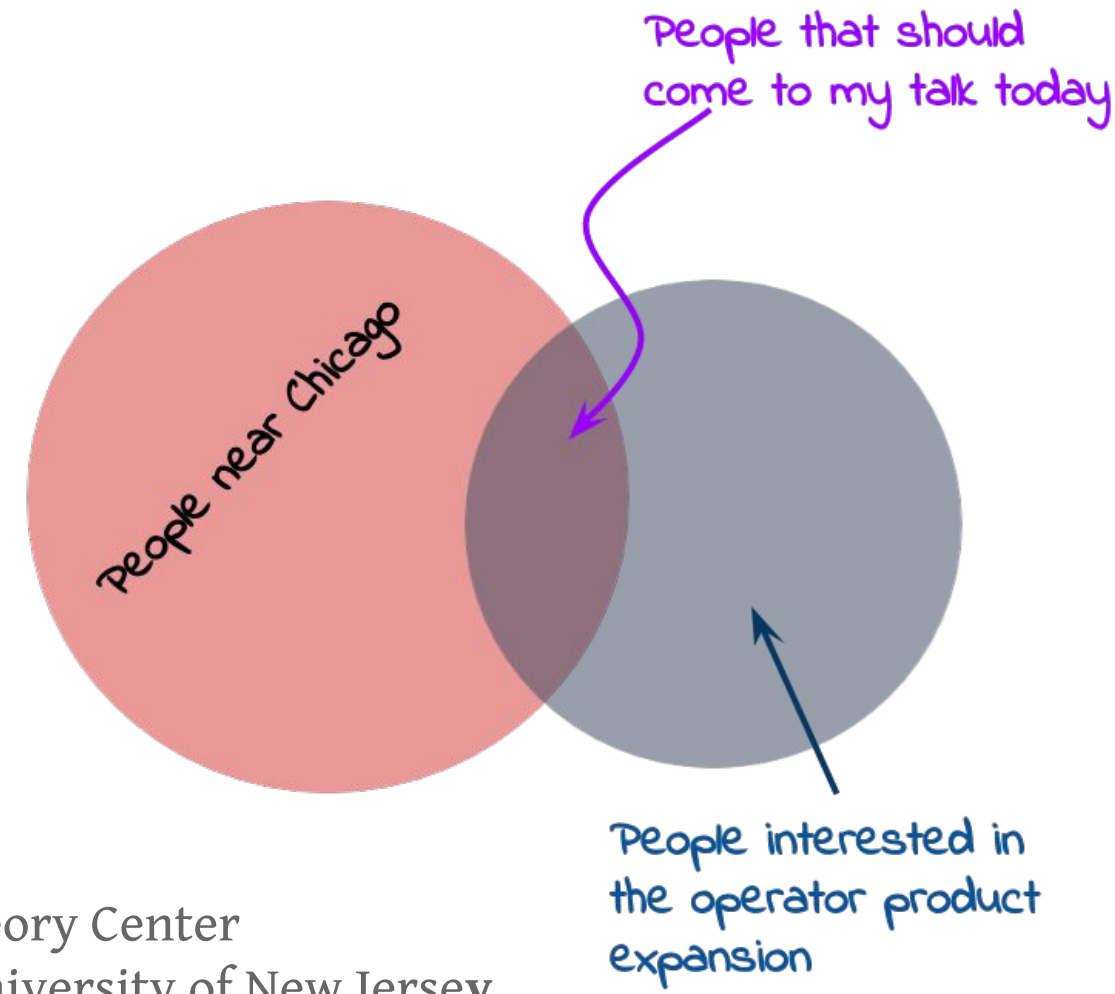
A new approach to the operator product expansion on the lattice

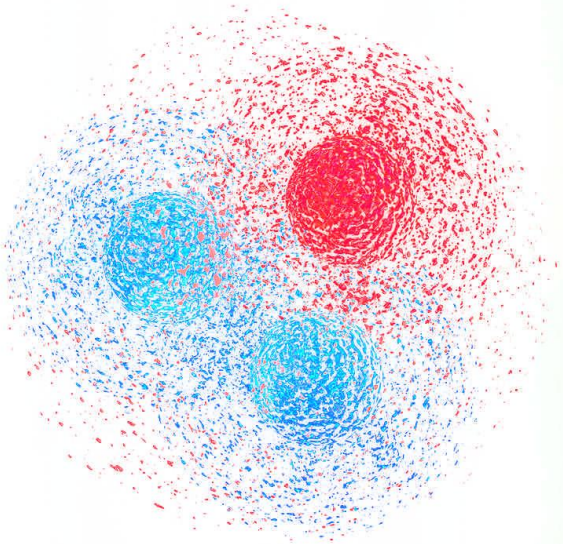


Chris Monahan

New High Energy Theory Center
Rutgers, The State University of New Jersey

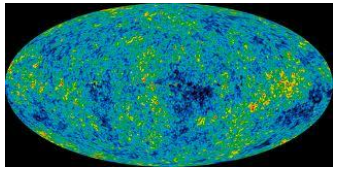
with Kostas Orginos

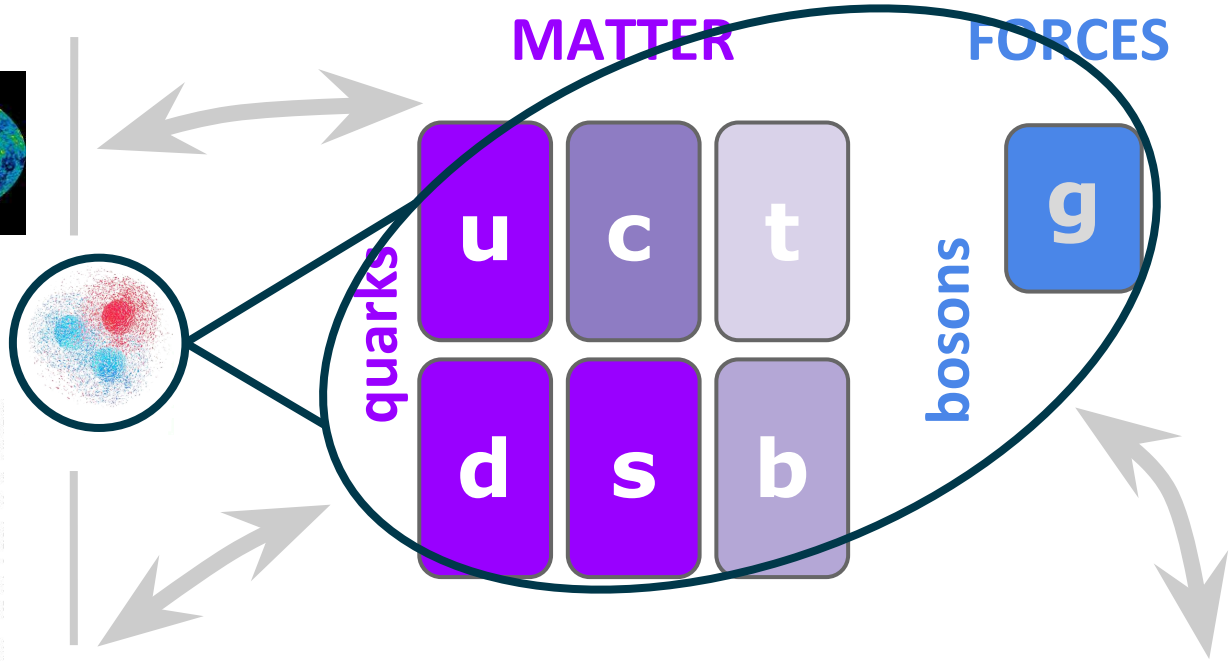
Fermilab

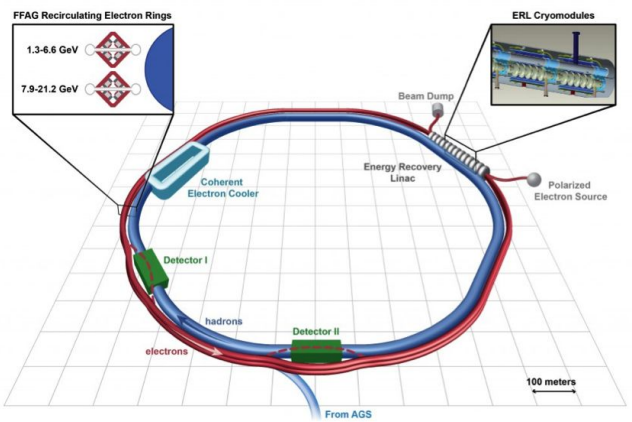


Aim:

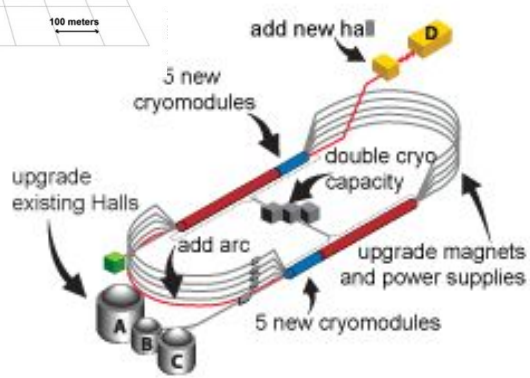
systematically determine nucleon structure
from ab initio quantum chromodynamics (QCD) calculations



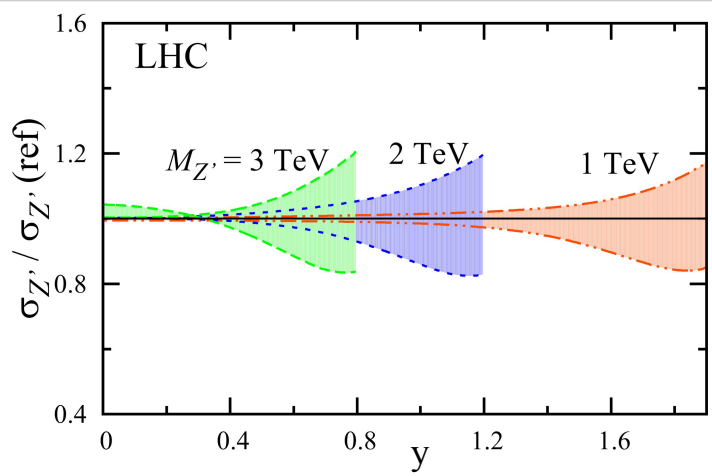




BNL EIC

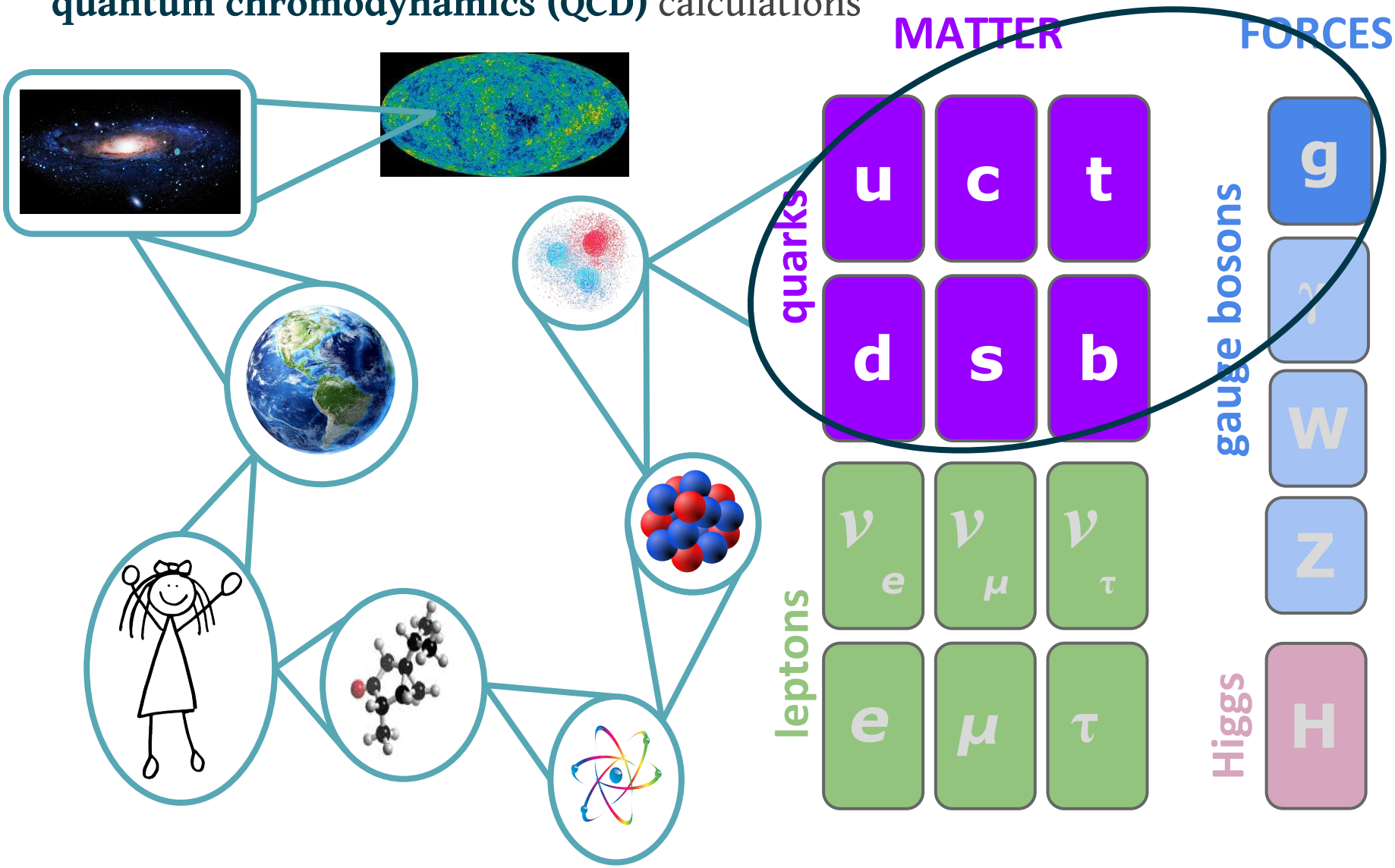


JLab 12 GeV

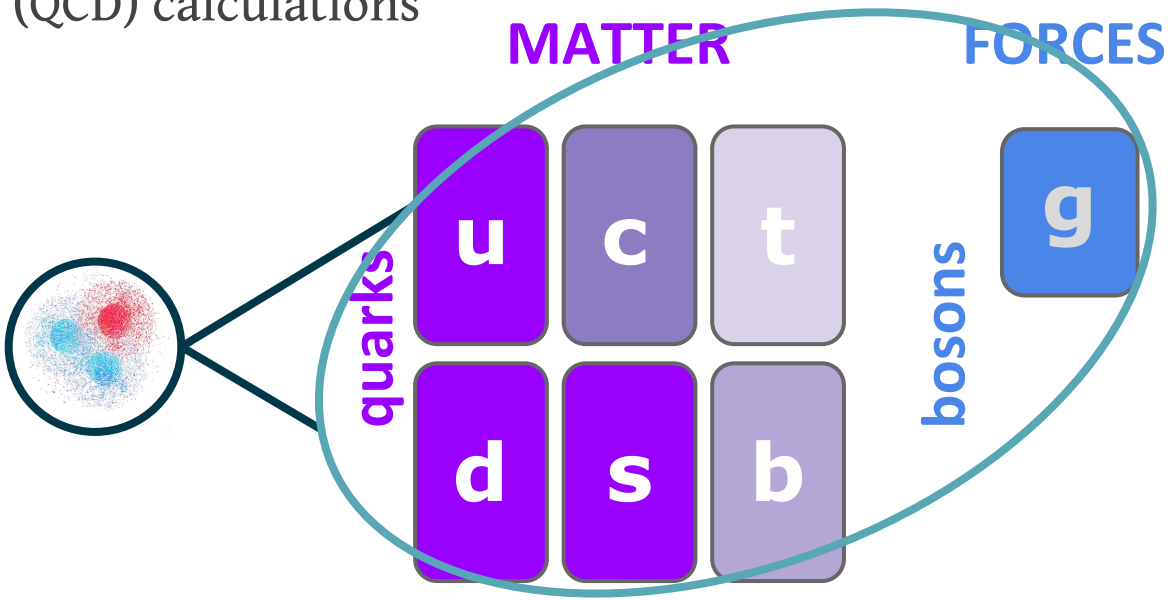


Brady *et al.*, JHEP 1206 (2012) 019

Aim: systematically determine nucleon structure from ab initio **quantum chromodynamics (QCD)** calculations



