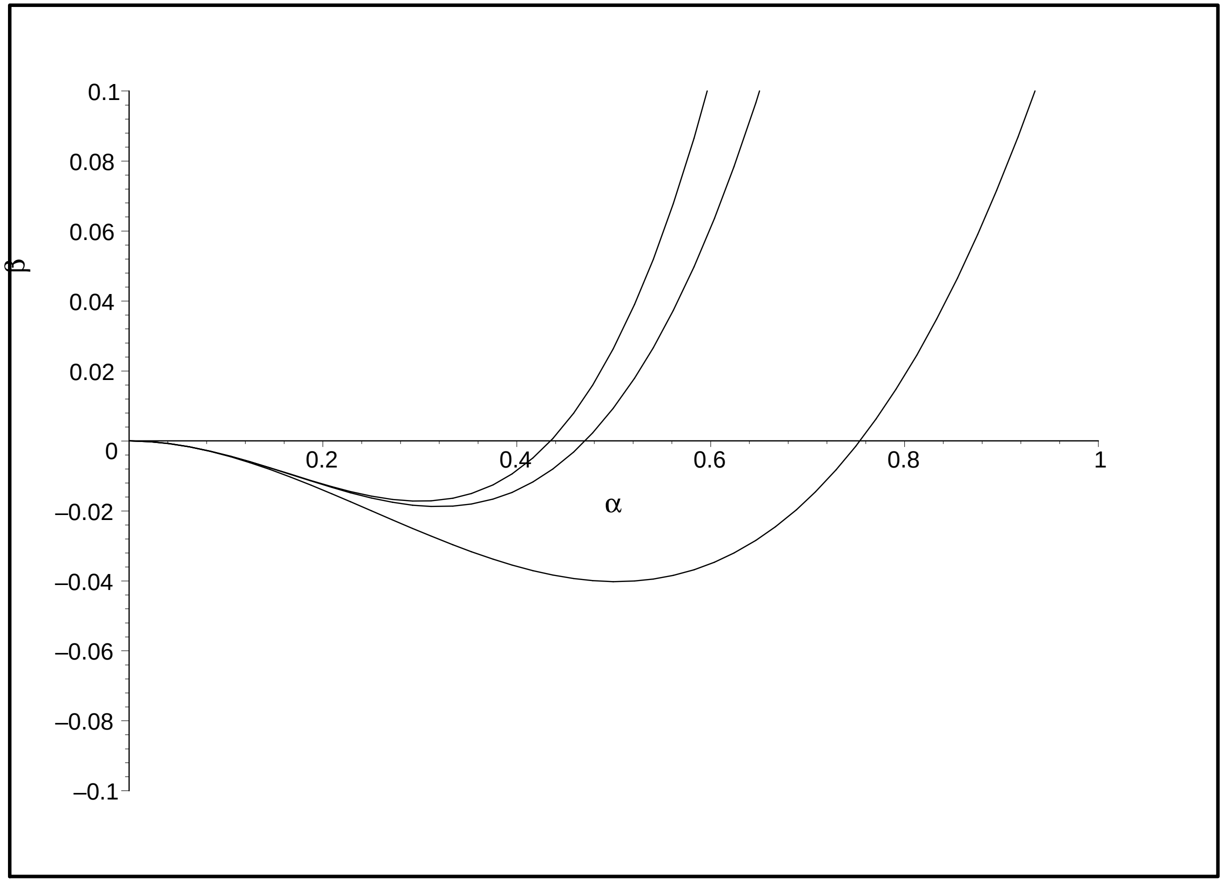

Figure 3: $\boldsymbol{\beta_{n\ell}}$ for SU(3), $\boldsymbol{N_f = 12}$, at $\boldsymbol{n = 2,\ 3,\ 4}$ loops. From bottom to top, curves are $\boldsymbol{\beta_{2\ell}}$, $\boldsymbol{\beta_{4\ell}}$, $\boldsymbol{\beta_{3\ell}}$.

It is interesting to study this UV to IR evolution in a SU(N_c) gauge theory with N_f fermions in the fundamental rep. in the limit $N_c \to \infty$, $N_f \to \infty$ with

$$r \equiv \frac{N_f}{N_c} \quad \text{and} \quad \alpha(\mu) N_c \equiv \xi(\mu) \quad \text{finite}$$

denoted the LNN (large N_c, N_f) limit.

We have carried out this study in RS, Phys. Rev. D87, 116007 (2013) [arXiv:1302.5434]. Define a rescaled beta function that is finite in the LNN limit:

$$\beta_\xi \equiv \frac{d\xi}{dt} = \lim_{LNN} \beta_\alpha N_c$$

Interval of r where $\beta_{\xi,2\ell}$ has an IR zero is

$$I_r: \quad \frac{34}{13} < r < \frac{11}{2}\,, \ \ i.e., \quad 2.615 < r < 5.500$$

2-loop IR zero of $\beta_{\xi,2\ell}$ is at

$$\xi_{IR,2\ell} = \frac{4\pi(11-2r)}{13r-34}$$

with more complicated expressions for the IR zero at 3-loop and 4-loop level (see paper).

Value of n-loop γ evaluated at n-loop $\xi_{IR,n\ell}$: $\gamma_{IR,n\ell} \equiv \gamma_{n\ell}\big|_{\xi=\xi_{IR,n\ell}}$;

$$\gamma_{IR,2\ell} = \frac{(11-2r)(1009-158r+40r^2)}{12(13r-34)^2}$$

with more complicated expressions for $\gamma_{IR,n\ell}$ at 3-loop and 4-loop level (see paper).
Numerical values:

r	$\gamma_{IR,2\ell}$	$\gamma_{IR,3\ell}$	$\gamma_{IR,4\ell}$
3.6	1.853	0.5201	0.3083
3.8	1.178	0.4197	0.3061
4.0	0.7847	0.3414	0.2877
4.2	0.5366	0.2771	0.2664
4.4	0.3707	0.2221	0.2173
4.6	0.2543	0.1735	0.1745
4.8	0.1696	0.1294	0.1313
5.0	0.1057	0.08886	0.08999
5.2	0.05620	0.05123	0.05156
5.4	0.01682	0.01637	0.01638

It is instructive to carry out a similar analysis in an asymptotically free $\mathcal{N} = 1$ supersymmetric gauge theory with vectorial chiral superfield content $\Phi_i, \ \tilde{\Phi}_i$, $i = 1, ..., N_f$ in the $R, \ \bar{R}$ reps., respectively.

