Multi-loop unitarity via computational algebraic geometry

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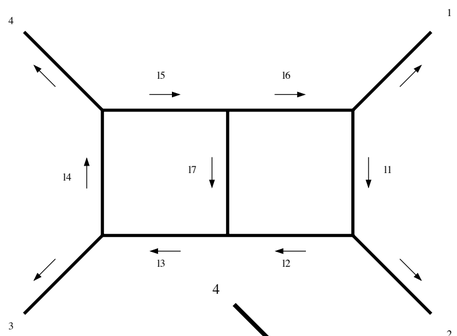
Yang Zhang

Niels Bohr Institute

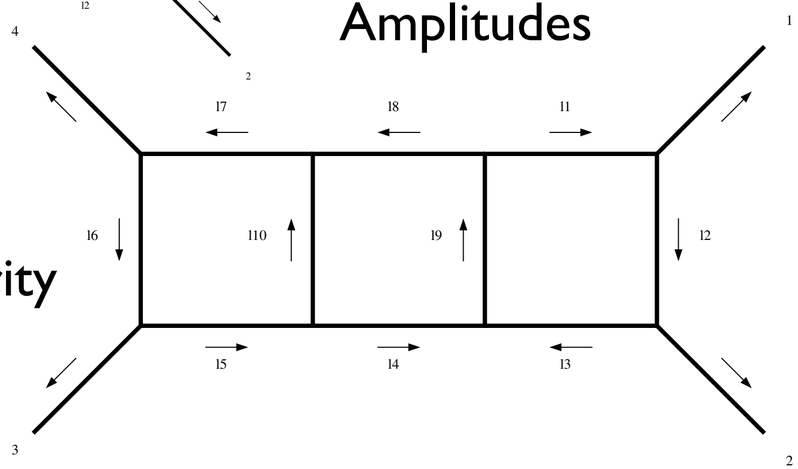
Based on

(2-loop 4-point)	arXiv:1202.2019, Simon Badger, Hjalte Frellesvig and YZ
(algebraic geometry methods)	arXiv:1205.5707, YZ
(3-loop 4-point)	arXiv:1207.2976, Simon Badger, Hjalte Frellesvig and YZ
(global structure)	arXiv:1302.1023, Rijun Huang and YZ
(2-loop 5-point QCD)	arXiv:1310.1051, Simon Badger, Hjalte Frellesvig and YZ
(maximal cut)	arXiv:1310.6006, Mads Sogaard and YZ

Outline

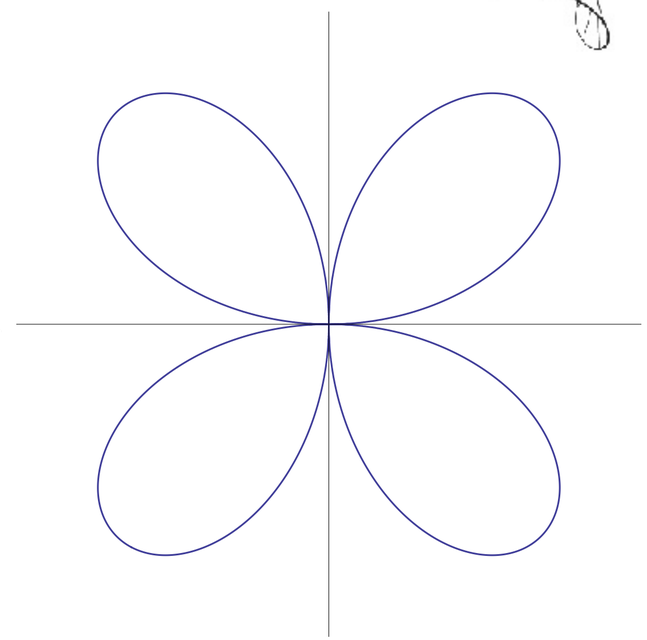


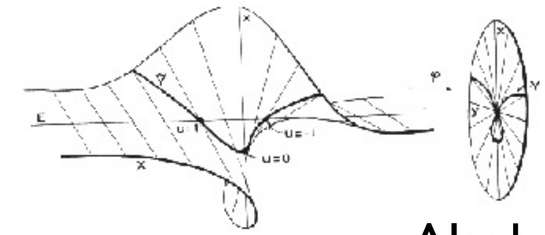
High-loop Amplitudes



Unitarity

$\Longrightarrow$





Algebraic geometry

Gröbner Basis
Primary Decomposition
Affine Variety Structure
Multivariate residue