Aim: systematically determine **nucleon structure** from ab initio quantum chromodynamics (QCD) calculations



nucleon mass

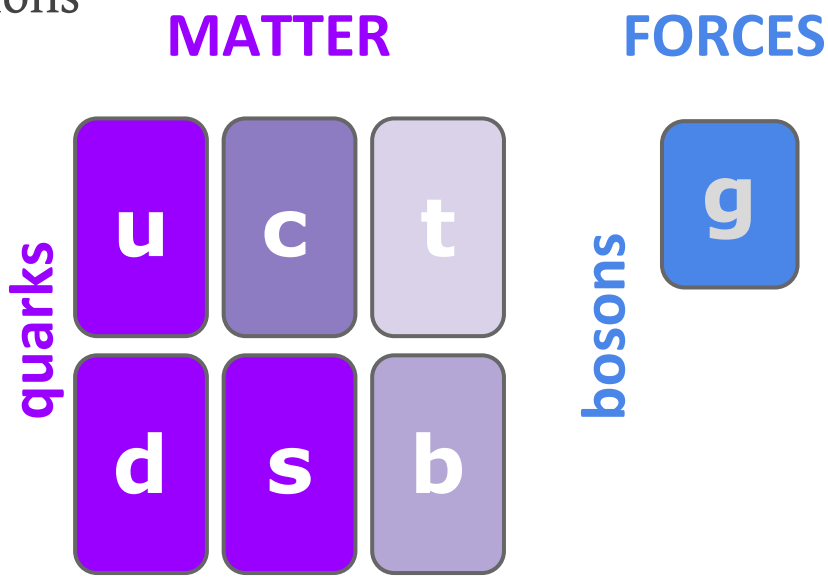
axial charge

form factors and structure functions

parton distributions

transverse momentum and generalised parton distributions

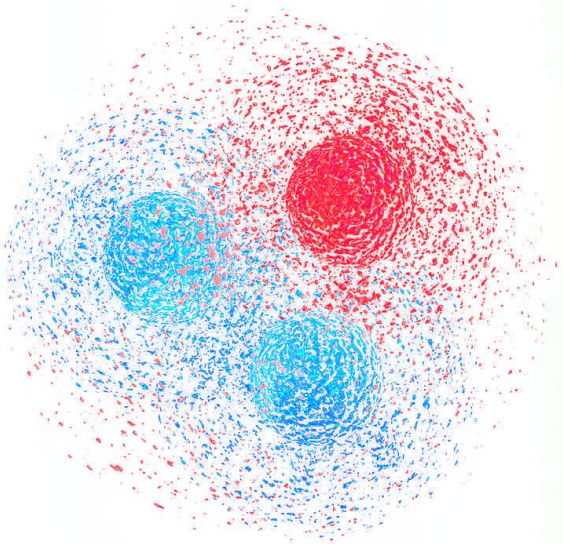
Aim: **systematically** determine nucleon structure from **ab initio** quantum chromodynamics (QCD) calculations



derived from QCD action

model-independent

uncertainties systematically improvable



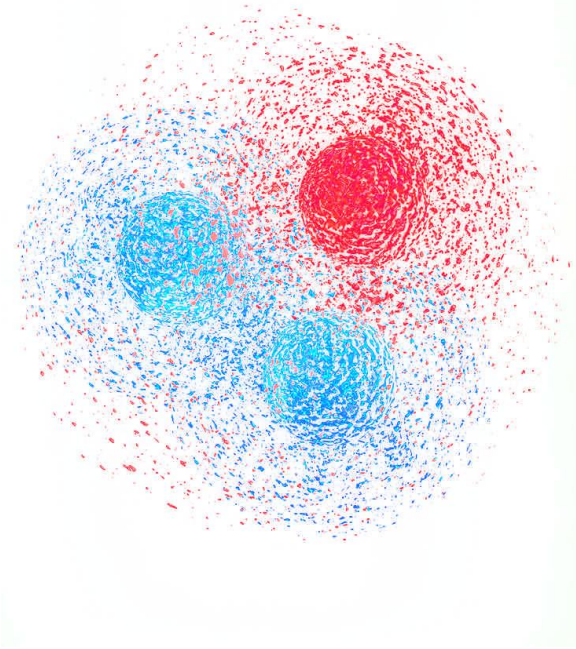
Aim:

systematically determine nucleon structure
from ab initio quantum chromodynamics (QCD) calculations

Proposal:

modify the operator product expansion

CJM & K. Orginos, PoS(Lattice2014) 330
CJM & K. Orginos, PRD 91 (2015) 074513



Physical motivation

- lattice quantum chromodynamics (QCD)
- parton distributions (on the lattice)

Modifying the operator product expansion

- Wilson coefficients
- Renormalisation group
- Nonperturbative calculations

QCD is the SU(3) gauge theory of the strong force

$$S_{\mathrm{QCD}} = \int \mathrm{d}^4x\, \overline{\psi}\,(i\not{D} - m)\,\psi - \frac{1}{4} G_{\mu\nu} G^{\mu\nu}$$

Here

$$D_\mu \psi = (\partial_\mu + i g A_\mu)\,\psi \qquad G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i g [A_\mu, A_\nu]$$

Phenomenologically rich and mathematically complicated

- confinement
- asymptotic freedom

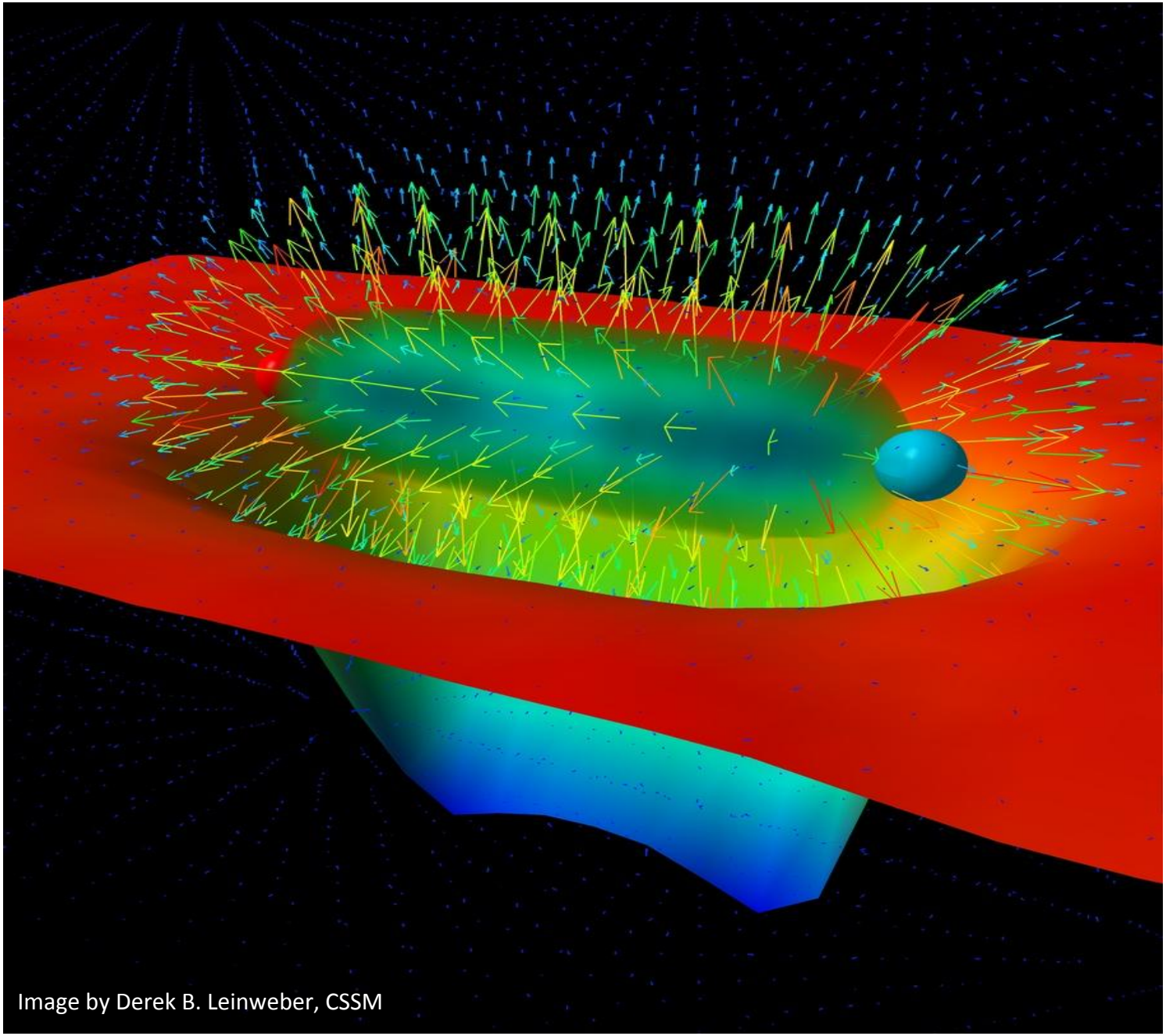
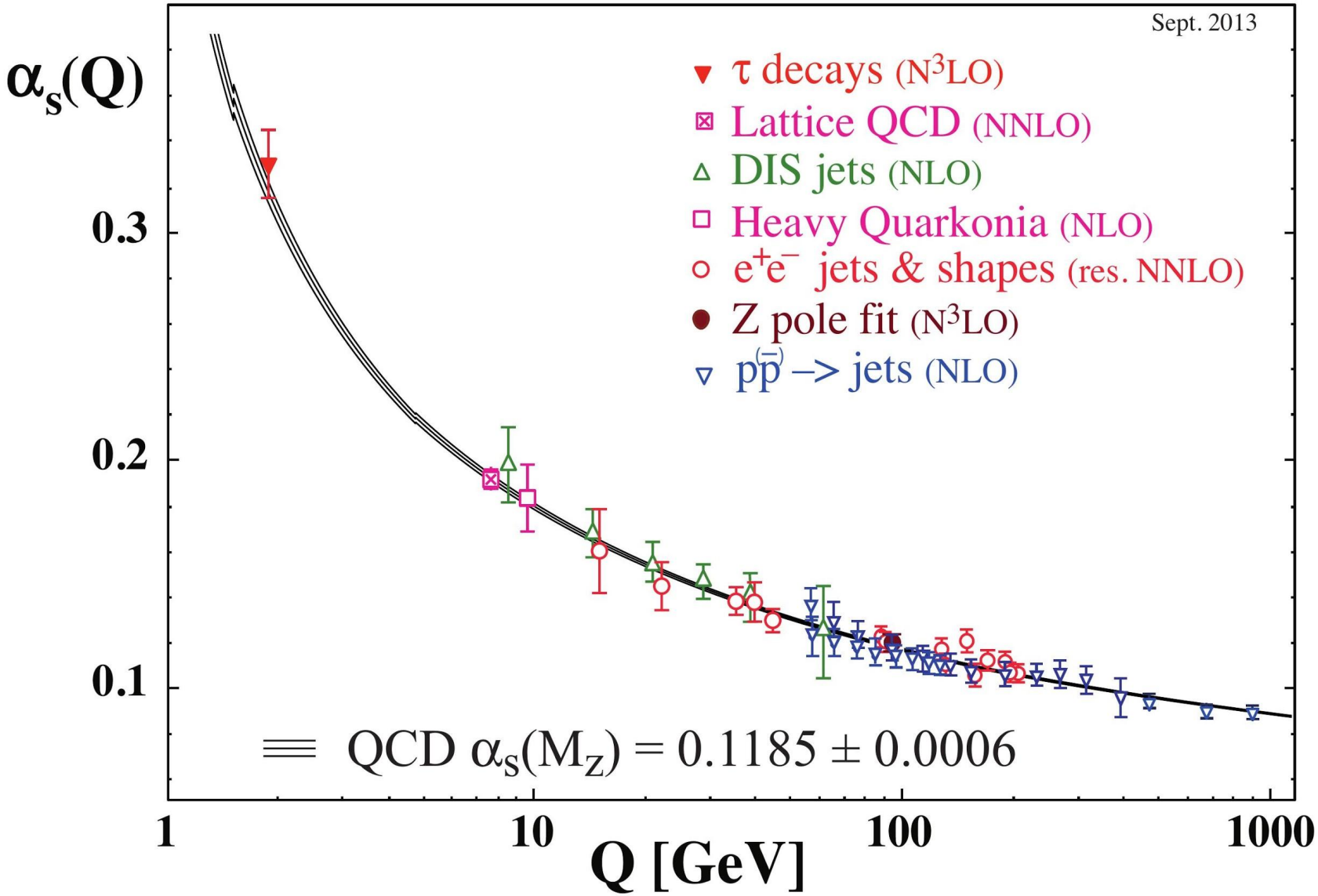