We have done this in Ryttov and Shrock, Phys. Rev. D 85, 076009 (2012) [arXiv:1202.1297]

An appeal of this analysis: exact results are known from work of Novikov, Shifman, Vainshtein, and Zakharov (1983, 1986) and Seiberg (1994).

One goal of this study: to compare results from higher-loop perturb. calcs. with exact results, in particular, for $N_{f,cr}$.

One lesson from the O(N) nonlinear σ model and the NSVZ beta function in SQCD: when one can sum a subset of diagrams contributing to the beta function to infinite-loop order, this provides a different and complementary approach to analyzing zeros of the beta function.

Further discussion of this in general: RS, Phys. Rev. D 91, 125039 (2015) [arXiv:1505.03588].

One can also apply similar RG methods to analyze the UV to IR evolution of asymptotically free chiral gauge theories. (The contents of chiral fermions in these theories are chosen so as to render them anomaly-free.)

However, there are important differences between vectorial and chiral gauge theories, VGT, χGT: fermion condensation in a VGT breaks global chiral symmetries, but not the gauge symmetry, while in a χGT it generically breaks both chiral global and gauge symmetries.

In an asymptotically free χGT that evolves from the UV to strong coupling in the IR, this typically leads to a sequence of self-breakings of the chiral gauge symmetry.

Some recent work: Appelquist and RS, Phys. Rev. D 88, 105012 (2013) [arXiv:1310.6076]; Y. Shi and RS, Phys. Rev. D D 91, 045004 (2015) [arXiv:1411.2042].

Study of Scheme Dependence in Calculation of IR Fixed Point

Since coeffs. b_n in $\beta_{n\ell}$, and hence also $\alpha_{IR,n\ell}$, are scheme-dependent for $n \geq 3$, it is important to assess the effects of this scheme dependence. Our earlier studies: Ryttov and RS, PRD 86, 065032 (2012) [arXiv:1206.2366] and PRD 86, 085005 (2012) [arXiv:1206.6895]; further: RS, PRD 88, 036003 (2013) [arXiv:1305.6524]; RS, PRD 90, 045011 (2014) [arXiv:1405.6244]; most recently: Choi and RS, PRD 90, 125029 (2014) [arXiv:1411.6645].

A scheme transformation (ST) is a map between α and α' or equivalently, a and a', where $a = \alpha/(4\pi)$ of the form

$$a = a' f(a')$$

with $f(0) = 1$ since ST has no effect in limit of zero coupling.

$$f(a') = 1 + \sum_{s=1}^{s_{max}} k_s (a')^s = 1 + \sum_{s=1}^{s_{max}} \bar{k}_s (\alpha')^s$$

where $\bar{k}_s = k_s/(4\pi)^s$, and s_{max} may be finite or infinite.

The Jacobian $J = da/da' = d\alpha/d\alpha' = 1 + \sum_{s=1}^{s_{max}} (s+1) k_s (a')^s$, satisfying $J = 1$ at $a = a' = 0$.

After the scheme transformation is applied, the beta function in the new scheme is given by

$$\beta_{\alpha'} \equiv \frac{d\alpha'}{dt} = \frac{d\alpha'}{d\alpha}\frac{d\alpha}{dt} = J^{-1}\,\beta_{\alpha}$$

with the expansion

$$\beta_{\alpha'} = -2\alpha' \sum_{\ell=1}^{\infty} b'_\ell (a')^\ell = -2\alpha' \sum_{\ell=1}^{\infty} \bar{b}'_\ell (\alpha')^\ell$$

where $\bar{b}'_\ell = b'_\ell/(4\pi)^\ell$.

We calculate the b'_ℓ as functions of the b_ℓ and k_s. At 1-loop and 2-loop, this yields the well-known results

$$b'_1 = b_1 \ , \quad b'_2 = b_2$$

We find

$$b'_3 = b_3 + k_1 b_2 + (k_1^2 - k_2) b_1 \ ,$$

$$b'_4 = b_4 + 2k_1 b_3 + k_1^2 b_2 + (-2k_1^3 + 4k_1 k_2 - 2k_3) b_1$$

$$b'_5 = b_5 + 3k_1 b_4 + (2k_1^2 + k_2) b_3 + (-k_1^3 + 3k_1 k_2 - k_3) b_2$$
$$+(4k_1^4 - 11k_1^2 k_2 + 6k_1 k_3 + 4k_2^2 - 3k_4) b_1$$

etc. at higher-loop order.

A physically acceptable ST must satisfy several conditions:

- C_1: the ST must map a (real positive) α to a real positive α', since a map taking $\alpha > 0$ to $\alpha' = 0$ would be singular, and a map taking $\alpha > 0$ to a negative or complex α' would violate the unitarity of the theory.
- C_2: the ST should not map a moderate value of α, where perturbation theory is applicable, to a value of α' so large that pert. theory is inapplicable.
- C_3: J should not vanish (or diverge) or else there would be a pole in $\beta_{\alpha'}$
- C_4: Existence of an IR zero of β is a scheme-independent property, so the ST should satisfy the condition that β_α has an IR zero if and only if $\beta_{\alpha'}$ has an IR zero.

These conditions can always be satisfied by an ST near the UVFP at $\alpha = \alpha' = 0$, but they are not automatic, and can be quite restrictive at an IRFP.