- Integrand reduction by algebraic geometry
- Maximal unitarity by algebraic geometry
- Examples: 2-loop 5-gluon planar QCD, 3-loop 4-point triple-box ...

Why high loops?

- Phenomenology: NNLO correction for theoretical prediction
- Theory： deep structure in gauge theories and gravity

Computation of two-loop Feynman diagrams is complicated.

Feynman rules,
Integration-by-parts identities

- two-loop massless QCD, 2→2 process
  Anastasiou, Glover, Tejeda-Yeomans and Oleari (2000)
  Bern, Dixon, Kosower (2002) Bern, De Freitas, Dixon (2002)
- two-loop, pp→H + 1 jet
  Gehrmann, Jaquier, Glover and Koukoutsakis (2011)
- NNLO, $e^+ e^- \to 3$ jets
  Gehrmann and Glover (2008)
- NNLO, q qbar→t tbar
  Bernreuther, Czakon, Mitov (2012)
- NNLO, g g→H g
  Boughezal, Caola, Melnikov, Petriello, Schulze (2013)

and etc.

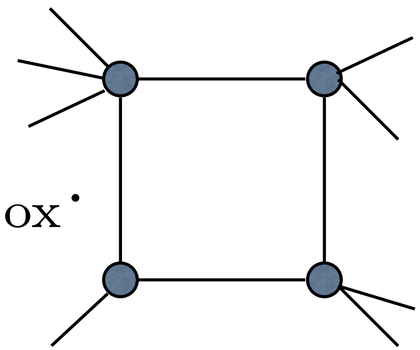
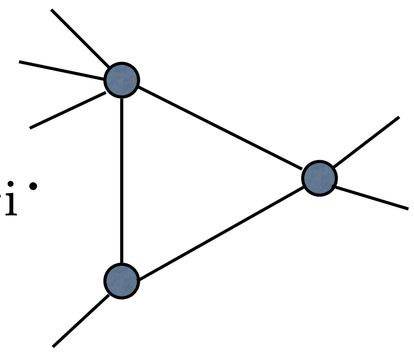
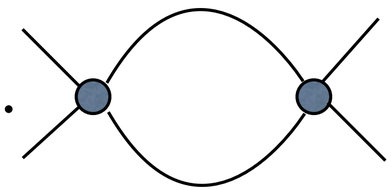
Unitarity

multi-loop integrand reduction...

multi-loop maximal unitarity ...

Unitarity at one-loop

$D = 4$

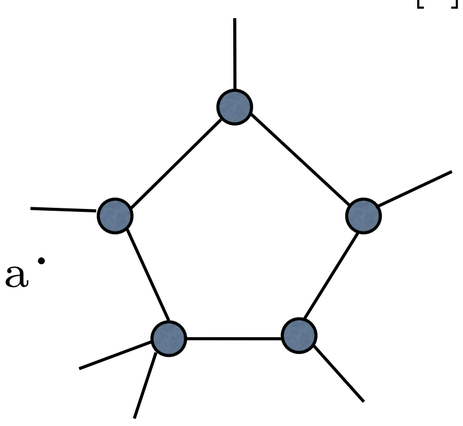
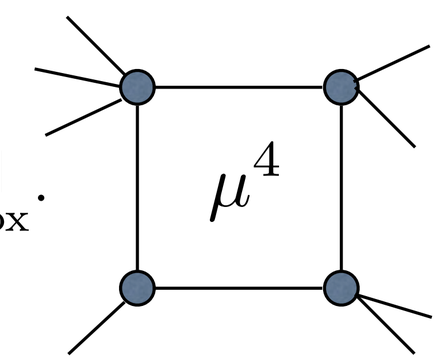
$A^{(1)} = c_{\text{box}} \cdot$  $+ c_{\text{tri}} \cdot$  $+ c_{\text{bub}} \cdot$ 