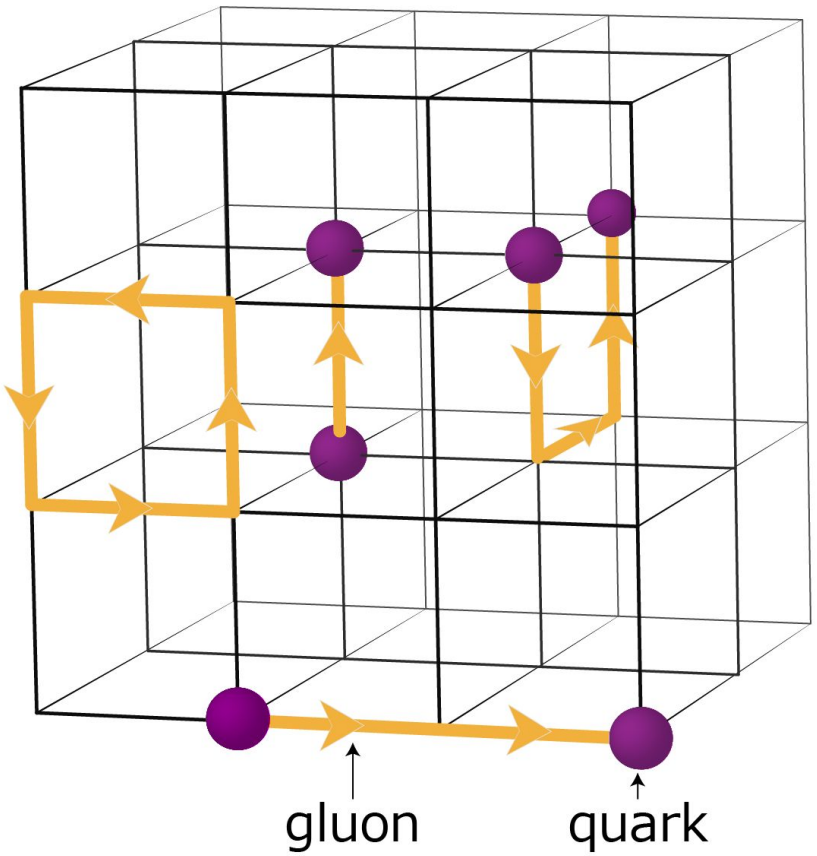
Image by Derek B. Leinweber, CSSM


Sept. 2013
$\alpha_s(Q)$
τ decays (N³LO)
Lattice QCD (NNLO)
DIS jets (NLO)
Heavy Quarkonia (NLO)
e⁺e⁻ jets & shapes (res. NNLO)
Z pole fit (N³LO)
p(p̄) –> jets (NLO)
QCD $\alpha_s(M_Z) = 0.1185 \pm 0.0006$
0.3
0.2
0.1
1
10
100
1000
Q [GeV]

PDG 2014

Nonperturbative QCD on the lattice

image: KEK lattice group/JICFuS



Nonperturbative QCD on the lattice

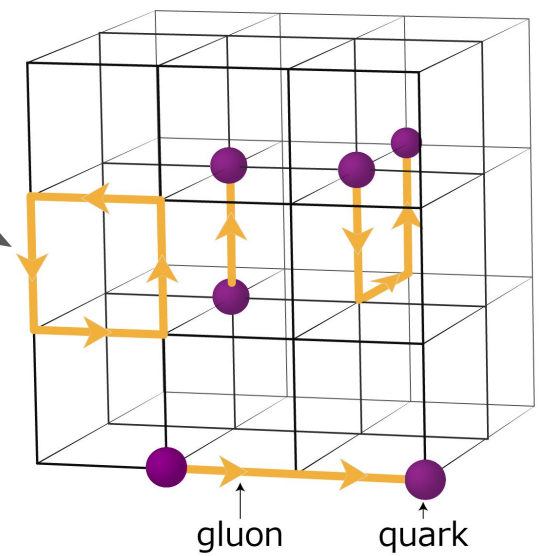
$$S_{\mathrm{QCD}} = \int \mathrm{d}^4x \left\{ \overline{\psi} \left(i \not{D} - m \right) \psi - \frac{1}{4} G_{\mu\nu} G^{\mu\nu} \right\}$$

$$S_{\mathrm{latt}} = \sum_x \left\{ \overline{\psi} \left(\sum_\mu \gamma_\mu \Delta_\mu + m \right) \psi + \sum_{\mu<\nu} \left(3 - P_{\mu\nu} \right) \right\}$$

Here

$$D_\mu \psi = \left(\partial_\mu + i g A_\mu \right) \psi$$

$$\Delta_\mu \psi = \frac{1}{2a} \left(U_\mu \psi(x + \hat{\mu}) - U_{-\mu} \psi(x - \hat{\mu}) \right)$$



Path integral

$$\langle\Omega|\mathcal{O}_1\mathcal{O}_2|\Omega\rangle_{\mathcal{M}_4} = \frac{1}{Z_{\mathcal{M}_4}}\int \mathcal{D}[\Phi]\, O_1[\Phi]\, O_2[\Phi]\, e^{iS[\Phi]} \qquad Z_{\mathcal{M}_4} = \int \mathcal{D}[\Phi]\, e^{iS[\Phi]}$$

Wick rotate

$$\langle\Omega|\mathcal{O}_1\mathcal{O}_2|\Omega\rangle_{\mathcal{E}_4} = \frac{1}{Z_{\mathcal{E}_4}}\int \mathcal{D}[\Phi]\, O_1[\Phi]\, O_2[\Phi]\, e^{-S[\Phi]} \qquad Z_{\mathcal{E}_4} = \int \mathcal{D}[\Phi]\, e^{-S[\Phi]}$$

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Boltzmann weight

Recall:

$$\boxed{P[s] = \frac{1}{Z_P} e^{-\frac{1}{k_B T} H[s]} \qquad Z_P = \sum_s e^{-H[s]} \qquad \langle\mathcal{O}\rangle = \frac{1}{Z_P}\sum_s O[s]\, e^{-H[s]}}$$

Path integral

$$\langle\Omega|\mathcal{O}_1\mathcal{O}_2|\Omega\rangle_{\mathcal{M}_4} = \frac{1}{Z_{\mathcal{M}_4}}\int \mathcal{D}[\Phi]\, O_1[\Phi]\, O_2[\Phi]\, e^{iS[\Phi]} \qquad Z_{\mathcal{M}_4} = \int \mathcal{D}[\Phi]\, e^{iS[\Phi]}$$

Wick rotate

$$\langle\Omega|\mathcal{O}_1\mathcal{O}_2|\Omega\rangle_{\mathcal{E}_4} = \frac{1}{Z_{\mathcal{E}_4}}\int \mathcal{D}[\Phi]\, O_1[\Phi]\, O_2[\Phi]\, e^{-S[\Phi]} \qquad Z_{\mathcal{E}_4} = \int \mathcal{D}[\Phi]\, e^{-S[\Phi]}$$

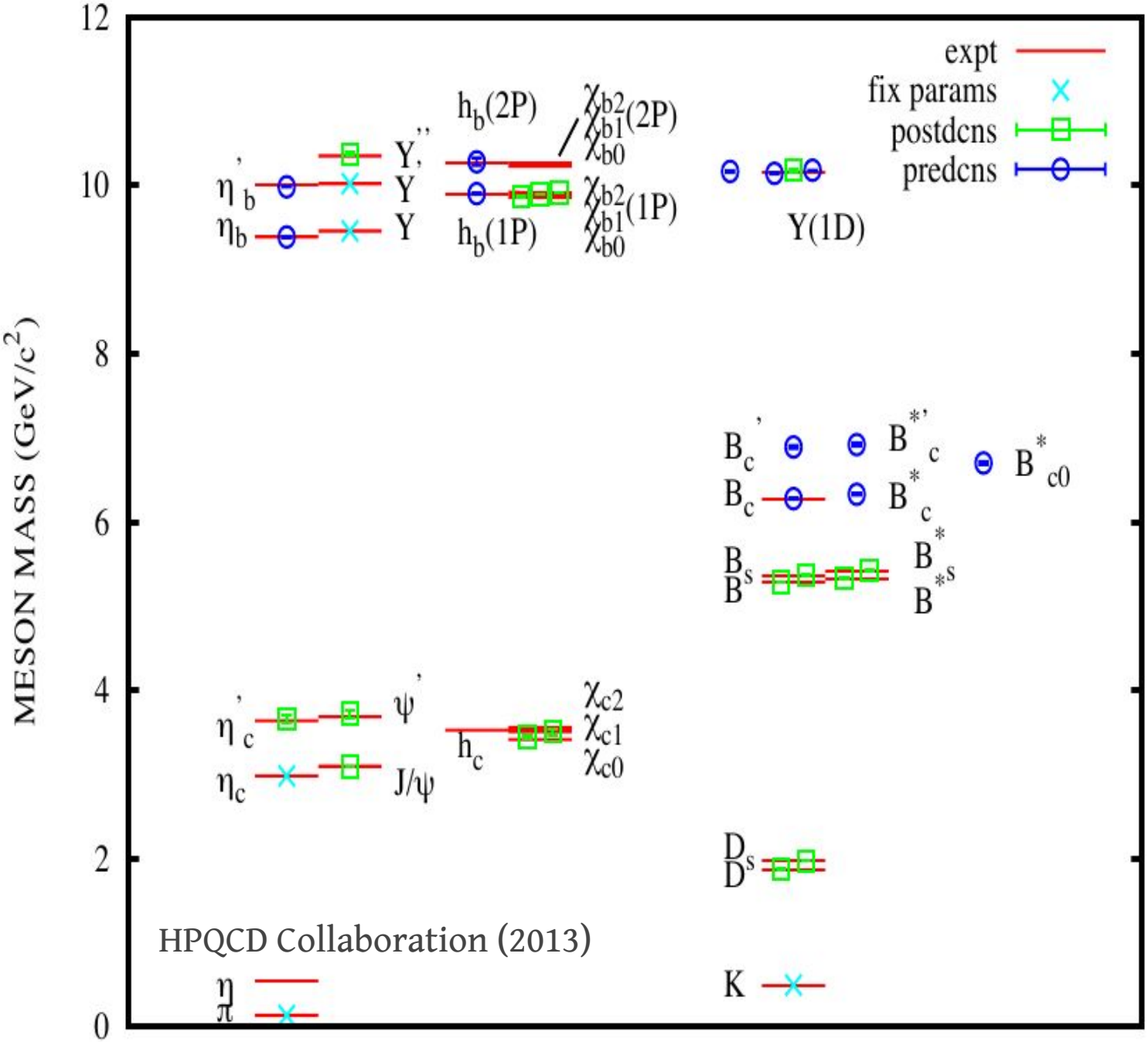
Generate field weighted configurations with probability $e^{-S[\Phi]}$

$$\langle\Omega|\mathcal{O}_1\mathcal{O}_2|\Omega\rangle_{\mathcal{E}_4} \simeq \frac{1}{N}\sum_{\Phi_{\text{weighted}}} O_1[\Phi_{\text{weighted}}]\, O_2[\Phi_{\text{weighted}}]$$

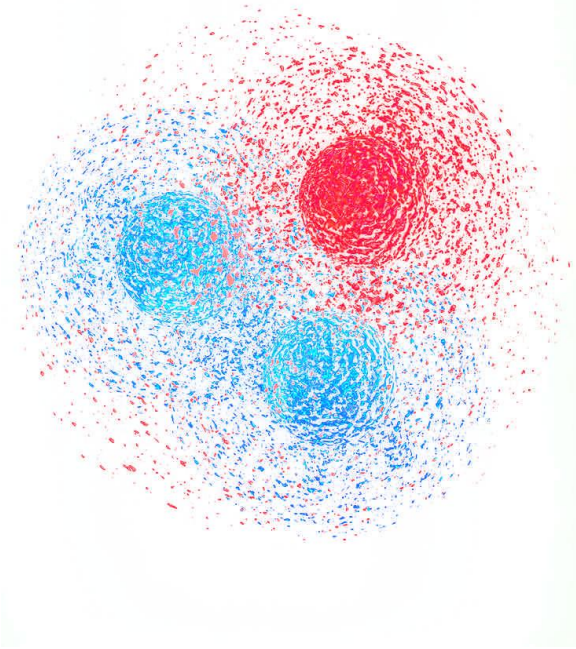
Measure correlation function N times

$$\langle\Omega|\mathcal{O}_\pi(t)\mathcal{O}_\pi^\dagger(0)|\Omega\rangle_{\mathcal{E}_4} \overset{at\gg 1}{\to} |\langle\pi|\mathcal{O}_\pi|\Omega\rangle|^2 e^{-m_\pi t} + \ldots$$

(With care) can extract excited spectrum



HPQCD Collaboration (2013)



Physical motivation

- lattice quantum chromodynamics (QCD)
- parton distributions (on the lattice)

Modifying the operator product expansion

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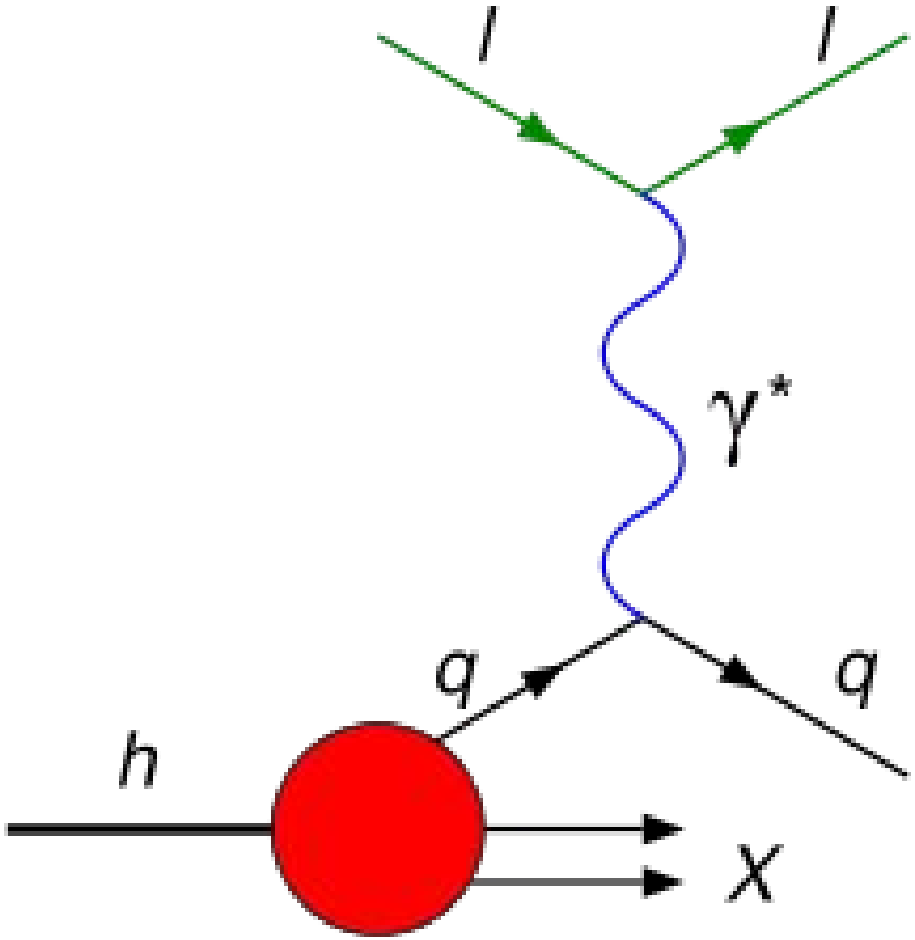
- Nonperturbative calculations