For example, consider the ST (dependent on a parameter r)

$$a = \frac{\tanh(ra')}{r}$$

with inverse

$$a' = \frac{1}{2r} \ln\left(\frac{1+ra}{1-ra}\right)$$

(e.g., for $r = 4\pi$, $\alpha = \tanh\alpha'$). This is acceptable for small a, but if $a > 1/r$, i.e., $\alpha > 4\pi/r$, it maps a real α to a complex α' and hence is physically unacceptable. For $r = 8\pi$, e.g., this pathology can occur at the moderate value $\alpha = 0.5$.

We have constructed several STs that are acceptable at an IRFP and have studied scheme dependence of the IR zero of $\beta_{n\ell}$ using these. For example, we have used a $\sinh$ transformation (depending on a parameter r):

$$a = \frac{\sinh(ra')}{r}$$

with inverse

$$a' = \frac{1}{r} \ln\left[ra + \sqrt{1+(ra)^2}\,\right]$$

Written in the form $a = a' f(a')$, this has the transformation function

$$f(a') = \frac{\sinh(ra')}{ra'}$$

This satisfies $f(0) = 1$ and also approaches the identity map as $r \to 0$. With no loss of generality, take $r \geq 0$.

The Jacobian is $J = \cosh(ra')$, which always satisfies C_3, i.e., is nonsingular.

Taylor series expansion of $f(a')$ has coefficients $k_s = 0$ for odd s and

$$k_2 = \frac{r^2}{6}, \quad k_4 = \frac{r^4}{120}, \quad k_6 = \frac{r^6}{5040}, \quad k_8 = \frac{r^8}{362880},$$

etc. for $s \geq 10$. Thus, for small $|r|a'$,

$$a = a' \left[1 + \frac{(ra')^2}{6} + O\Big((ra')^4\Big) \right]$$

so (for $a \neq 0$) $a' < a$ for $|r| > 0$.

Illustrative results with this $\sinh$ scheme transformation follow. We denote the IR zero of $\beta_{\alpha'}$ at the n-loop level as $\alpha'_{IR,n\ell} \equiv \alpha'_{IR,n\ell,r}$.

For SU(3) gauge theory with $N_f = 12$, $\alpha_{IR,2\ell} = 0.754$, and:

$$\alpha_{IR,3\ell,\overline{\rm MS}} = 0.435, \quad \alpha'_{IR,3\ell,r=3} = 0.434, \quad \alpha'_{IR,3\ell,r=6} = 0.433,$$
$$\alpha_{IR,4\ell,\overline{\rm MS}} = 0.470, \quad \alpha'_{IR,4\ell,r=3} = 0.470, \quad \alpha'_{IR,4\ell,r=6} = 0.467,$$

Thus, we find moderately small scheme dependence in the value of the IR zero at 3-loop and 4-loop level for moderate α and r.

Recently, in Choi and RS, PRD 90, 125029 (2014) [arXiv:1411.6645], we have constructed and applied two new scheme transformations to study the scheme dependence of an IR zero of the beta function in an AF gauge theory. Our new work confirms and extends our earlier studies.

S_{L_r} scheme transformation:

$$S_{L_r}: \quad a = \frac{\ln(1+ra')}{r}$$

where r is a (real) parameter; corresponding transformation function:

$$f(a') = \frac{\ln(1+ra')}{ra'}$$

$$\text{Inverse}: \quad a' = \frac{e^{ra}-1}{r}, \quad \text{Jacobian}: J = \frac{1}{1+ra'} = e^{-ra}$$

Here $f(a')$ has the Taylor series expansion

$$f(a') = 1 + \sum_{s=1}^{\infty} \frac{(-ra')^s}{s+1},$$

i.e., coefficients are $k_s = (-r)^s/(s+1)$.

So for small $|r|a'$,

$$a = a'\Big[\,1 - \frac{ra'}{2} + O\Big((ra')^2\Big)\,\Big]$$

so (for $a \neq 0$), $a' > a$ if $r > 0$ and $a' < a$ if $r < 0$.

Note that for a given s, these k_s are much larger than those for the $\sinh$ ST, so for a given value of r, the S_{L_r} ST is farther from the identity than the $\sinh$ ST. Allowed range of r determined by conditions C_1-C_4.

Illustrative results with this S_{L_r} scheme transformation: We again denote the IR zero of $\beta_{\alpha'}$ at the n-loop level as $\alpha'_{IR,n\ell} \equiv \alpha'_{IR,n\ell,r}$.

For SU(3) with $N_f = 12$, $\alpha_{IR,2\ell} = 0.754$, and:

$$\alpha'_{IR,3\ell,r=-2} = 0.429, \quad \alpha'_{IR,3\ell,r=-1} = 0.432, \quad \alpha'_{IR,3\ell,r=0} = \alpha_{IR,3\ell,\overline{\rm MS}} = 0.435,$$
$$\alpha'_{IR,3\ell,r=1} = 0.438, \quad \alpha'_{IR,3\ell,r=2} = 0.441,$$

$$\alpha'_{IR,4\ell,r=-2} = 0.450, \quad \alpha'_{IR,4\ell,r=-1} = 0.460, \quad \alpha'_{IR,4\ell,r=0} = \alpha_{IR,4\ell,\overline{\rm MS}} = 0.470,$$
$$\alpha'_{IR,4\ell,r=1} = 0.482, \quad \alpha'_{IR,4\ell,r=2} = 0.496$$

Again, we find rather small scheme dependence in the value of the IR zero of beta at $n = 3$ and $n = 4$ loop level with this scheme transformation for moderate α and r.

We have also considered scheme transformation involving rational transformation functions; for example,

$$S_{Q_r}: \quad a = \frac{a'}{1 - ra'}$$

where r is a (real) parameter; corresponding transformation function:

$$f(a') = \frac{1}{1 - ra'}$$

$$\text{Inverse}: \quad a' = \frac{a}{1 + ra}, \quad \text{Jacobian}: J = \frac{1}{(1 - ra')^2} = (1 + ra)^2$$

Here $f(a')$ has the Taylor series expansion

$$f(a') = 1 + \sum_{s=1}^{\infty} (ra')^s \,,$$

i.e., coefficients are $k_s = r^s$. So for small $|r|a'$,

$$a = a'\Big[\, 1 + ra' + O\Big((ra')^2\Big) \,\Big] \,.$$

Here, $a' < a$ if $r > 0$ and $a' > a$ if $r < 0$. Again, allowed range of parameter r determined by the conditions C_i, $i = 1, ..4$.