- no pentagon, hexagon ...
- scalar integral (numerator is one.)
- c coefficients are independent of loop momenta

$D = 4 - 2\epsilon$

$$l = l_{[4]} + l_{\perp}, \quad (l_{\perp})^2 \equiv -\mu^2$$

Also contains

$c_{\text{penta}} \cdot$  $+ c^{[4]}_{\text{box}} \cdot$  $+ \ldots$

no hexagon ...

Unitarity:

Determine 'c' coefficients from on-shell cut solutions and tree amplitudes

quadruple cut $\rightarrow c_{\text{box}}$

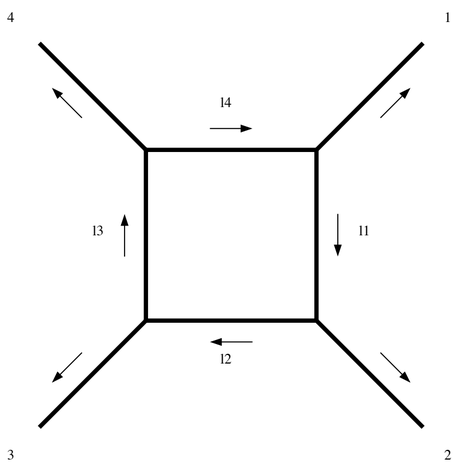
triple cut $\rightarrow c_{\text{tri}}$

double cut $\rightarrow c_{\text{bub}}$

Integrand reduction: box

Integrand-level reduction, Ossola, Papadopoulos and Pittau (OPP), 2006
Giele, Kunszt, Melnikov, 2008

$$A^{(1)} = \int \frac{d^4k}{(2\pi)^4} \frac{N(k)}{D_1 D_2 D_3 D_4}$$



$$N(k) = \Delta_{1234}(k) + \sum_{i_1<i_2<i_3} \Delta_{i_1 i_2 i_3}(k) \prod_{i \neq i_1,i_2,i_3} D_i + \sum_{i_1<i_2} \Delta_{i_1 i_2}(k) \prod_{i \neq i_1,i_2} D_i$$

$$= \Delta_{1234}(k) + O(D_1, D_2, D_3, D_4)$$

$\Delta_{1234}(k)$ is a polynomial in scalar products (SP). $\mathbb{SP} = \{k \cdot P_1, k \cdot P_2, k \cdot P_3, k \cdot \omega\}$

ω is auxiliary, $(\omega \cdot P_i) = 0$, $i = 1, 2, 3, 4$

$$\begin{aligned} 2(k \cdot P_1) &= D_4 - D_1 - P_1^2 \\ 2(k \cdot P_2) &= D_1 - D_2 + P_2^2 \\ 2(k \cdot P_3) &= D_2 - D_3 + 2P_2 \cdot P_3 + P_3^2 \end{aligned}$$

3 reducible scalar products $\mathbb{RSP} = \{k \cdot P_1, k \cdot P_2, k \cdot P_3\}$

1 irreducible scalar product $\mathbb{ISP} = \{k \cdot \omega\}$

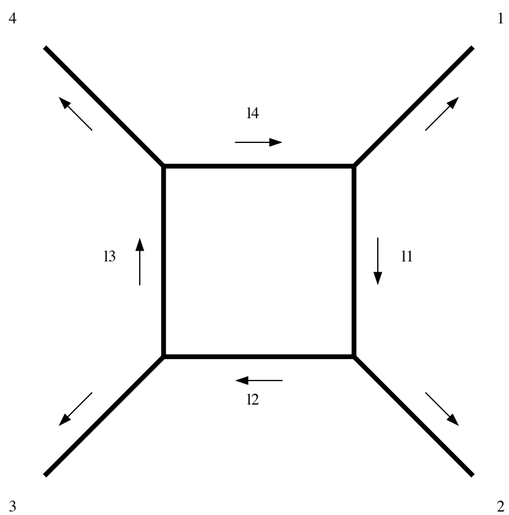
$$\Delta_{1234}(k) = \sum_i c_i (k \cdot \omega)^i$$

$\Delta_{1234}(k)$ is a polynomial in ISP only.