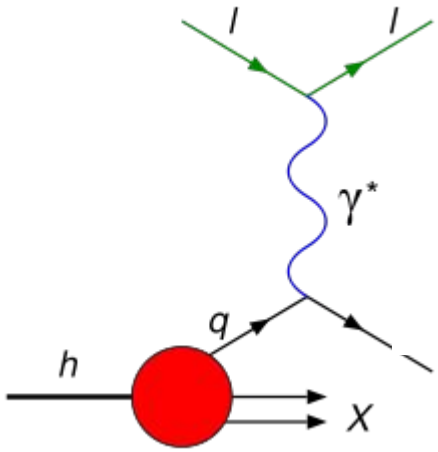
Deep inelastic scattering

$$q^2 = -Q^2$$

$$x = \frac{Q^2}{2P \cdot q}$$



Deep inelastic scattering



Decompose cross-section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega \mathrm{d}E'} = \frac{e^4}{16\pi^2 Q^4}\, \ell^{\mu\nu}\, W_{\mu\nu}$$

Hadronic tensor

$$W_{\mu\nu}(p,q) = \frac{1}{4\pi}\int \mathrm{d}^4x\, e^{iq\cdot x} \langle\, p, \lambda' \,|\, [j_\mu(x), j_\nu(x)] \,|\, p, \lambda \,\rangle$$

Express in terms of structure functions F_1, F_2, g_1, g_2

$$F(x,Q^2) = \int \mathrm{d}y\, C\left(\frac{x}{y}, \frac{Q^2}{\mu^2}\right) f_{q/N}(x,\mu^2)$$

(Light cone) parton distributions universal

$$f_{q/N}(x,Q^2) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \mathrm{d}y^-\, e^{-iy^-p^+} \langle\, N \,|\, \overline{\psi}(0^+, y^-, 0_\mathrm{T})\gamma_+ U(y^-, 0)\, \psi(0) \,|\, N \,\rangle$$

Relate hadronic tensor to forward Compton amplitude

$$W_{\mu\nu} = \frac{1}{2\pi}\mathrm{Im}\{T_{\mu\nu}\}$$

Operator product expansion generates “twist” (dimension - spin) expansion

Twist-2 operators dominate in Bjorken limit

$$\overline{\psi}\gamma_{\{\mu_1}\overleftrightarrow{D}_{\mu_2}\ldots\overleftrightarrow{D}_{\mu_n\}}\psi - \mathrm{traces}$$

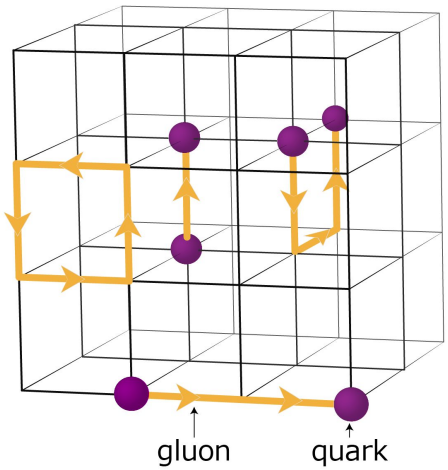
Mellin moments

$$\langle x^n\rangle_{f_{q/N}} = \int_{-1}^{1}\mathrm{d}x\, x^n f_{q/N}(x)$$

Wick rotation of moments is trivial

... however...

Rotational symmetry broken to cubic symmetry



Physical results require continuum limit

$$\langle \mathcal{O}_{\text{cont}}(\mu, x)\rangle = \lim_{a\to 0} \langle \mathcal{O}_{\text{latt}}(a\mu, x/a)\rangle$$

Operators **mix under renormalisation** on the lattice

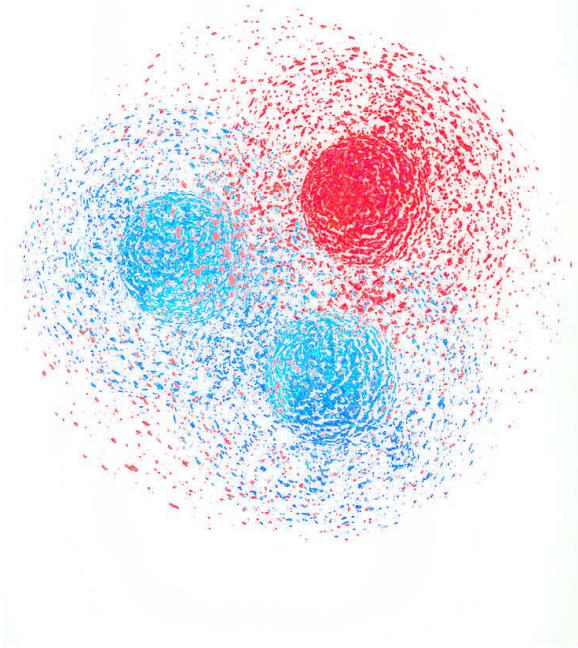
$$\langle \mathcal{O}_{\text{cont}}(\mu, x)\rangle = \lim_{a\to 0} \sum_j \langle \mathcal{O}^{(j)}_{\text{latt}}(a\mu, x/a)\rangle$$

Power divergent mixing between operators of different mass dimension

$$\langle \mathcal{O}_{\text{cont}}\rangle \sim \frac{1}{a^2}\langle \overline{\psi}\,\psi\rangle + \frac{1}{a}\langle \overline{\psi}\gamma^\mu D_\mu \psi\rangle + \langle \overline{\psi} D^\mu D_\mu \psi\rangle + \ldots$$

Limits lattice calculations to first four moments

Detmold *et al.*, Eur. Phys. J. C 3 (2001) 1
Detmold *et al.*, Phys. Rev. D 68 (2001) 034025
Detmold *et al.*, Mod. Phys. Lett. A 18 (2003) 2681



Aim:

systematically determine nucleon structure
from ab initio QCD calculations

Clarified aim:

systematically determine parton distributions
from ab initio lattice QCD calculations with a formalism that
removes power-divergent mixing in the continuum limit

Proposal:

modify the operator product expansion

Dawson *et al.*, Nucl. Phys. B 514 (1998) 313
Detmold & Lin, Phys. Rev. D 73 (2006) 014501
Lohmayer & Neuberger, PoS(LATTICE 2011)

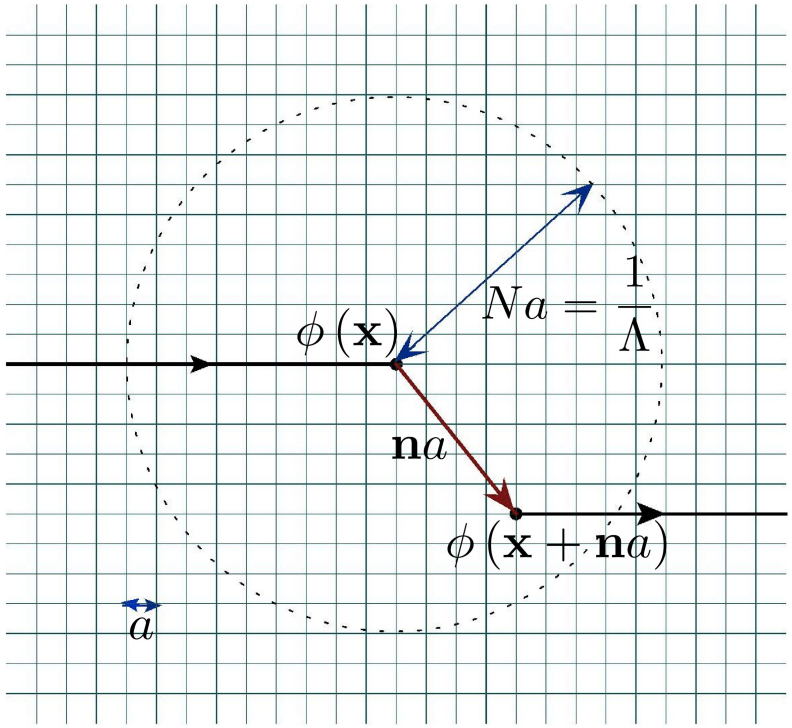
“Smearing” partially restores rotational symmetry: **suppresses operator mixing**

Widely used lattice technique

Construct operators with improved continuum limits, *i.e.* reduced systematic uncertainties

Precisely identify hadronic excited states

Dudek *et al.*, Phys. Rev. Lett. 103 (2009) 262001
Dudek *et al.*, Phys. Rev. D 82 (2010) 034508
Edwards *et al.*, Phys. Rev. D 84 (2011) 074508
Meinel, Phys. Rev. D 85 (2012) 114510



Davoudi & Savage, Phys. Rev. D 86 (2012) 054505

Smearing implemented via the **gradient flow**

Narayanan & Neuberger, JHEP 0603 (2006) 064
Lüscher, Commun. Math. Phys. 293 (2010) 899

Deterministic evolution of fields in "flow time" τ toward classical minimum

$$\frac{\partial}{\partial\tau}\overline{\phi}(\tau,x) = \partial^2\overline{\phi}(\tau,x) \qquad \overline{\phi}(\tau=0,x) = \phi(x) \qquad \widetilde{\overline{\phi}}(\tau,p) = e^{-\tau p^2}\widetilde{\phi}(p)$$

Here exact solution possible with Dirichlet boundary conditions

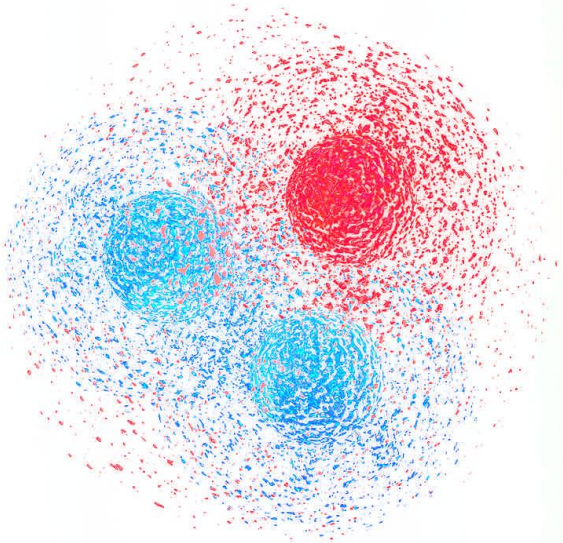
$$\overline{\phi}(\tau,x) = \int \mathrm{d}^4y \int \frac{\mathrm{d}^4p}{(2\pi)^4} e^{ip\cdot(x-y)} e^{-\tau p^2}\phi(y) = \frac{1}{16\pi^2\tau^2}\int \mathrm{d}^4y\, e^{-(x-y)^2/(4\tau)}\phi(y)$$

Smearing radius $s_{\mathrm{rms}} = \sqrt{8\tau}$

Interactions occur at zero flow time (*i.e.* in the original "boundary" theory)

Renormalised boundary theory requires no further renormalisation

Lüscher & Weisz, JHEP 1102 (2011) 51
Makino & Suzuki, arXiv:1410.7538



Smearing implemented via the **gradient flow**:

removes power-divergent mixing provided the flow time is kept fixed in physical units in the continuum limit

Consider twist-2 operators

$$\mathcal{T}_{\mu_1 \ldots \mu_n}(x) = \phi(x)\partial_{\mu_1} \ldots \partial_{\mu_n}\phi(x) - \text{traces}$$

Example: continuum matrix element

$$\langle\, \Omega \,|\, \phi^2(0) \cdot \phi(0)\partial_\mu\partial_\nu\phi(0) \,|\, \Omega \,\rangle = 0$$

On the lattice

$$\langle\, \Omega \,|\, \phi^2(0) \cdot \phi(0)\nabla_\mu\nabla_\nu\phi(0) \,|\, \Omega \,\rangle = -\frac{\delta_{\mu\nu}}{48a^2} + \mathcal{O}(a^0, \lambda)$$

[quantum corrections do not affect scaling]

Davoudi & Savage, Phys. Rev. D 86 (2012) 054505

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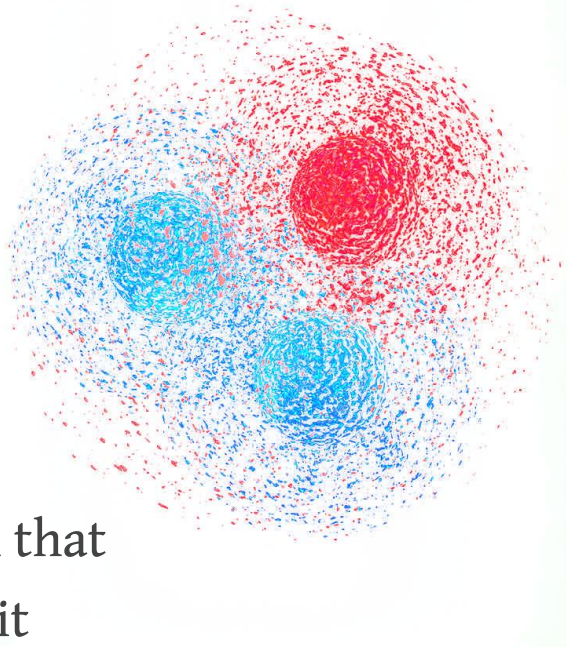
With smeared degrees of freedom

$$\langle\, \Omega \,|\, \overline{\phi}^2(\tau, 0) \cdot \overline{\phi}(\tau, 0)\nabla_\mu\nabla_\nu\overline{\phi}(\tau, 0) \,|\, \Omega \,\rangle = -\frac{\delta_{\mu\nu}}{256\pi^2\tau} + \mathcal{O}(a^0, \lambda)$$

Aim:

systematically determine nucleon structure
from ab initio QCD calculations



Clarified aim:

systematically determine parton distributions
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Proposal:

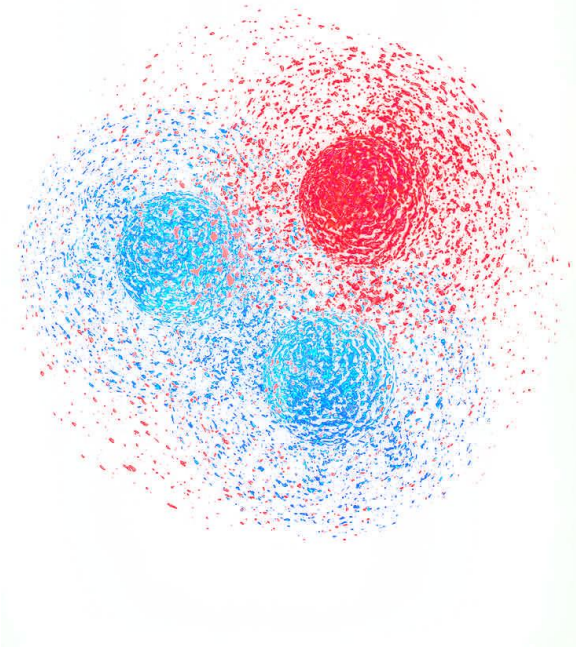
modify the operator product expansion

Dawson *et al.*, Nucl. Phys. B 514 (1998) 313
Detmold & Lin, Phys. Rev. D 73 (2006) 014501
Lohmayer & Neuberger, PoS(LATTICE 2011)

Clarified proposal:

systematically connect nonperturbative, smeared lattice
calculations to continuum physics by modifying the
operator product expansion to account for new scale