For the S_{Q_r} scheme transformation, as with the S_{L_r} ST, we find that the shift in the IR zero of the beta function at 3-loop and 4-loop level is small for moderate α and r.

These results are in agreement with our previous ones for the $\sinh$ scheme transformation.

Our studies provide a quantitative evaluation of scheme-dependent effects in calculations of the IR zero in the beta function. We have found reasonably small scheme-dependence in the value of the IR zero of β for moderate α_{IR} and ST-parameter r.

Since the coefficients b_ℓ at loop order $\ell \geq 3$ in the beta function are scheme-dependent, one might expect that it would be possible, at least in the vicinity of zero coupling (UVFP in an asymp. free theory; IRFP in an IR-free theory) to construct a scheme transformations that would set $b'_\ell = 0$ for some range of $\ell \geq 3$, and, indeed a ST that would do this for all $\ell \geq 3$, so that $\beta_{\alpha'}$ would consist only of the 1-loop and 2-loop terms ('t Hooft scheme).

We have constructed an explicit scheme transformation that can do this in the vicinity of zero coupling constant. However, we have also shown that it is much more difficult to try to do this at a zero of β away from the origin (IR zero for an asymp. free theory; UV zero for an IR-free theory).

Specifically, we construct a scheme transformation, denoted S_{R,m,k_1}, that removes the terms in the beta function from loop order 3 up to $m+1$, inclusive, for small coupling. In the limit $m \to \infty$, this transforms to the 't Hooft scheme.

To construct this ST, first, we take advantage of the property that in b'_ℓ, the ST coefficient $k_{\ell-1}$ appears only linearly. For example, $b'_3 = b_3 + k_1 b_2 + (k_1^2 - k_2) b_1$, etc. for higher-$\ell$ b'_ℓ. So solve eq. $b'_3 = 0$ for k_2, obtaining

$$k_2 = \frac{b_3}{b_1} + \frac{b_2}{b_1} k_1 + k_1^2$$

This determines $S_{R,2,k_1}$.

To get $S_{R,3,k_1}$, substitute this k_2 into expression for b'_4 and solve eq. $b'_4 = 0$, obtaining

$$k_3 = \frac{b_4}{2b_1} + \frac{3b_3}{b_1} k_1 + \frac{5b_2}{2b_1} k_1^2 + k_1^3$$

This determines $S_{R,3,k_1}$. We continue this procedure iteratively to calculate S_{R,m,k_1} for higher m. In general, the equation $b'_\ell = 0$ is a linear equation for $k_{\ell-1}$, so one is guaranteed a unique solution.

So the ST S_{R,m,k_1} has nonzero k_s, $s = 1, ..., m$ and in the transformed beta function, sets $b'_\ell = 0$ for $\ell = 3, ..., m+1$. The coefficients k_s for this ST depend on the b_n in the original beta function for $n = 1, ..., m+1$, and on the parameter k_1.

In addition to the successful application near the origin, $\alpha = 0$, we have shown that this ST S_{R,m,k_1} can be applied over part, but not all, of the interval I where the 2-loop beta function has an IR zero.

Study of RG Flows in Infrared-Free Gauge Theories

If the β function of a theory is positive near zero coupling, then this theory is IR-free; as μ increases from the IR to the UV, the coupling grows. It is of interest to investigate whether an IR-free theory might have a UV fixed point (UV zero of β).

In addition to performing perturbative calculations of β to search for such a UVFP in an IR-free theory, one can use large-N methods. An explicit example is the O(N) nonlinear σ model in $d = 2 + \epsilon$ spacetime dimensions. From an exact solution of this model in the limit $N \to \infty$ in 1976, we found that (for small ϵ)

$$\beta(\lambda) = \frac{d\lambda}{dt} = \epsilon\lambda\Big(1 - \frac{\lambda}{\lambda_c}\Big) \ , \ i.e., \quad \beta(x) = \frac{dx}{dt} = \epsilon x\Big(1 - \frac{x}{x_c}\Big)$$

where λ is the effective coupling, $\lambda_c = 2\pi\epsilon/N$; $x = \lim_{N\to\infty} \lambda N$, $x_c = 2\pi\epsilon$ (W. Bardeen, B. W. Lee, and R. Shrock, Phys. Rev. D 14, 985 (1976); E. Brézin and J. Zinn-Justin, Phys. Rev. B 14, 3110 (1976)). Thus this theory has a UVFP at x_c, so that if initial value of $x < x_c$, then $x \nearrow x_c$ as $\mu \to \infty$.

There has long been interest in RG properties of $d = 4$ QED and, more generally, U(1) gauge theory (early work: Gell-Mann and Low; Johnson, Baker, and Willey; Adler; Yamawaki, Miransky,..).

Consider a vectorial U(1) theory with N_f massless Dirac fermions of charge q. With no loss of generality, set $q = 1$. Write β function as

$$\beta_\alpha = 2\alpha \sum_{\ell=1}^{\infty} b_\ell \, a^\ell$$

The 1-loop and 2-loop coefficients are

$$b_1 = \frac{4N_f}{3}\,, \qquad b_2 = 4N_f$$

These coefficients have the same sign, so the two-loop beta function, $\beta_{\alpha,2\ell}$, does not have a UV zero, and this is the maximal scheme-independent information about it. The coefficients have been calculated up to five loops in the $\overline{\mathrm{MS}}$ scheme.