Integrand basis for box

$$\Delta_{1234}(k) = \sum_i c_i (k\cdot\omega)^i$$

How many terms are there?



Renormalizability $\quad i = 0, 1, 2, 3, 4$

Cut-equations for ISP $\quad k^2 = D_1$

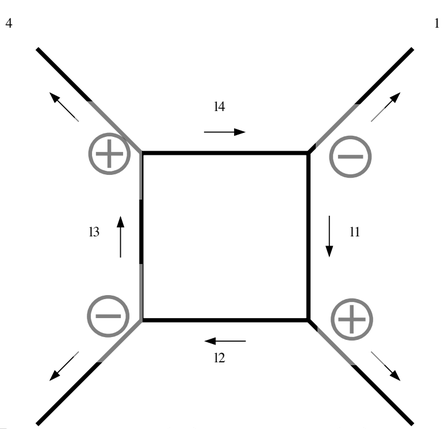
$$(k\cdot\omega)^2 = t^2/4 + O(D_1, D_2, D_3, D_4)$$

Reducible

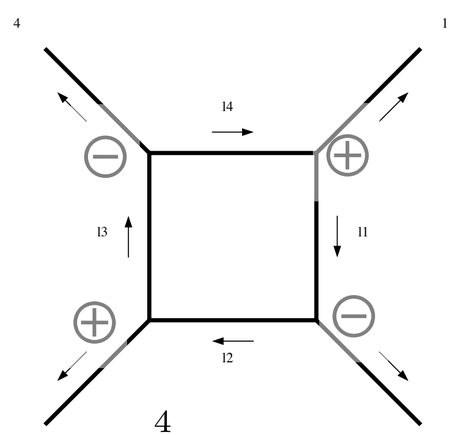
$$i = 0, 1$$

integrand basis $\quad \boxed{\Delta_{1234}(k) = c_0 + c_1(k\cdot\omega)}$ $\quad \boxed{c_{\text{box}} = c_0}$

(Generalized-) Unitarity Cuts $\quad D_1 = D_2 = D_3 = D_4 = 0$



$$\prod_{i=1}^{4} A^i_{\text{tree}}(k^{(1)}) = N^{(1)}$$



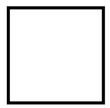
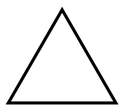
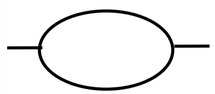
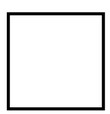
$$\prod_{i=1}^{4} A^i_{\text{tree}}(k^{(2)}) = N^{(2)}$$

Spurious term: $\quad \displaystyle\int \frac{d^4k}{(2\pi)^4} \frac{k\cdot\omega}{D_1 D_2 D_3 D_4} = 0$

But c_1 is crucial for low-point functions!

$$N(k) - c_0 - c_1(k\cdot\omega) = \sum_{i_1<i_2<i_3} \Delta_{i_1 i_2 i_3}(k) \prod_{i\neq i_1,i_2,i_3} D_i + \ldots$$

One loop, other diagrams

Dimension	Diagram	# SP (ISP+RSP)	#terms in integrand basis (non-spurious + spurious)	# Solutions (dimension)
4		4 (1+3)	2 (1+1)	2 (0)
4		4 (2+2)	7 (1+6)	1 (1)
4		4 (3+1)	9 (1+8)	1(2)
4-2ε		5 (2+3)	5 (3+2)	1(1)