CJM & K. Orginos, PoS(Lattice2014) 330
CJM & K. Orginos, PRD 91 (2015) 074513



Physical motivation

- lattice quantum chromodynamics (QCD)
- parton distributions (on the lattice)

Modifying the operator product expansion

- Wilson coefficients
- Renormalisation group
- Nonperturbative calculations

Wilson's idea: operator product expansion (OPE)

Wilson, Phys. Rev. 179 (1969) 1499

nonlocal operator ~ (perturbative) coefficients x local operators

For example, in free scalar field theory

$$\phi(x)\phi(0) = \frac{1}{x^2} c_{\mathbb{I}} \mathbb{I} + c_{\phi^2} \phi^2(0) + \mathcal{O}(x)$$

[here the OPE is just a Laurent expansion]

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Wilson, Phys. Rev. 179 (1969) 1499

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Interactions modify Wilson coefficients

$$\phi(x)\phi(0) = \frac{1}{4\pi x^2} \left(1 + a_{\mathbb{I}} \log(x^2\mu^2) \ldots\right)\mathbb{I} + \left(1 + a_{\phi^2} \log(x^2\mu^2) \ldots\right)\phi^2(0) + \ldots$$

... but not their leading-x behaviour

(Formally) convenient to separate leading-x behaviour

$$O(x) \overset{x\to 0}{\sim} \sum_k d_k(x^2)\, c_k(x,\mu)\, \mathcal{O}_R^{(k)}(0,\mu)$$

Operator relation

$$\langle\Omega| O(x)\tilde{\phi}(p_1)\ldots\tilde{\phi}(p_N)|\Omega\rangle \overset{x\to 0}{\sim} \sum_k d_k(x^2)\, c_k(x,\mu) \langle\Omega|\mathcal{O}_R^{(k)}(0,\mu)\tilde{\phi}(p_1)\ldots\tilde{\phi}(p_N)|\Omega\rangle$$

Replace local operators

nonlocal operator ~ (perturbative) coefficients x local operators

$$O(x) \overset{x\to 0}{\sim} \sum_k d_k(x^2)\, c_k(x,\mu)\mathcal{O}_R^{(k)}(0,\mu)$$

with **locally smeared operators**

nonlocal operator ~ (perturbative) coefficients x locally smeared operators

$$O(x) \overset{x\to 0}{\sim} \sum_k d_k(x^2)\overline{c}_k(x,\mu,\tau)\overline{\mathcal{O}}_R^{(k)}(0,\mu,\tau)$$

Replace local operators

nonlocal operator ~ (perturbative) coefficients x local operators

$$O(x) \overset{x\to 0}{\sim} \sum_k d_k(x^2)\, c_k(x,\mu)\mathcal{O}_R^{(k)}(0,\mu)$$

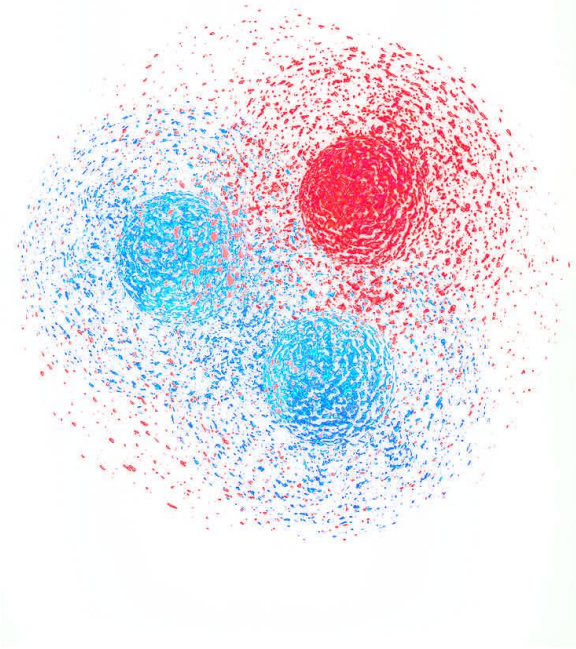
with **locally smeared operators**

nonlocal operator ~ (perturbative) coefficients x locally smeared operators

$$O(x) \overset{x\to 0}{\sim} \sum_k d_k(x^2)\overline{c}_k(x,\mu,\tau)\overline{\mathcal{O}}_R^{(k)}(0,\mu,\tau)$$

Our example

$$\phi(x)\phi(0) = \frac{1}{x^2}c_{\mathbb{I}}\mathbb{I} + c_{\phi^2}\,\phi^2(0) + \mathcal{O}(x) \longrightarrow \phi(x)\phi(0) = \frac{1}{x^2}\overline{c}_{\mathbb{I}}\mathbb{I} + \overline{c}_{\phi^2}\,\overline{\phi}^2(\tau,0) + \mathcal{O}(x)$$



Physical motivation

- lattice quantum chromodynamics (QCD)
- parton distributions (on the lattice)

Modifying the operator product expansion

- Wilson coefficients
- Renormalisation group
- Nonperturbative calculations

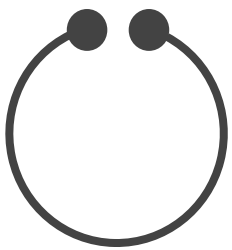
Calculate Wilson coefficients in standard manner:

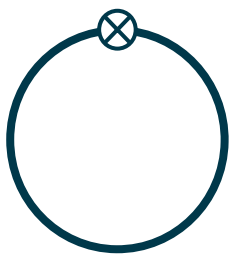
$$\phi(x)\phi(0) = \frac{1}{x^2}\overline{c}_{\mathbb{I}}\mathbb{I} + \overline{c}_{\phi^2}\,\overline{\phi}^2(\tau,0) + \mathcal{O}(x)$$

Rearrange sOPE and work at tree-level and expand to order m^2

$$\overline{c}_{\mathbb{I}}^{(0)} = x^2\Big\{\,\langle\,\Omega\,|\,\phi(x)\phi(0)\,|\,\Omega\,\rangle^{(0)} - \langle\,\Omega\,|\,\overline{\phi}^2(\tau,0)\,|\,\Omega\,\rangle^{(0)}\Big\}_{\text{to }\mathcal{O}(m^2)}$$

[Or graphically]





So

$$\overline{c}_{\mathbb{I}}^{(0)} = x^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\left\{\frac{e^{ik\cdot x}}{k^2+m^2} - \frac{e^{-2k^2\tau}}{k^2+m^2}\right\}_{\text{to }\mathcal{O}(m^2)}$$

We obtain

$$\bar{c}_{\mathbb{I}}^{(0)} = \frac{1}{4\pi^2}\left[1 - \frac{x^2}{8\tau} + \frac{m^2x^2}{4}\left[\gamma_{\mathrm{E}} - 1 + \log\left(\frac{x^2}{8\tau}\right)\right]\right]$$

Compare to the Wilson coefficient in the original OPE

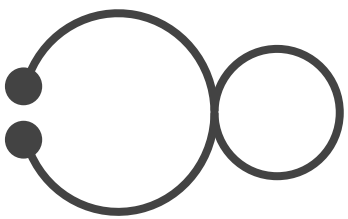
$$c_{\mathbb{I}}^{(0)} = \frac{1}{4\pi^2}\left[1 + \frac{m^2x^2}{4}\left[1 + 2\gamma_{\mathrm{E}} + \log\left(\frac{\mu^2x^2}{16}\right)\right]\right]$$

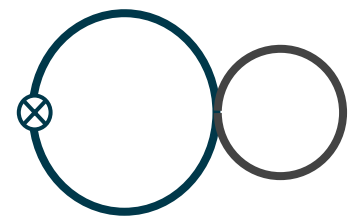
Beyond tree-level things get only slightly trickier...

One loop calculation proceeds similarly

$$\bar{c}_{\mathbb{I}}^{(1)} = x^2 \Big\{ \langle\, \Omega \,|\, \phi(x)\phi(0) \,|\, \Omega \,\rangle^{(1)} - \langle\, \Omega \,|\, \overline{\phi}^2(\tau, 0) \,|\, \Omega \,\rangle^{(1)} \Big\}_{\text{to } \mathcal{O}(m^2)}$$

[Or graphically]





Thus

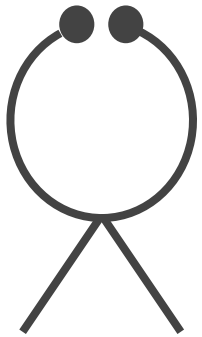
$$\bar{c}_{\mathbb{I}}^{(1)} = x^2 \left\{ \frac{1}{2} \int \frac{\mathrm{d}^4 k_1}{(2\pi)^4} \frac{1}{k_1^2 + m^2} \int \frac{\mathrm{d}^4 k_2}{(2\pi)^4} \left[\frac{e^{ik_2 \cdot x}}{(k_2^2 + m^2)^2} - \frac{e^{-2k_2^2 \tau}}{(k_2^2 + m^2)^2} \right] \right\}_{\text{to } \mathcal{O}(m^2)}$$

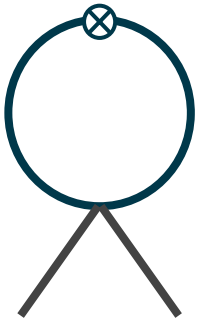
$$\bar{c}_{\mathbb{I}}^{(1)} = \frac{1}{4\pi^2} \left[1 - \frac{x^2}{8\tau} + \frac{m_{\mathrm{R}}^2 x^2}{4} \left[\gamma_{\mathrm{E}} - 1 + \log\left(\frac{x^2}{8\tau} \right) \right] \right]$$

For the leading connected contribution

$$\overline{c}^{(1)}_{\phi^2} = \left\{ \left\langle \Omega \,|\, \phi(x)\phi(0)\, \widetilde{\phi}_{\mathrm{R}}(p_1)\widetilde{\phi}_{\mathrm{R}}(p_2) \,|\, \Omega \right\rangle^{(1)} - \left\langle \Omega \,|\, \overline{\phi}^2(\tau,0)\, \widetilde{\phi}_{\mathrm{R}}(p_1)\widetilde{\phi}_{\mathrm{R}}(p_2) \,|\, \Omega \right\rangle^{(1)} \right\}_{\text{to } \mathcal{O}(m^0)}$$

[Or graphically]





$$\overline{c}^{(1)}_{\phi^2} = \frac{1}{(p_1^2+m^2)(p_2^2+m^2)} \left\{ -\frac{\lambda}{2} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \left[\frac{e^{ik\cdot x}}{(k^2+m^2)(q^2+m^2)} - \frac{e^{-(k^2+q^2)\tau}}{(k^2+m^2)(q^2+m^2)} \right] \right\}_{\text{to } \mathcal{O}(m^0)}$$

$$\bar{c}^{(1)}_{\phi^2} = \frac{1}{(p_1^2+m^2)(p_2^2+m^2)}\left\{-\frac{\lambda}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\left[\frac{e^{ik\cdot x}}{(k^2+m^2)(q^2+m^2)}-\frac{e^{-(k^2+q^2)\tau}}{(k^2+m^2)(q^2+m^2)}\right]\right\}_{\text{to } \mathcal{O}(m^0)}$$

Derivative:

$$\lambda\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{q\left(e^{ik\cdot x}-e^{-(k^2+q^2)\tau}\right)}{(k^2+m^2)(q^2+m^2)}$$

Convergent: small spacetime limit is well-defined and vanishes if

$$\lim_{x\to 0}\lambda\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{q\left(e^{ik\cdot x}-e^{-(k^2+q^2)\kappa^2x^2}\right)}{(k^2+m^2)(q^2+m^2)} = 0$$

or

$$\lim_{x\to 0}\lambda\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{q\left(e^{ik\cdot x}-1\right)}{(k^2+m^2)(q^2+m^2)} = 0+\mathcal{O}(\tau)$$

$$\bar{c}^{(1)}_{\phi^2} = \frac{1}{(p_1^2+m^2)(p_2^2+m^2)}\left\{-\frac{\lambda}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\left[\frac{e^{ik\cdot x}}{(k^2+m^2)(q^2+m^2)}-\frac{e^{-(k^2+q^2)\tau}}{(k^2+m^2)(q^2+m^2)}\right]\right\}_{\text{to } \mathcal{O}(m^0)}$$

Derivative:

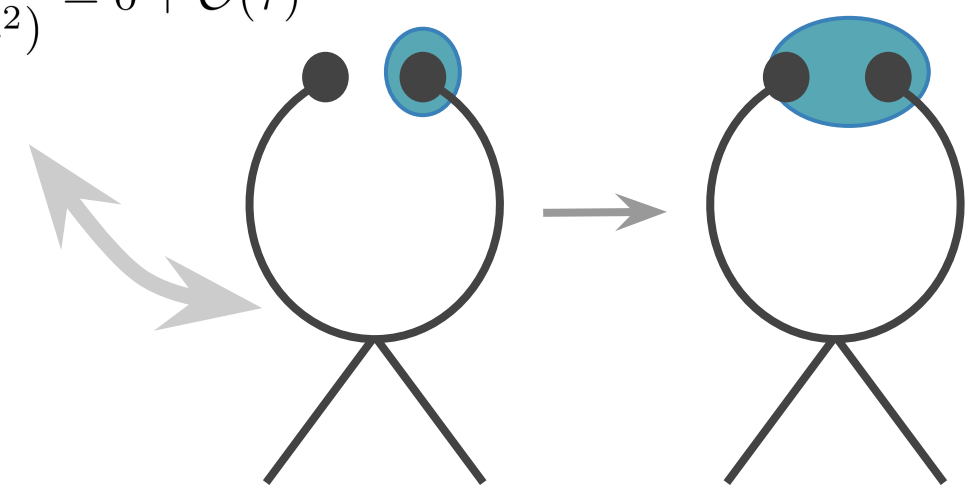
$$\lambda\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{q\left(e^{ik\cdot x}-e^{-(k^2+q^2)\tau}\right)}{(k^2+m^2)(q^2+m^2)}$$

Convergent: small spacetime limit is well-defined and vanishes if

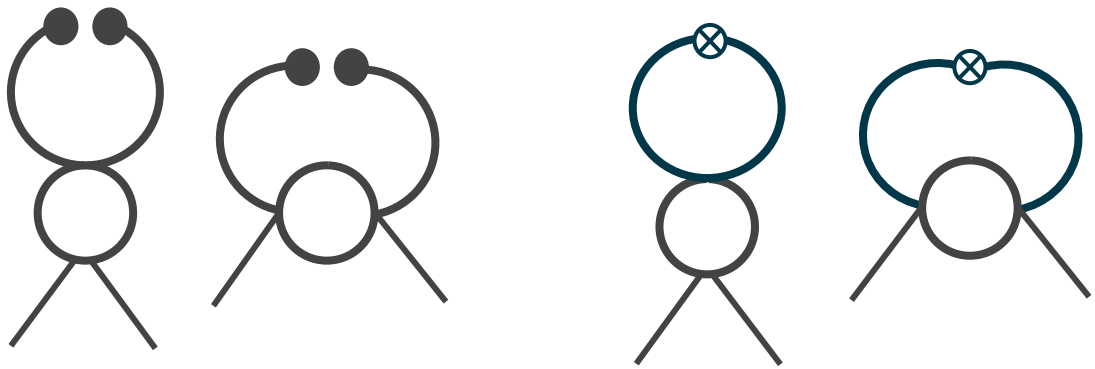
$$\lim_{x\to 0}\lambda\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{q\left(e^{ik\cdot x}-e^{-(k^2+q^2)\kappa^2x^2}\right)}{(k^2+m^2)(q^2+m^2)}=0$$

or

$$\lim_{x\to 0}\lambda\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{q\left(e^{ik\cdot x}-1\right)}{(k^2+m^2)(q^2+m^2)}=0+\mathcal{O}(\tau)$$