The 3-loop coefficient (deRafael and Rosner) is negative:

$$b_3 = -2N_f\Big(1 + \frac{22N_f}{9}\Big)$$

Hence, $\beta_{\alpha,3\ell}$ has a UV zero, namely,

$$\alpha_{UV,3\ell} = 4\pi a_{UV,3\ell} = \frac{4\pi[9 + \sqrt{3(45 + 44N_f)}\ ]}{9 + 22N_f}$$

The 4-loop coefficient is (Gorishny et al.)

$$b_4 = N_f\Big[-46 + \Big(\frac{760}{27} - \frac{832\zeta(3)}{9}\Big)N_f - \frac{1232}{243}N_f^2\Big]$$

Numerically,

$$b_4 = -N_f\,(46 + 82.97533N_f + 5.06996N_f^2]$$

This is negative for all $N_f > 0$.

Recently, b_5 has been calculated (Kataev, Larin; Baikov et al., 2012, 2013). Numerically,

$$b_5 = N_f(846.6966 + 798.8919N_f - 148.7919N_f^2 + 9.22127N_f^3$$

which is positive for all $N_f > 0$.

In RS, PRD 89, 045019 (2014) [arXiv:1311.5268], we have investigated whether the n-loop beta function for this U(1) gauge theory has a UV zero for n up to 5 loops, for a large range of N_f. Our results are given in the table (dash means no UV zero).

N_f	$\alpha_{UV,2\ell}$	$\alpha_{UV,3\ell}$	$\alpha_{UV,4\ell}$	$\alpha_{UV,5\ell}$
1	—	10.2720	3.0400	—
2	—	6.8700	2.4239	—
3	—	5.3689	2.0776	—
4	—	4.5017	1.8463	—
5	—	3.9279	1.67685	2.5570
10	—	2.5871	1.2135	1.3120
20	—	1.7262	0.8483	—
100	—	0.7081	0.33265	—
500	—	0.3038	0.1203	—
10^3	—	0.2127	0.07678	—
10^4	—	0.016614	0.016965	—

A necessary condition for the perturbatively calculated β function to yield evidence for a stable UV zero is that it should remain present when one increases the loop order and the fractional change in the value should decrease going from n to $n+1$ loops.

As is evident from the table, we do not find that the UV zeros that we have calculated at $\ell = 3,\ 4,\ 5$ loop order for a large range of N_f values satisfy this necessary condition. Hence, our results do not give evidence for a UVFP in U(1) gauge theory for general N_f.

We have also carried out an analysis in the limit

$$N_f \to \infty \quad \text{with finite} \quad y(\mu) \equiv N_f\, a(\mu) = \frac{N_f\, \alpha(\mu)}{4\pi}$$

We denote this as the LNF (large-N_f) limit; analogous to $N \to \infty$ limit in nonlinear σ model.

We set $b_1 = b_{1,1} N_f$ with $b_{1,1} = 4/3$. Further,

$$b_\ell = \sum_{k=1}^{\ell-1} b_{\ell,k}\, N_f^k \quad \text{for } \ell \geq 2 \ ,$$

where the $b_{\ell,k}$ are independent of N_f.

Hence,

$$b_\ell \propto N_f^{\ell-1} \quad \text{for} \ \ \ell \geq 2 \quad \text{as} \ \ N_f \to \infty$$

We thus define the finite quantities

$$\check{b}_\ell \equiv \frac{b_\ell}{N_f^{\ell-1}} \quad \text{for } \ell \geq 2$$

so

$$\lim_{N_f \to \infty} \check{b}_\ell = b_{\ell,\ell-1} \quad \text{for } \ell \geq 2$$

We define a rescaled β function that is finite in the LNF limit as $\beta_y \equiv \beta_\alpha N_f$. Then

$$\beta_y = 8\pi b_{1,1}\, y^2 \left[1 + \frac{1}{b_{1,1} N_f} \sum_{\ell=2}^{\infty} b_\ell\, y^{\ell-1}\right]$$

The condition that the n-loop β_y, $\beta_{y,n\ell}$, has a zero at $y \neq 0$ is the equation

$$1 + \frac{1}{b_{1,1} N_f} \sum_{\ell=2}^{n} b_\ell\, y^{\ell-1} = 0$$

In the LNF limit, of the $n-1$ roots of this equation, the relevant one has the approximate form

$$y_{UV,n\ell} \sim \left(-\frac{b_{1,1} N_f}{b_{n,n-1}}\right)^{\frac{1}{n-1}}$$

Hence, $\beta_{y,n\ell}$ has a zero for $y \neq 0$ in the LNF limit if and only if $b_{n,n-1} < 0$, which is not, in general true. Further, even if it were true for a given loop order n, in the LNF limit, $\lim_{N_f \to \infty} y_{UV,n\ell} = \infty$.

One can reexpress β_y as a series in powers of $\nu \equiv 1/N_f$:

$$\beta_y = 8\pi b_{1,1}\, y^2 \left[1 + \sum_{s=1}^{\infty} F_s(y) \nu^s\right]$$

An exact integral representation of $F_1(y)$ is known (cf. Holdom, 2010). We have used this representation to determine the signs of $b_{n,n-1}$ up to $n = 24$ loops. We find that these signs are scattered, and show no indication of an onset of negative signs. This confirms our earlier discussion.

Thus, we do not find evidence of a UVFP in a U(1) gauge theory with N_f massless charged fermions for large N_f.

We have also studied an SU(N) non-Abelian gauge theory with N_f massless fermions in a given representation, for large N_f. This theory is again IR-free, and we again we do not find evidence of a UVFP.

RG Flows in the O(N) $\lambda|\vec{\phi}|^4$ Theory

We have carried out a similar study, again up to 5-loop order, of another IR-free theory, namely O(N) $\lambda|\vec{\phi}|^4$ theory (in $d = 4$) to search for a possible UV zero of the beta function, in RS, Phys. Rev. D 90, 065023 (2014) [arXiv:1408.3141].