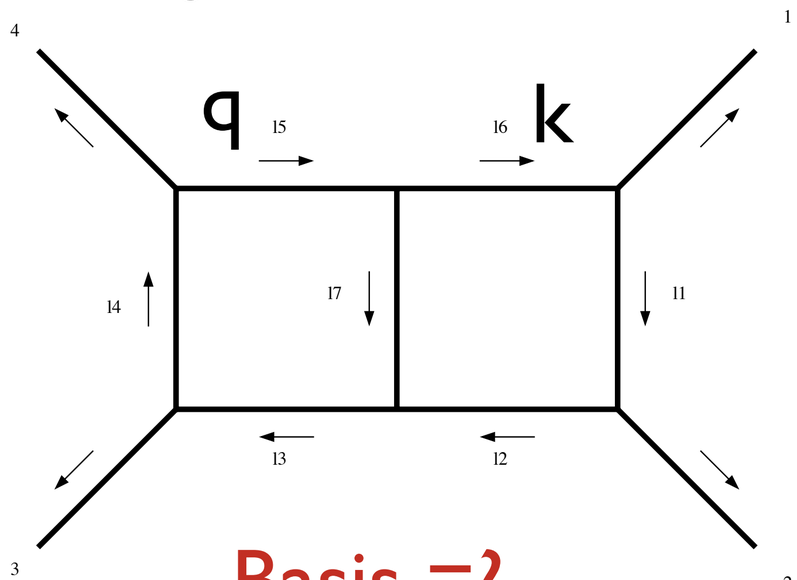
- straightforward to obtain integrand basis, unitarity cut solutions
- all one-loop master integrals are known
- c coefficients can be automatically computed by public codes

  - 'NGluon', Badger, Biedermann, and Uwer
  - 'CutTools', Ossola, Papadopoulos, and Pittau
  - 'GoSam', Cullen, Greiner, Heinrich, Luisoni, and Mastrolia
  - ...

Generalization to higher loops?

Example: 4D massless two-loop hepta cut

P. Mastrolia, G. Ossola, 2011
S. Badger, H. Frellesvig, YZ, 2012



Basis =?

7 cut-equations in 8 SP's

$$\mathbb{SP} = \{k \cdot P_1, k \cdot P_2, k \cdot P_4, k \cdot \omega, q \cdot P_1, q \cdot P_2, q \cdot P_4, q \cdot \omega\}$$

4 cut-equations to identify 4 RSP's

4 ISP's $\quad \mathbb{ISP} = \{k \cdot P_4, k \cdot \omega, q \cdot P_1, q \cdot \omega\}$

3 cut-equations for ISP's

$$(k \cdot \omega)^2 = (k \cdot P_4 - t/2)^2 \quad (1)$$

$$(q \cdot \omega)^2 = (q \cdot P_1 - t/2)^2 \quad (2)$$

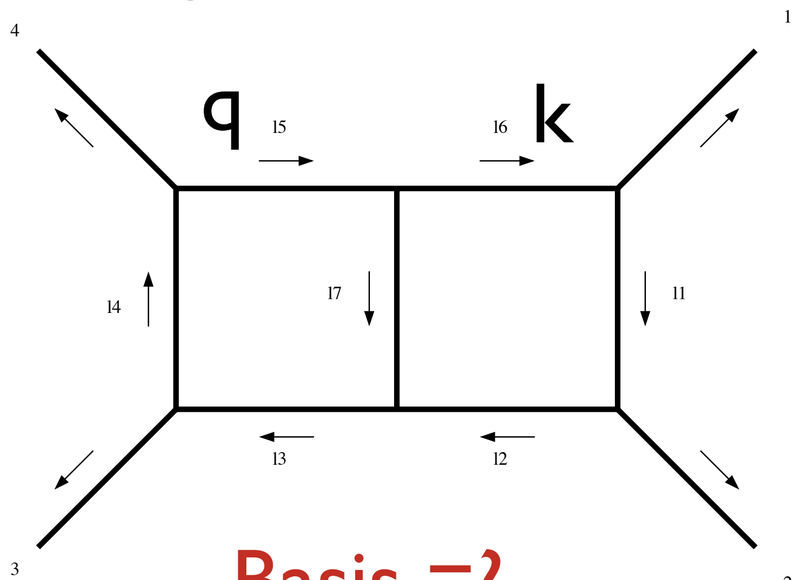
$$(k \cdot \omega)(q \cdot \omega) = -\frac{t^2}{4} + \frac{t(k \cdot P_4)}{2} + \frac{t(q \cdot P_1)}{2} + \left(1 + \frac{2t}{s}\right)(k \cdot P_4)(q \cdot P_1) \quad (3)$$