At two loops



Leading to

$$d_{\phi^2}(x^2/\tau, \mu^2 x^2) = 1 + \frac{\lambda_{\mathrm{R}}}{32\pi^2}\left[\gamma_{\mathrm{E}} - 1 + \log\left(\frac{x^2}{8\tau}\right)\right]$$

Renormalised boundary theory remains renormalised at non-zero flow time

Makino & Suzuki, PTEP (2015) 033B08
Makino *et al.*, PTEP (2015) 043B07
Aoki *et al.*, JHEP 1504 (2015) 156
Kikuchi & Onogi, JHEP 1411 (2014) 094

Gradient flow in 2D O(3) model

$$\frac{\partial n^i(\tau,x)}{\partial \tau} = \left[\delta^{ij} - n^i(\tau,x)n^j(\tau,x)\right]\partial^2 n^j(\tau,x) \qquad n^i(\tau=0,x) = n^i(x)$$

$$n^i(\tau,x) = \pi^i(\tau,x) \quad \text{for} \quad i=1,2 \qquad n^3(\tau,x) = \sqrt{1-\pi^i(\tau,x)\pi^i(\tau,x)}$$

Exact solution no longer possible: generate iterative tree-level expansion

$$n^i(\tau,x) = \int \mathrm{d}^2 y \int \frac{\mathrm{d}^2 p}{(2\pi)^2} e^{ip\cdot(x-y)} \left[e^{-\tau p^2} n^i(y) - \int_0^\tau \mathrm{d}s\, e^{-sp^2} R^i(s,y) \right]$$

$$R^i(s,y) = n^i(s,y) n^j(s,x) \partial^2 n^j(s,x)$$

Interactions occur in the bulk, *i.e.* at non-zero flow time, but no closed loops

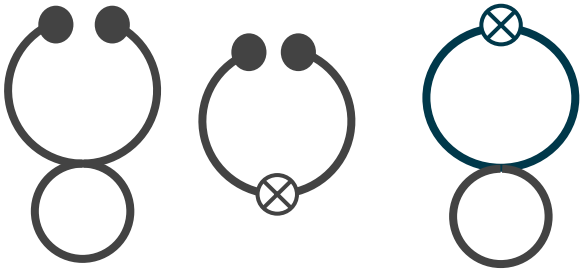
Consider

$$\pi(x)\pi(0) = \frac{b_{\mathbb{I}}}{4\pi}\mathbb{I} + b_{\pi^2}\pi^2(\tau,0) + b_{\partial_\mu}x^\mu\partial_\mu\pi^2(\tau,0) + \dots$$

One loop calculations (almost) as straightforward as 4D φ^4 scalar field theory

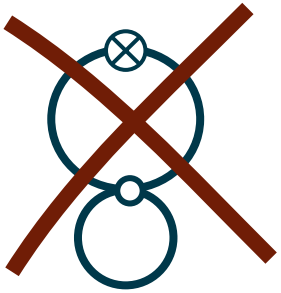
Two loops - interactions complicate the picture

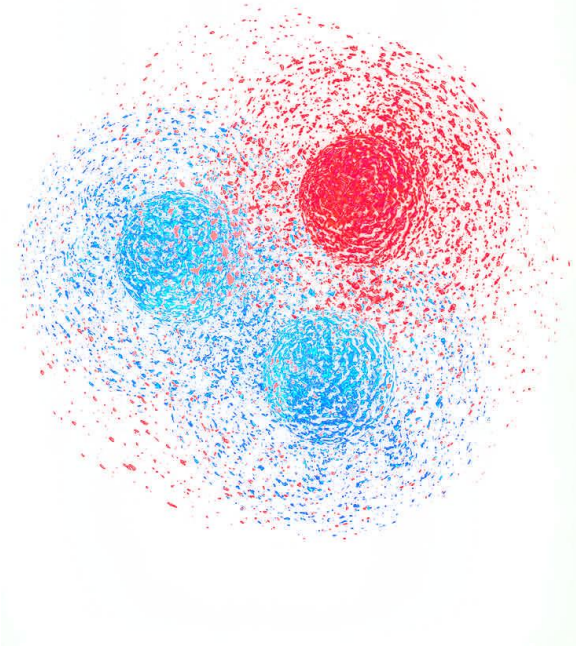
I. quantum interactions



2. tree interactions

but still no flow loops





Physical motivation

- lattice quantum chromodynamics (QCD)
- parton distributions (on the lattice)

Modifying the operator product expansion

- Wilson coefficients
- Renormalisation group
- Nonperturbative calculations

Renormalisation group equations for connected Green functions

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu} = \mu\frac{\partial}{\partial\mu}\bigg|_{\lambda,m_\mathrm{R}} + \beta\frac{\partial}{\partial\lambda}\bigg|_{\mu,m_\mathrm{R}} - \gamma_m m_\mathrm{R}\frac{\partial}{\partial m_\mathrm{R}}\bigg|_{\mu,\lambda}$$

Then

$$\left[\mu\frac{\mathrm{d}}{\mathrm{d}\mu} + N\gamma\right]\left\langle\,\Omega\,|\,\widetilde{\phi}_\mathrm{R}(p_1)\ldots\widetilde{\phi}_\mathrm{R}(p_N)\,|\,\Omega\,\right\rangle = 0$$

$$\left[\mu\frac{\mathrm{d}}{\mathrm{d}\mu} + N\gamma - \gamma_m\right]\left\langle\,\Omega\,|\,[\phi^2(0)]_\mathrm{R}\,\widetilde{\phi}_\mathrm{R}(p_1)\ldots\widetilde{\phi}_\mathrm{R}(p_N)\,|\,\Omega\,\right\rangle = 0$$

Applying $\left[\mu\frac{\mathrm{d}}{\mathrm{d}\mu} + (N+2)\gamma\right]$ to our example OPE

$$\left\langle\,\Omega\,|\,\widetilde{\phi}_\mathrm{R}(p_1)\ldots\widetilde{\phi}_\mathrm{R}(p_{N+2})\,|\,\Omega\,\right\rangle = c_{\phi^2}\left\langle\,\Omega\,|\,[\phi^2(0)]_\mathrm{R}\,\widetilde{\phi}_\mathrm{R}(p_1)\ldots\widetilde{\phi}_\mathrm{R}(p_N)\,|\,\Omega\,\right\rangle + \mathcal{O}(x)$$

we obtain

$$\left[\mu\frac{\mathrm{d}}{\mathrm{d}\mu} + 2(\gamma+\gamma_m)\right]c_{\phi^2} = 0$$

For the sOPE we first write

Lüscher & Weisz, JHEP 1102 (2011) 51
Suzuki, PTEP (2013) 083B03
Makino & Suzuki, PTEP (2014) 063B02

$$[\phi^2(0)]_{\mathrm{R}} = \mathcal{Z}_{\phi^2}(\tau,\mu^2)\overline{\phi}^2(\tau,0) + \mathcal{O}(\tau)$$

where

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\log\left(\mathcal{Z}_{\phi^2}(\tau,\mu^2)\right) = 2\gamma_m$$

To one loop

$$\mathcal{Z}_{\phi^2}(\tau,\mu^2) = 1 - \frac{\lambda}{32\pi^2}\left[1+\gamma_{\mathrm{E}}+\log(2\tau\mu^2)\right] + \mathcal{O}(\lambda^2,\tau)$$

We now apply $\left[\mu\frac{\mathrm{d}}{\mathrm{d}\mu} - \kappa\frac{\mathrm{d}}{\mathrm{d}\kappa} + (N+2)\gamma\right]$ to our example sOPE

$$\left\langle\, \Omega \,|\, \widetilde{\phi}_{\mathrm{R}}(p_1)\ldots\widetilde{\phi}_{\mathrm{R}}(p_{N+2}) \,|\, \Omega \,\right\rangle = \overline{c}_{\phi^2}\left\langle\, \Omega \,|\, \overline{\phi}^2(\tau,0)\,\widetilde{\phi}_{\mathrm{R}}(p_1)\ldots\widetilde{\phi}_{\mathrm{R}}(p_N) \,|\, \Omega \,\right\rangle + \mathcal{O}(x)$$

to obtain

$$\left[\mu\frac{\mathrm{d}}{\mathrm{d}\mu} - \kappa\frac{\mathrm{d}}{\mathrm{d}\kappa} + 2(\gamma - \zeta_{\phi^2})\right]\overline{c}_{\phi^2} = 0$$

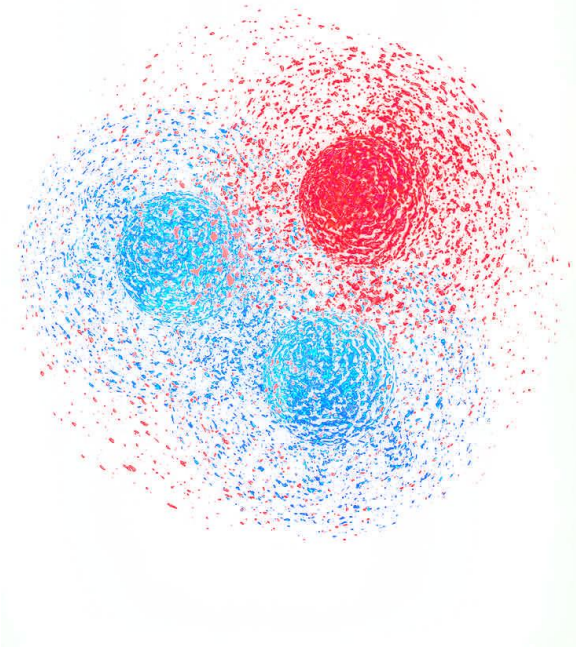
where

$$\zeta_{\phi^2} = \kappa\frac{\mathrm{d}}{\mathrm{d}\kappa}\log\left(\mathcal{Z}_{\phi^2}(\kappa^2x^2,\mu^2)\right) = 2\gamma_m$$

[Recall]

$$\left[\mu\frac{\mathrm{d}}{\mathrm{d}\mu} + 2(\gamma + \gamma_m)\right]c_{\phi^2} = 0 \qquad\qquad \gamma_m = -\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}\mu}\log(m_{\mathrm{R}})$$

Analogous equations apply to the matrix elements



Physical motivation

- lattice quantum chromodynamics (QCD)
- parton distributions (on the lattice)

Modifying the operator product expansion

- Wilson coefficients
- Renormalisation group
- Nonperturbative calculations

Nonperturbative calculation in 2D φ^4 scalar field theory

- implemented in C++

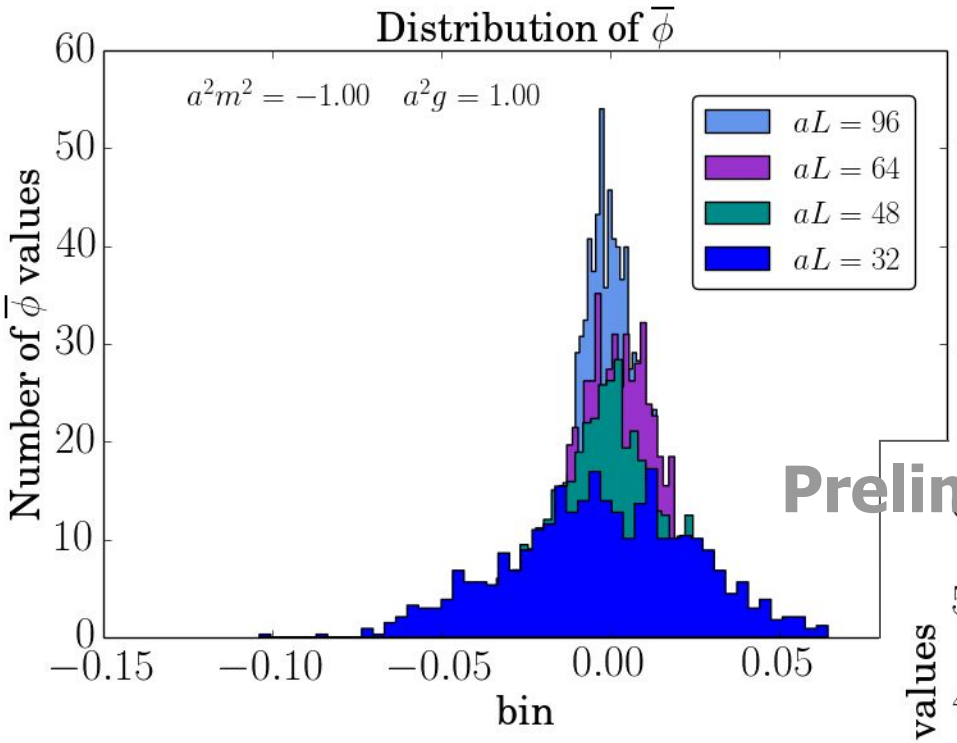
- experimented with 3 update algorithms and different combinations
  - Metropolis
  - microcanonical
  - embedded Wolff cluster

  and might implement multigrid method for fun

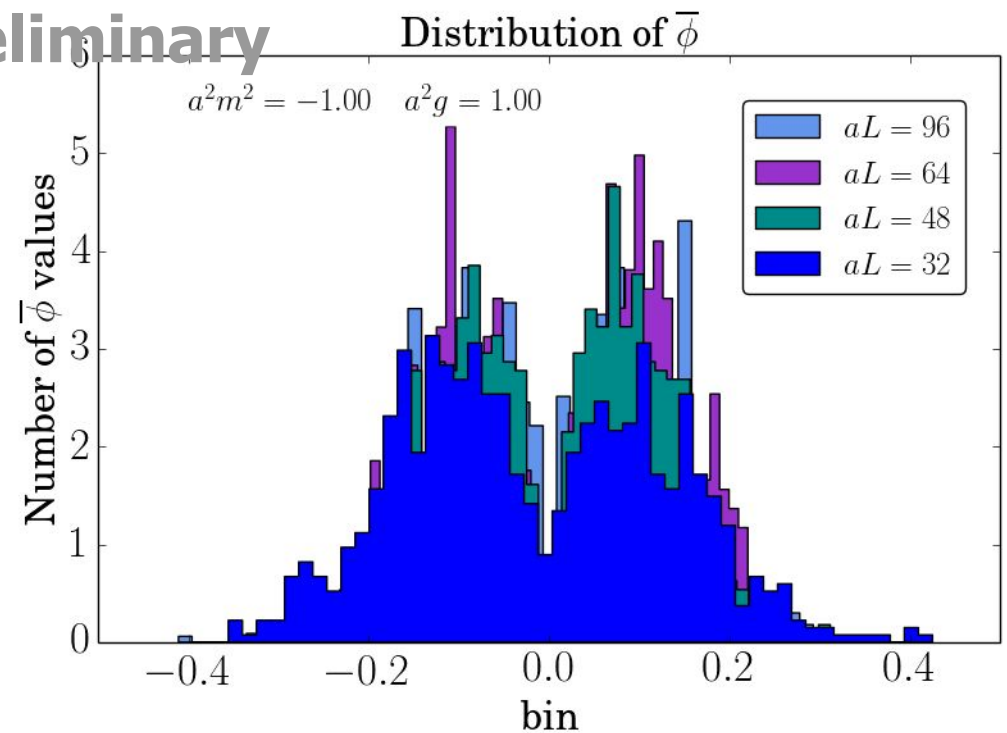
Morningstar (2007) hep-lat/0702020
Wolff, PRL 62 (1989) 361