Interaction term: $\mathcal{L}_{int} = -\frac{\lambda}{4!}(\vec{\phi}^{\,2})^2$

$$\beta \text{ function}: \quad \beta_a = \frac{da}{dt} = a \sum_{\ell=1}^{\infty} b_\ell \, a^\ell \quad \text{where} \quad a = \frac{\lambda}{16\pi^2}$$

Coefficients:

$$b_1 = \frac{1}{3}(N+8) \ , \quad b_2 = -\frac{1}{3}(3N+14)$$

$$b_3 = \frac{11}{72}N^2 + \left(\frac{461}{108} + \frac{20\zeta(3)}{9}\right)N + \frac{370}{27} + \frac{88\zeta(3)}{9}$$

Numerically,

$$b_3 = 0.15278N^2 + 6.93976N + 24.4571$$

and so forth for b_4 and b_5 (calculated in $\overline{\rm MS}$ scheme)

Although the two-loop beta function has a UV zero, it occurs at too large a value of the coupling for the perturbative calculation to be reliable, as shown by the fact that when one calculates to higher-loop order, the 3-loop beta function has no UV zero, and the 4-loop and 5-loop beta functions differ considerably from the 2-loop and 3-loop beta functions where the 2-loop function has a zero.

We have studied this further with scheme transformations and Padé approximants.

We thus conclude that in the range of λ where the perturbative calculation of the n-loop beta function is reliable, the theory does not exhibit evidence of a UV zero up to the level of $n = 5$ loops.

See figure with beta function curves and table of values of the UV zero $a_{UV,n\ell}$ of the n-loop beta function. Dash — means no physical UV zero.

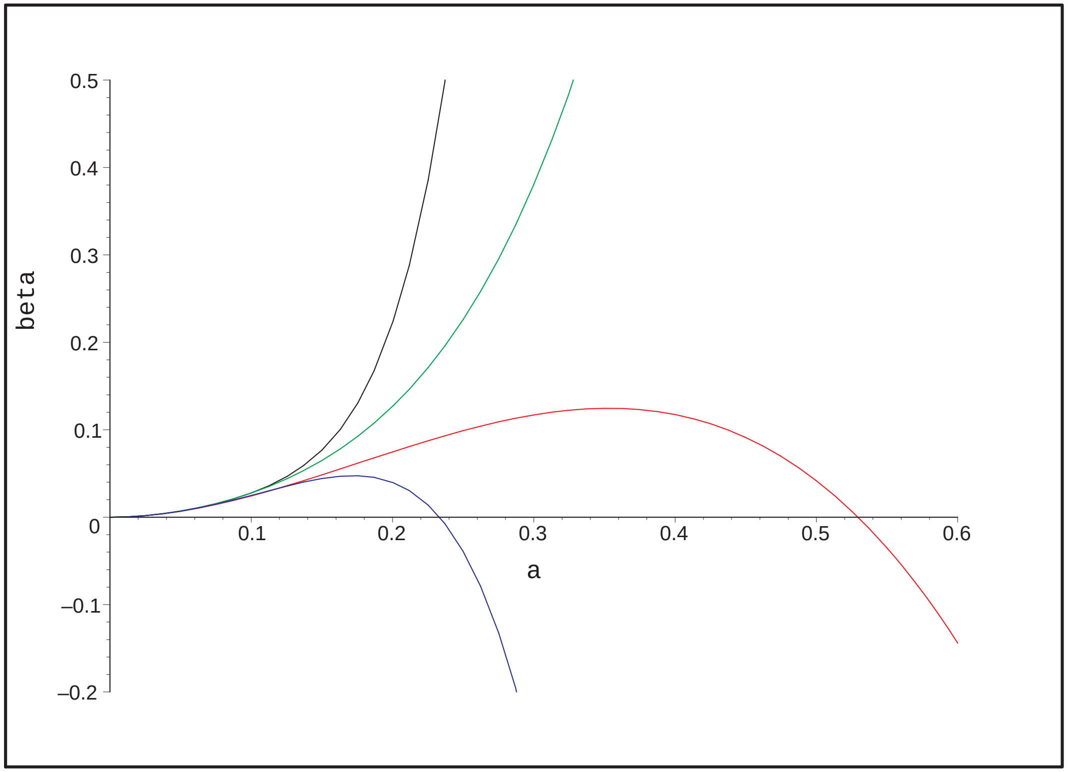

Figure 4: Plot of the n-loop β function $\beta_{a,n\ell}$ as functions of a for $N = 1$ and (i) $n = 2$ (red), (ii) $n = 3$ (green), (iii) $n = 4$ (blue), and $n = 5$ (black). At $a = 0.18$, going from bottom to top, the curves are for $n = 4$, $n = 2$, $n = 3$, and $n = 5$.

N	$a_{UV,2\ell}$	$a_{UV,3\ell}$	$a_{UV,4\ell}$	$a_{UV,5\ell}$
1	0.5294	—	0.2333	—
2	0.5000	—	0.2217	—
3	0.4783	—	0.2123	—
4	0.4615	—	0.2044	—
5	0.4483	—	0.1978	—
6	0.4375	—	0.1920	—
7	0.4286	—	0.1869	—
8	0.42105	—	0.1823	—
9	0.4146	—	0.1783	—
10	0.4091	—	0.1746	—
100	0.3439	—	0.1012	—
1000	0.3344	—	0.07241	0.02276
3000	0.3337	—	0.5475	0.008850
10^4	0.3334	—	—	0.003460

RG Flows in a Scalar-Fermion Theory

With E. Mølgaard, we have calculated RG flows for scalar-fermion theories in Mølgaard and RS, PR D 89, 105007 (2014) [arXiv:1403.3058].
To study flows in simple context, use the (one-gen.) leptonic sector of the SM with the gauge fields turned off . This has a global chiral symmetry group: $\mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y$.

fermions: ψ_L: fund. rep. of SU$(2)_L$ with U$(1)_Y$ charge Y_ψ; χ_R: singlet of SU$(2)_L$ with U$(1)_Y$ charge Y_χ; scalar ϕ: fund. rep. of SU(2) with U$(1)_Y$ charge Y_ϕ.