Naive guessing: all renormalizable monomials which do NOT contain $(k \cdot \omega)^2$, $(q \cdot \omega)^2$ or $(k \cdot \omega)(q \cdot \omega)$.

$$\Delta_{\text{dbox}} = (k \cdot P_4)^m (q \cdot P_1)^n (k \cdot \omega)^\alpha (q \cdot \omega)^\beta$$
$$m + \alpha \leq 4, n + \beta \leq 4, m + n + \alpha + \beta \leq 6$$
$$(\alpha, \beta) = (0,0), (1,0), (0,1)$$

Example: 4D massless two-loop hepta cut

P. Mastrolia, G. Ossola, 2011
S. Badger, H. Frellesvig, YZ, 2012



Basis =?

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$$\mathbb{SP} = \{k \cdot P_1, k \cdot P_2, k \cdot P_4, k \cdot \omega, q \cdot P_1, q \cdot P_2, q \cdot P_4, q \cdot \omega\}$$

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$$\Delta_{\text{dbox}} = (k \cdot P_4)^m (q \cdot P_1)^n (k \cdot \omega)^\alpha (q \cdot \omega)^\beta$$
$$m + \alpha \leq 4, n + \beta \leq 4, m + n + \alpha + \beta \leq 6$$
$$(\alpha, \beta) = (0,0), (1,0), (0,1)$$

56 terms? wrong...

Example: 4D massless two-loop hepta cut

S. Badger, H. Frellesvig, YZ, 2012

3 cut-equations for ISP's, and their combinations

$$(k\cdot\omega)^2 = (k\cdot P_4 - t/2)^2 \quad (1)$$
$$(q\cdot\omega)^2 = (q\cdot P_1 - t/2)^2 \quad (2)$$
$$(k\cdot\omega)(q\cdot\omega) = -\frac{t^2}{4} + \frac{t(k\cdot P_4)}{2} + \frac{t(q\cdot P_1)}{2} + \left(1+\frac{2t}{s}\right)(k\cdot P_4)(q\cdot P_1) \quad (3)$$

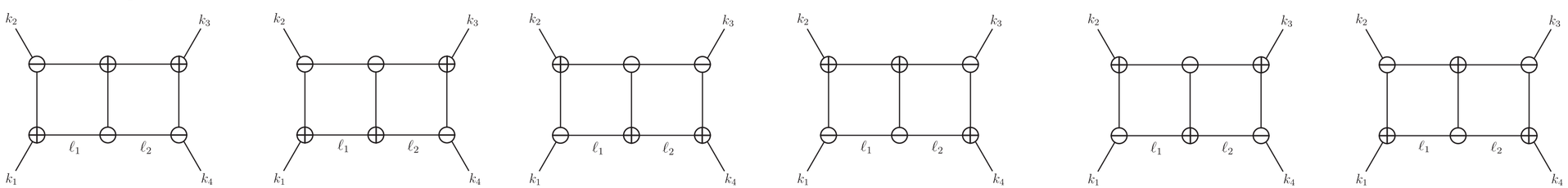
reduced

$$\boxed{(1)\times(2)-(3)^2} \quad \text{or} \quad G\begin{pmatrix} P_1 & P_2 & P_4 & k & q \\ P_1 & P_2 & P_4 & k & q \end{pmatrix} = 0$$

$$4(k\cdot P_4)^2(q\cdot P_1)^2 = -2s(k\cdot P_4)^2(q\cdot P_1) - 2s(k\cdot P_4)(q\cdot P_1)^2 - st(k\cdot P_4)(q\cdot P_1)$$

We have to "exhaust" all combinations...