- currently running on scalar processor
  - few seconds to tens of minutes for $L = 2^n$ ($n = 4 \ldots 8$) for configurations
  - measurements < few seconds

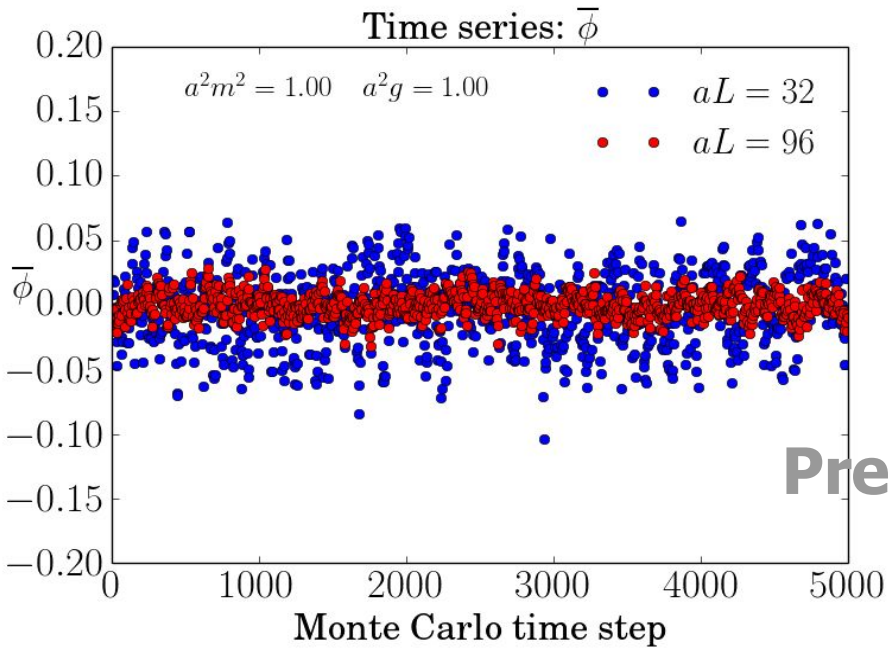
- primarily a dry-run test case for 2D O(3) model (code sort of exists)

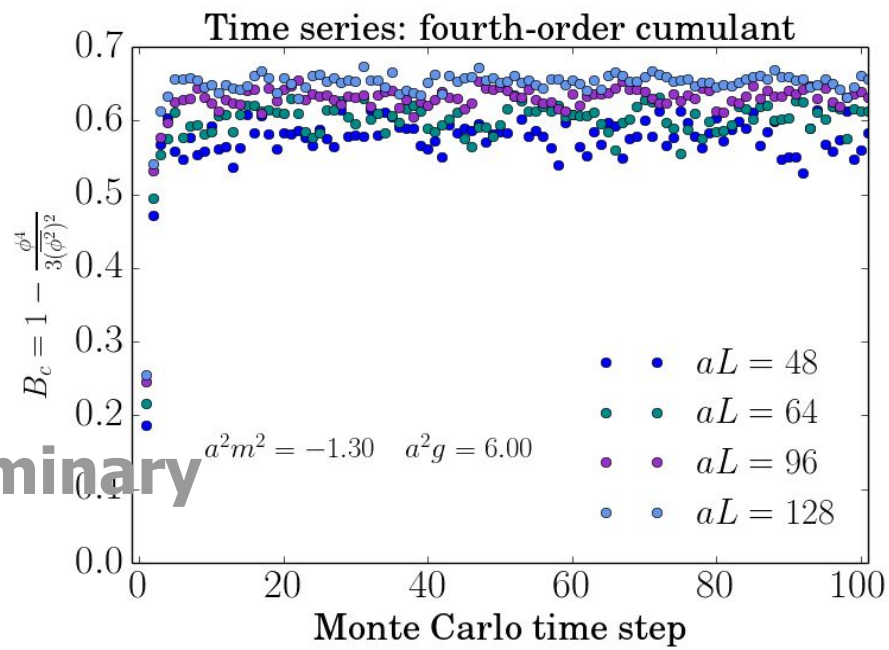


Preliminary

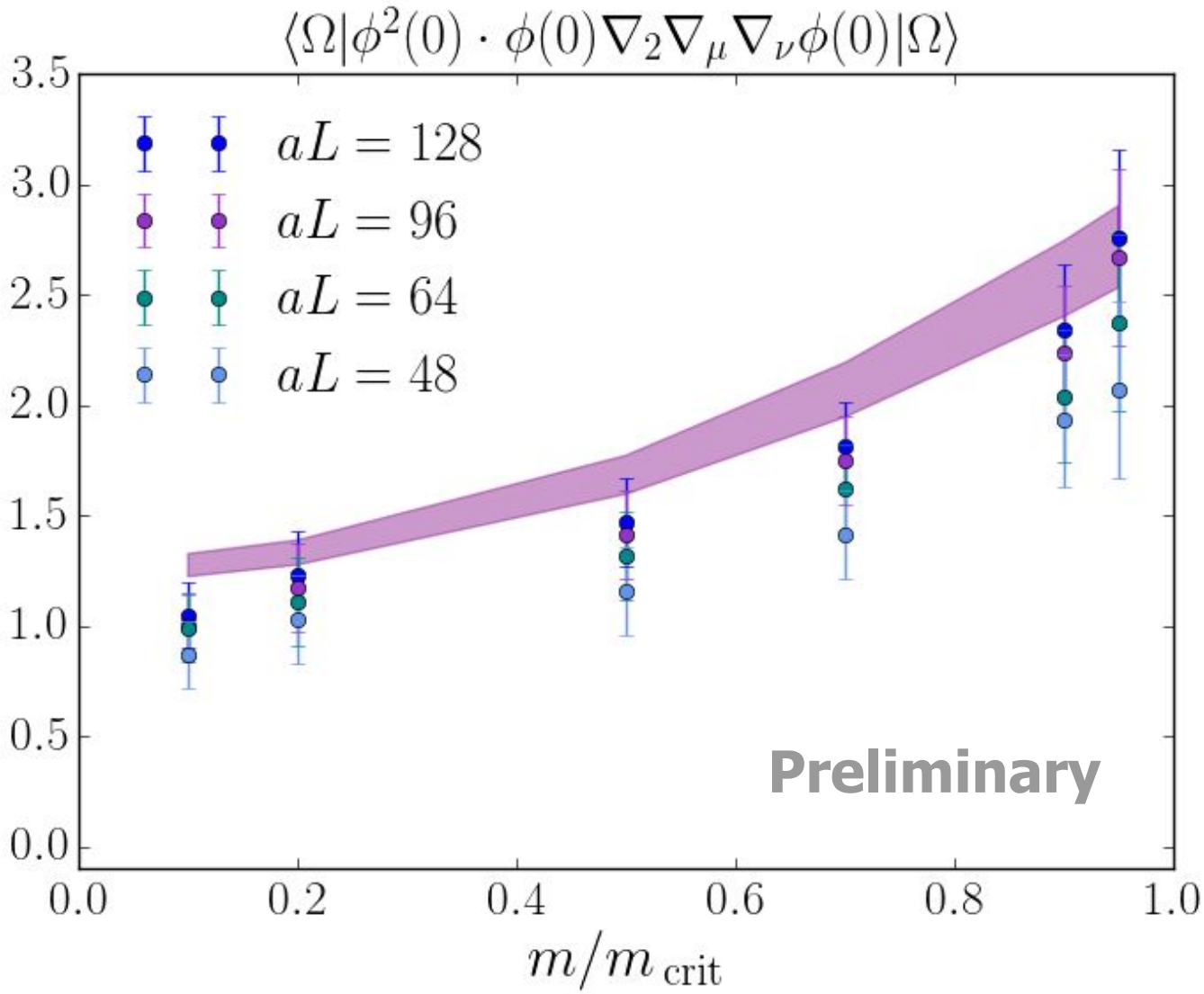


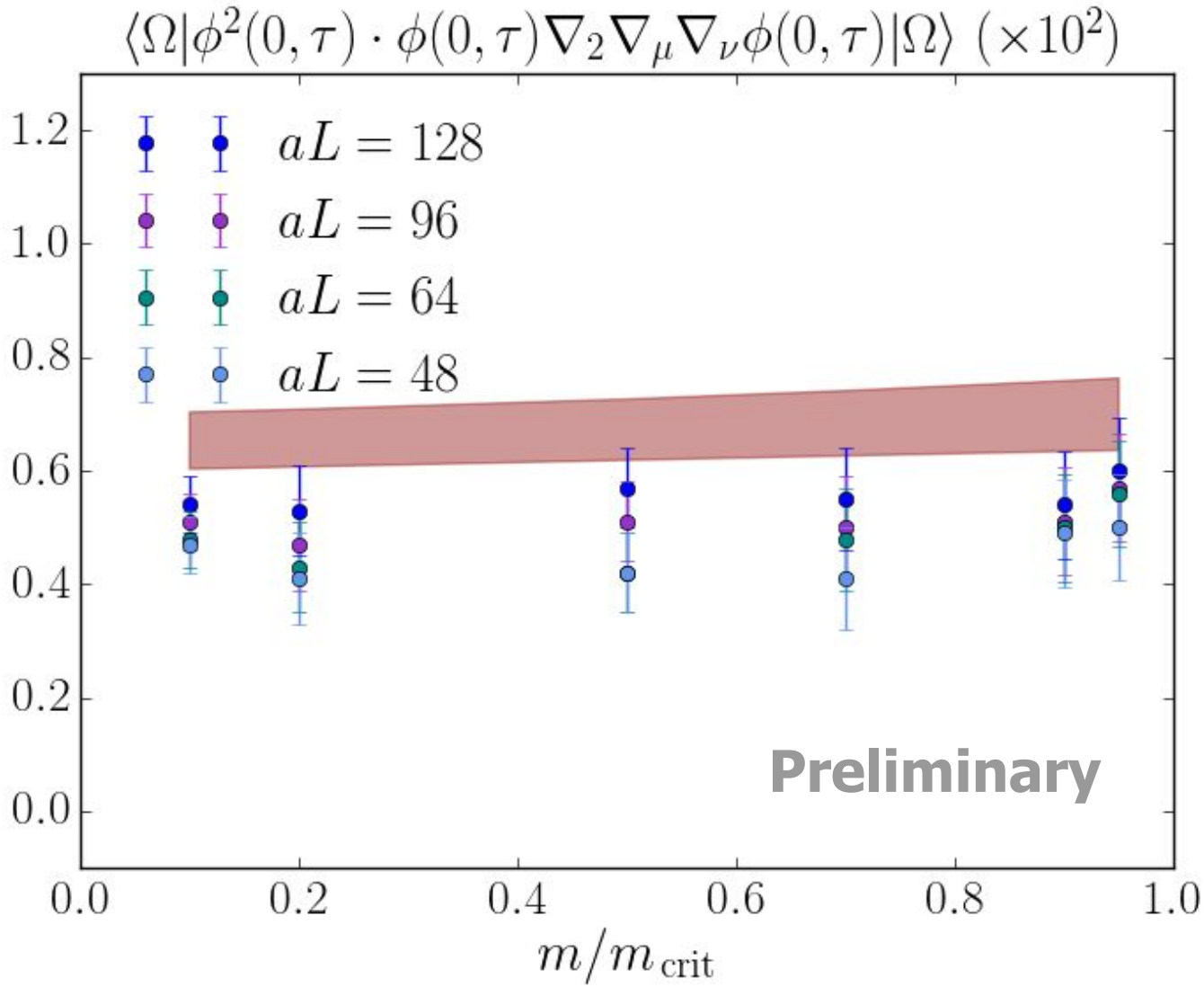
Loinaz & Willey, PRD 58 (1998) 076003

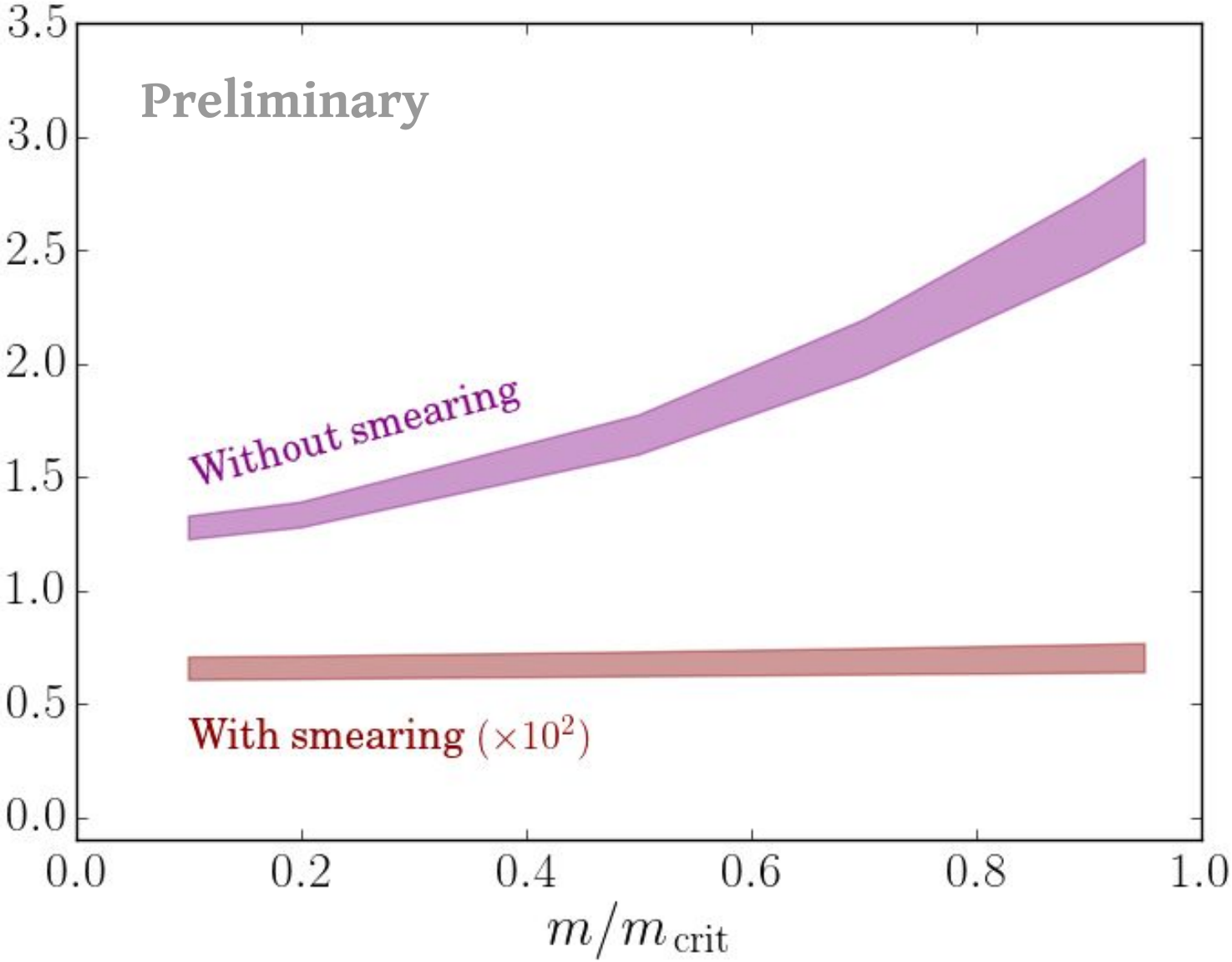




Preliminary


$\langle\Omega|\phi^2(0)\cdot\phi(0)\nabla_2\nabla_\mu\nabla_\nu\phi(0)|\Omega\rangle$
$aL = 128$
$aL = 96$
$aL = 64$
$aL = 48$
Preliminary
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0.0
0.2
0.4
0.6
0.8
1.0
$m/m_{\rm crit}$