Set $Y_\phi = Y_\psi - Y_\chi$ so Yukawa term $y\bar{\psi}_L \chi_R \phi + h.c.$ allowed.

Fermions in Lagrangian are massless; chiral symmetry forbids fermion masses.

Now the RG flows depend on two couplings rather than one, namely y and the quartic scalar coupling λ, so they are more complicated than the ones we have discussed so far. Two beta functions (with $dt = d \ln \mu$):

$$\beta_y = \frac{dy}{dt}\,, \qquad \beta_\lambda = \frac{d\lambda}{dt}$$

Convenient variables: $a_y = y^2/(4\pi)^2$ and $a_\lambda = \lambda/(4\pi)^2$. Corresponding beta functions: $\beta_{a_y} = da_y/dt = (2y)(4\pi)^{-2}\,\beta_y$ and $\beta_{a_\lambda} = da_\lambda/dt = (4\pi)^{-2}\,\beta_\lambda$.

As before, a condition for perturbation theory to be reliable is that calculating the beta functions to given loop orders and then comparing with beta functions calculated to one higher loop should give similar results.

With β_y and β_λ calculated to various loop orders (1,1), (1,2), (2,1), (2,2), we have integrated to get the RG flows. See plots.

For small a_y and a_λ, the RG flow is to the IR-free zero of both beta functions at $a_y = a_\lambda = 0$, i.e., $y = \lambda = 0$.

For larger y and λ, the flows show further structure.

Comparison of these different loop-order RG flows yields information on the extent of the region in a_y and a_λ where the perturbative calculations agree with each other and hence may be reliable.

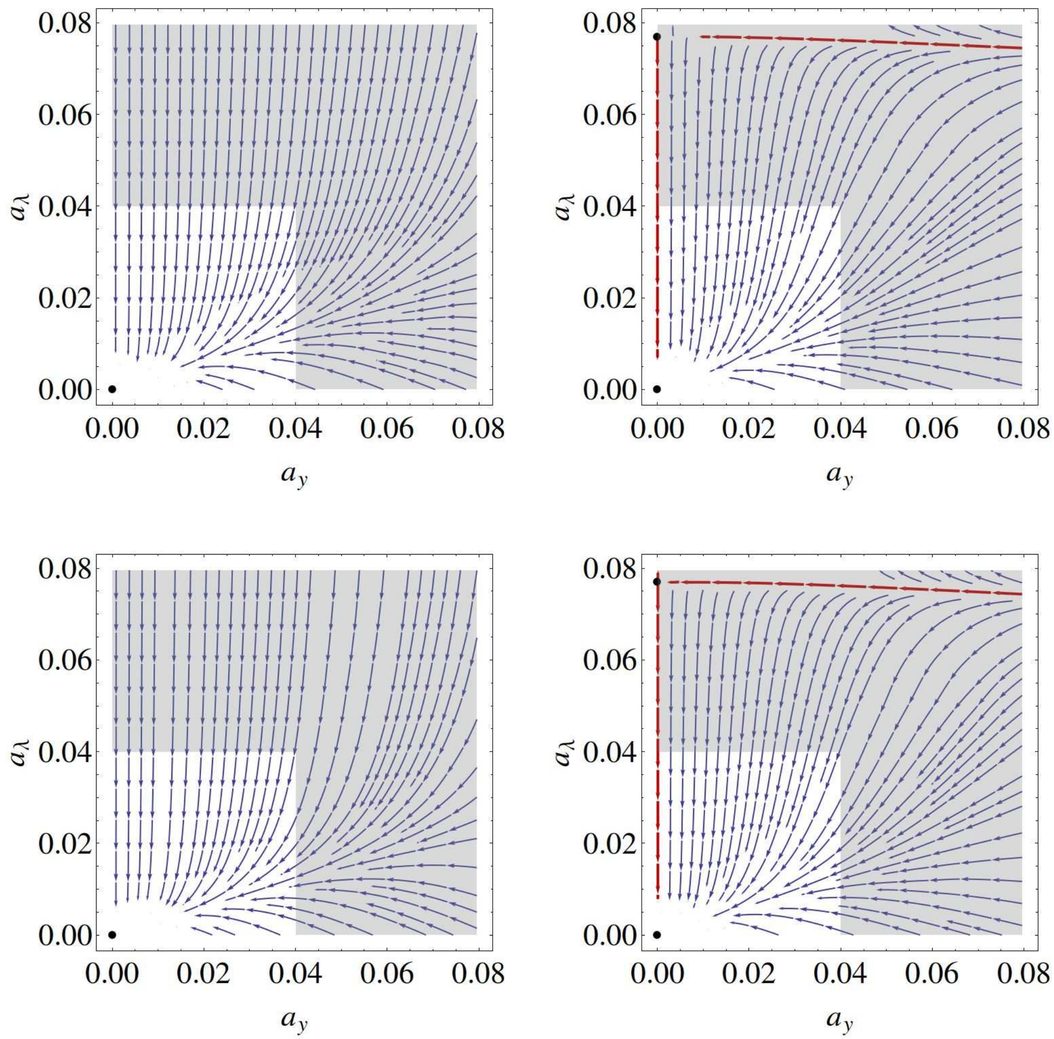

Figure 5: RG flows obtained via integration of beta functions $\boldsymbol{\beta_{a_y,\ell}}$ and $\boldsymbol{\beta_{a_\lambda,\ell'}}$ for small $\boldsymbol{a_y}$ and $\boldsymbol{a_\lambda}$, calculated for loop orders $\boldsymbol{(\ell,\ell')}$: (1,1) (upper left); (1,2) (upper right); (2,1) (lower left); and (2,2) (lower right). Arrows are flows from UV to IR.