Finally, we determine that the basis contains 32 terms



6 families of hepta-cut solutions, Laurant series contains 38 terms

Solving 38 linear equations for 32 coefficients, done!

Messy, not automatic!

Gröbner basis and integrand basis

arXiv:1205.5707, YZ
arXiv:1205.7087, Mastrolia, Mirabella, Ossola and Peraro

$$I = \langle D_1, \dots D_k \rangle = \{ \sum_{i=1}^{k} g_i D_i | \quad \forall g_i \in R \}$$

$$\int \frac{d^4k}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \frac{N}{D_1 D_2 \dots D_7}, \quad N = Q + \Delta_{\text{dbox}}, \quad Q \in I$$

Synthetic polynomial division

N divided by $\{D_1, \dots D_k\}$:

Define a monomial order, and recursively preform $N/D_1, \dots, N/D_k$. Finally, the division process will stop and we have

$$N = f_1 D_1 + \dots f_k D_k + r'$$

where r' is the remainder. $\Delta_{\text{dbox}} = r'$???

$$\begin{array}{r}
x^2 - 9x - 27 \\
x-3\overline{)x^3 - 12x^2 + 0x - 42} \\
\underline{x^3 - 3x^2} \\
-9x^2 + 0x \\
\underline{-9x^2 + 27x} \\
-27x - 42 \\
\underline{-27x + 81} \\
-123
\end{array}$$

In most cases, it does not work since it stops too early,
unless we are using Gröbner basis.

$$I = \langle D_1, \dots, D_k \rangle = \langle g_1, \dots, g_m \rangle$$

Gröbner basis
'good' generators

$$N = q_1 g_1 + \dots q_k g_k + r$$

- r is uniquely determined.
- If $N \in I$, $r = 0$.

$$\boxed{\Delta_{\text{dbox}} = r}$$

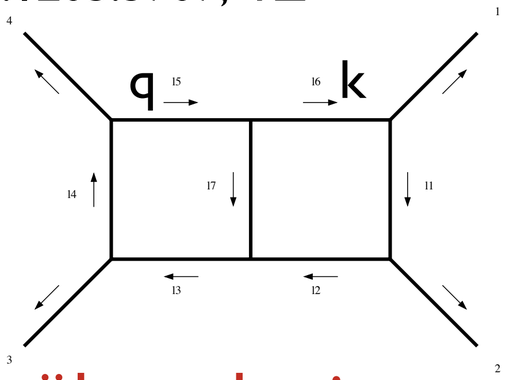
$$\begin{pmatrix} y^3 & x - 2y^2 \end{pmatrix} = \begin{pmatrix} x^3 - 2xy & x^2y - 2y^2 + x \end{pmatrix} \begin{pmatrix} -\frac{1}{4} - \frac{1}{4}xy - \frac{1}{2}y^3 & y^2 \\ \frac{1}{4}x^2 - \frac{1}{2}y + \frac{1}{2}xy^2 & 1 - xy \end{pmatrix}$$

Toy Model: $N = xy^3$, $I = \langle x^3 - 2xy, x^2y - 2y^2 + x \rangle$. Direct synthetic division of N towards $\{x^3 - 2xy, x^2y - 2y^2 + x\}$ gives $r' = xy^3$.

But the Gröbner basis is $I = \langle y^3, x - 2y^2 \rangle$, and the synthetic division of N on Gröbner basis gives $r = 0$. So $N \in I$.