$\langle\Omega|\phi^2(0,\tau)\cdot\phi(0,\tau)\nabla_2\nabla_\mu\nabla_\nu\phi(0,\tau)|\Omega\rangle\ (\times 10^2)$
$aL = 128$
$aL = 96$
$aL = 64$
$aL = 48$
Preliminary
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
$m/m_{\rm crit}$


Preliminary
Without smearing
With smearing ($\times 10^2$)
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0.0
0.2
0.4
0.6
0.8
1.0
m/m_{crit}

2D and 4D φ^4 scalar field theory

- illustrative perturbative and nonperturbative calculations complete

2D non-linear sigma model

- code (sort of) written (not debugged!)
- determine:
  - finite volume running coupling
  - twist-2 matrix elements
  - finite volume step-scaling procedure to match to...
- 2-loop perturbation theory for twist-2 matrix elements (underway)

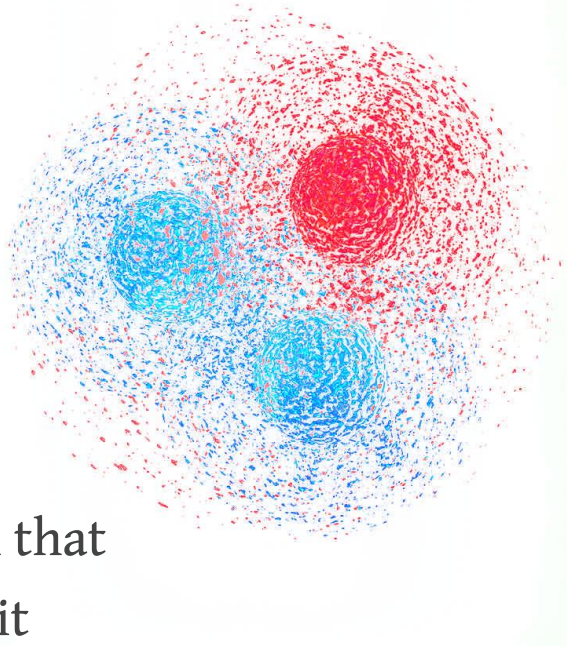
QCD

- nonlinear flow equations
- extra fermion renormalisation required
- flow time evolution still classical

Narayanan & Neuberger, JHEP 0603 (2006) 064
Lüscher, Commun. Math. Phys. 293 (2010) 899
Lüscher & Weisz, JHEP 1102 (2011) 51
Lüscher, JHEP 1304 (2013) 123

Aim:

systematically determine nucleon structure
from ab initio QCD calculations



Clarified aim:

systematically determine parton distributions
from ab initio lattice QCD calculations with a formalism that
removes power-divergent mixing in the continuum limit

Proposal:

modify the operator product expansion

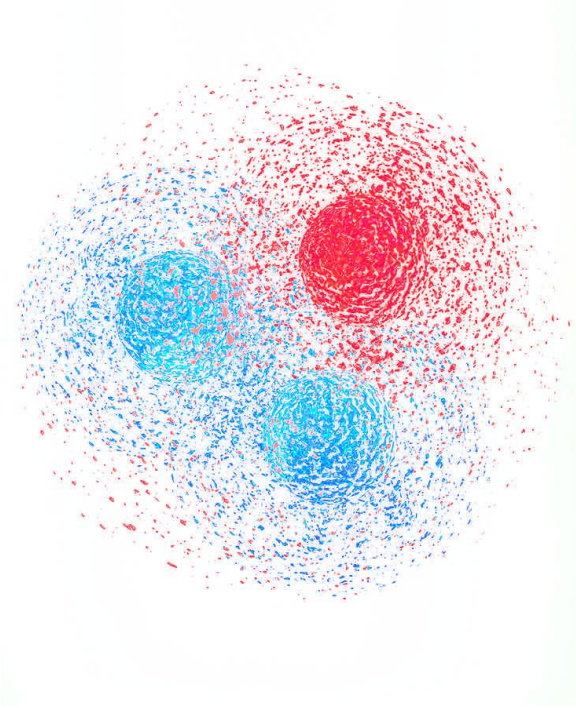
Clarified proposal:

systematically connect nonperturbative, smeared lattice
calculations to continuum physics by modifying the
operator product expansion to account for new scale

CJM & K. Orginos, PoS(Lattice2014) 330
CJM & K. Orginos, PRD 91 (2015) 074513

Operator product expansion	Smeared OPE
perturbative coefficients x local operators	coefficients x locally smeared operators
spacetime dependence of perturbative coefficients dictated by operator mass dimension	spacetime dependence of perturbative coefficients dictated by operator mass dimension
single scale: spacetime separation	two scales: spacetime separation smearing radius
renormalisation group equations govern scale-dependence	for small smearing radius renormalisation group equations govern scale-dependence
	still requires renormalisation beyond leading order

Thank you



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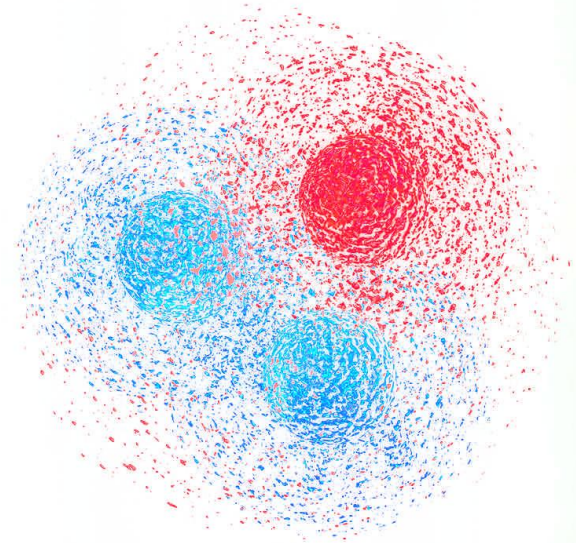
Extras



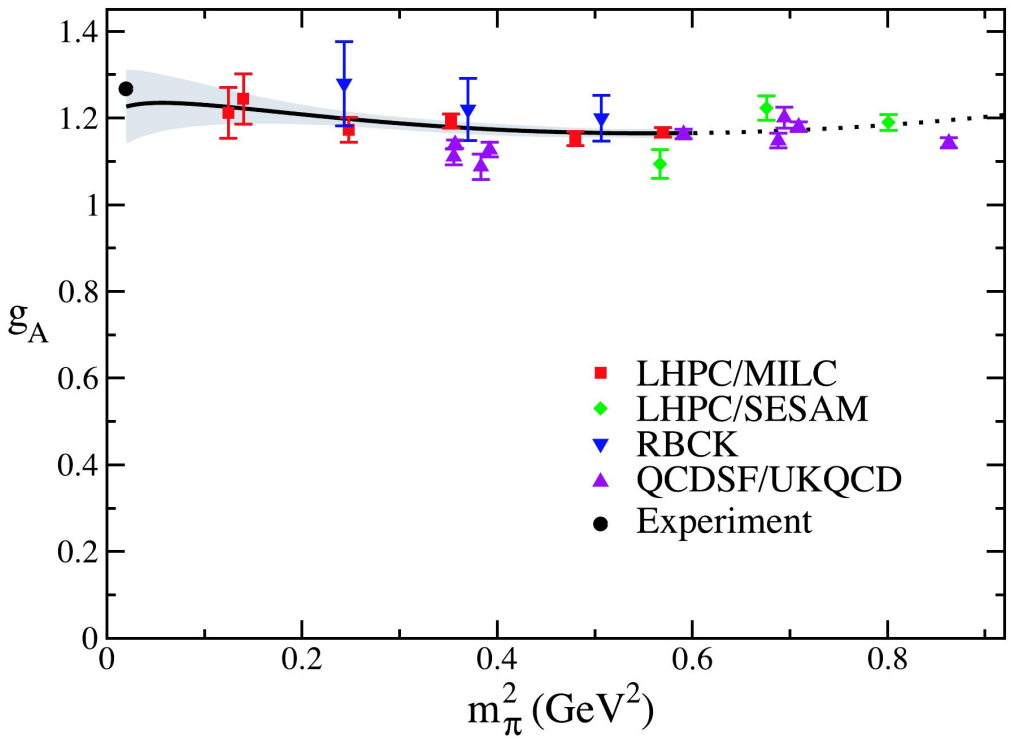
Range of lattice calculations

- Meson distribution amplitudes
  - quenched — Martinelli & Sachrajda, Phys. Lett. B 1 (1987) 184; Martinelli & Sachrajda, Nucl. Phys. B 306 (1988) 805
  - unquenched — Best *et al*, Phys. Rev. D 56 (1997) 2743
- Nucleon
  - axial charge — Edwards *et al*, Phys. Rev. Lett. 96 (2006) 052001; Capitani *et al*, Phys. Rev. D 86 (2012) 074502; Horsley *et al*, Phys. Lett. B 732 (2014) 41
  - unpolarised — Gockeler *et al*, Phys. Rev. D 53 (1996) 2317
  - polarised — Gockeler *et al*, Phys. Rev. D 53 (1996) 2317
  - higher twist contributions — Capitani *et al*, Nucl. Phys. B (Proc. Suppl.) 79 (1999) 179
  - transverse momentum distributions — Musch *et al*, Phys. Rev. D 83 (2011) 094507
  - generalised parton distributions — Gockeler *et al*, Nucl. Phys. B (Proc. Suppl.) 140 (2005) 399

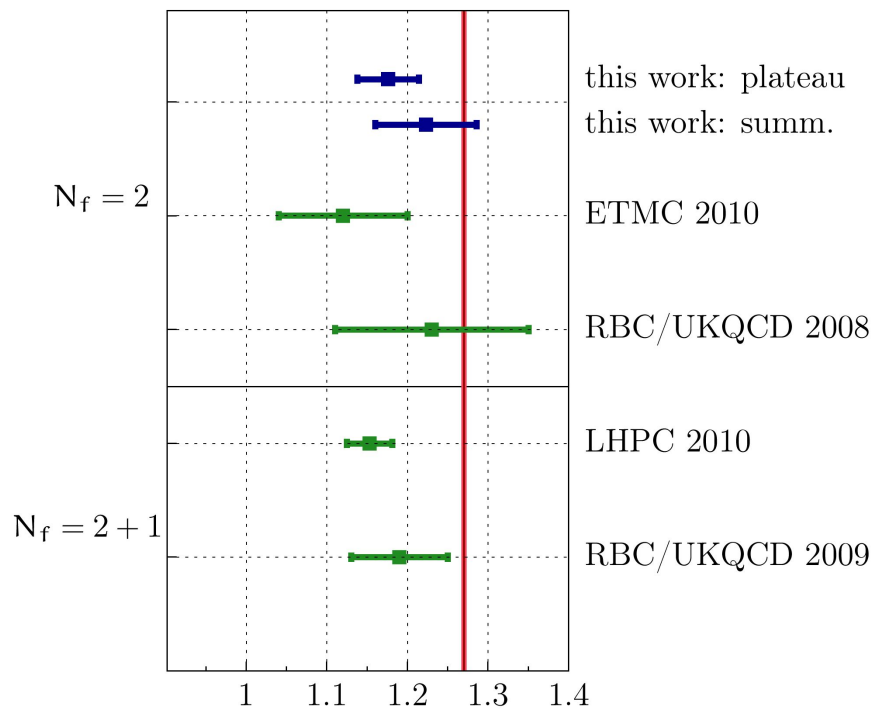
Nucleon axial charge

$$\langle x^0 \rangle_{\Delta q} = \int_0^1 \mathrm{d}x \, [\Delta q(x) + \Delta \bar{q}(x)]$$

$$\Delta q(x) = q_\uparrow(x) - q_\downarrow(x)$$



Edwards *et al*, Phys. Rev. Lett. 96 (2006) 052001

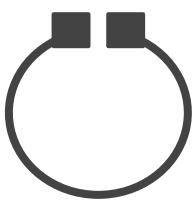


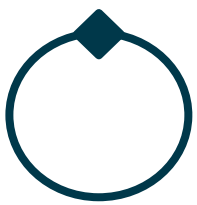
Capitani *et al*, Phys. Rev. D 86 (2012) 074502

Consider

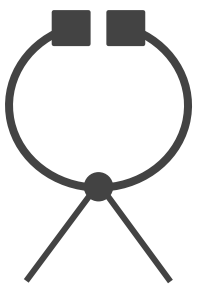
$$\pi(x)\pi(0) = \frac{b_{\mathbb{I}}}{4\pi}\mathbb{I} + b_{\pi^2}\pi^2(\tau,0) + b_{\partial_\mu}x^\mu\partial_\mu\pi^2(\tau,0) + \dots$$

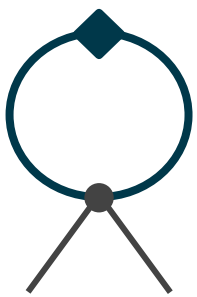
One loop calculations (almost) as straightforward as 4D φ^4 scalar field theory





$$b_{\mathbb{I}}(x^2/\tau) = -g_0^2\delta^{ij}\left[\gamma_{\mathrm{E}} + \log\left(\frac{x^2}{8\tau}\right)\right]$$





$$b_{\pi^2}(x^2/\tau) = 1 - \frac{g_0^2}{4\pi}\delta^{ik}\delta^{jl}\left[\gamma_{\mathrm{E}} + \log\left(\frac{x^2}{8\tau}\right)\right]$$