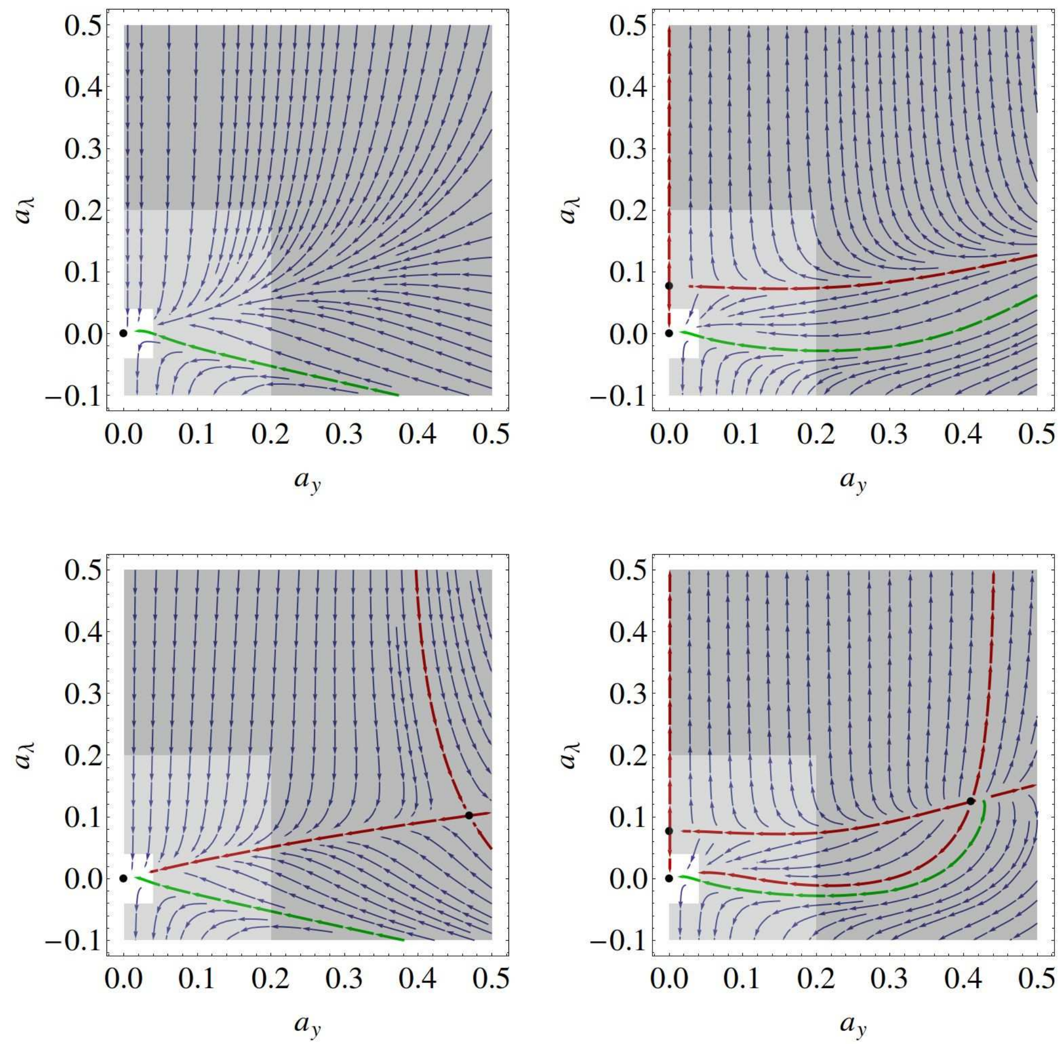

Figure 6: RG flows obtained via integration of beta functions $\boldsymbol{\beta_{a_y,\ell}}$ and $\boldsymbol{\beta_{a_\lambda,\ell'}}$ for moderate $\boldsymbol{a_y}$ and $\boldsymbol{a_\lambda}$, calculated for loop orders $\boldsymbol{(\ell,\ell')}$: (1,1) (upper left); (1,2) (upper right); (2,1) (lower left); and (2,2) (lower right). Arrows are flows from UV to IR.

Conclusions

- Understanding the UV to IR evolution of an asymptotically free gauge theory and the nature of the IR behavior is of fundamental field-theoretic interest.

- Our higher-loop calcs. give info. on this UV to IR flow and on determination of $\alpha_{IR,n\ell}$ and $\gamma_{IR,n\ell}$; interesting comparison with γ_{IR} from lattice.

- We have investigated effects of scheme-dependence of IR zero in the beta function in higher-loop calculations and have constructed explicit scheme transformations that remove higher-loop terms in beta.

- RG flows in IR-free theories: U(1) gauge theory, nonabelian g.t. with $N_f > N_{f,b1z}$, and $\lambda|\vec{\phi}|^4$; general evidence against a UV zero in beta functions.

- RG flows in Yukawa models.

This talk covered material from:

T. A. Ryttov and R. Shrock, “Higher-Loop Corrections to the Infrared Evolution of a Gauge Theory with Fermions”, Phys. Rev. D 83, 056011 (2011) [arXiv:1011.4542].

T. A. Ryttov and R. Shrock, “Scheme Transformations in the Vicinity of an Infrared Fixed Point” Phys. Rev. D 86, 065032 (2012) [arXiv:1206.2366].

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R. Shrock, “Higher-Loop Structural Properties of the β Function in Asymptotically Free Vectorial Gauge Theories”, Phys. Rev. D 87, 105005 (2013) [arXiv:1301.3209].

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E. Mølgaard and R. Shrock, “Renormalization-Group Flows and Fixed Points in a Theory with Yukawa Interactions”, Phys. Rev. D 89, 105007 (2014) [arXiv:1403.3058].

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Y. Shi and R. Shrock, “Renormalization-Group Evolution of Chiral Gauge Theories”, Phys. Rev. D 91, 045004 (2015) [arXiv:1411.2042].

R. Shrock, “Analysis of a Zero of a Beta Function Using All-Orders Summation of Diagrams”, Phys. Rev. D 91, 125039 (2015) [arXiv:1505.03588].