Grobner basis: dbox example

arXiv:1205.5707, YZ



4 ISP's $\mathbb{ISP} = \{k \cdot P_4, k \cdot \omega, q \cdot P_1, q \cdot \omega\}$

$$N = q_1 g_1 + \ldots q_k g_k + \Delta_{\text{dbox}}$$

N contains 160 terms where Δ_{dbox} contains 32 terms.

Gröbner basis contains the Gram-matrix relation

$$G\begin{pmatrix} P_1 & P_2 & P_4 & k & q \\ P_1 & P_2 & P_4 & k & q \end{pmatrix} = 0 \qquad 4(k\cdot P_4)^2(q\cdot P_1)^2 + 2s(k\cdot P_4)^2(q\cdot P_1) + 2s(k\cdot P_4)(q\cdot P_1)^2 + st(k\cdot P_4)(q\cdot P_1) \in G(I)$$

In principle, it works for arbitrary number of loops, any dimension
Automated by the public code: 'BasisDet'

http://www.nbi.dk/~zhang/BasisDet.html, YZ 2012

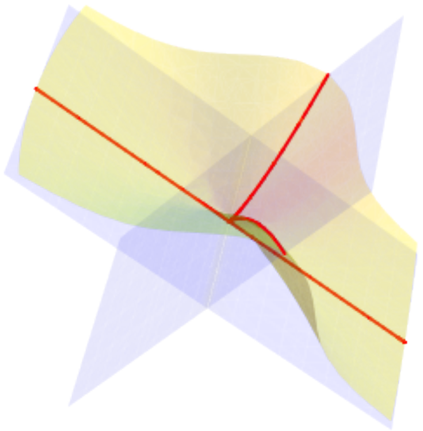
Dimension propagators, kinematics $\longrightarrow$ Integrand basis

Can also find ISP automatically!

Primary decomposition

arXiv:1205.5707, YZ

Find the number of branches of unitarity solutions



$I = \langle x^2 - y^2, x^3 + y^3 - z^2 \rangle$. How many (irreducible) curves are there in $\mathcal{Z}(I)$.

Primary decomposition:

$$I = I_1 \bigcap I_2 \quad I_1 = \langle x + y, z^2 \rangle, \quad I_2 = \langle x - y, 2y^3 - z^2 \rangle$$

- AG software 'Macaulay 2'
- Numeric Algebraic geometry methods

$$I = I_1 \cap I_2 \cap I_3 \cap I_4 \cap I_5 \cap I_6$$

4D massless dbox hepta-cut: 6 families of solutions

dictionary

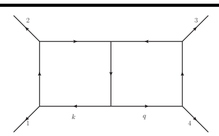
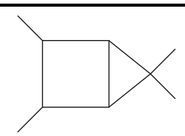
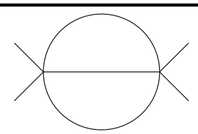
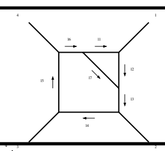
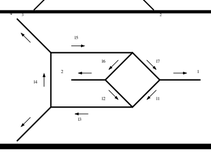
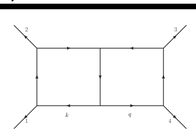
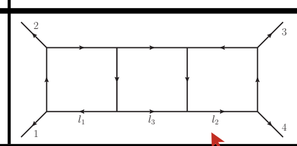
Algebra	Geometry	
height I	$\dim \mathcal{Z}(I) = n - \text{height } I$	(# free parameters)
arithmetic genus	(geometric) genus	(topology)

4D massless dbox hepta-cut: each family is one-dimensional with genus 0 (Riemann sphere)

High genus examples: arXiv:1302.1203, Rijun and YZ

works for arbitrary number of loops, any dimension

More examples

Dimension	Diagram	# SP (ISP+RSP)	#terms in integrand basis (non-spurious + spurious)	# Solutions (dimension)
4		8 (4+4)	32 (16+16)	6 (1)
4		8 (5+3)	69 (18+51)	4 (2)
4		4 (3+1)	42 (12+30)	1(5)
4		8 (3+5)	20 (10+10)	2 (2)
4		8 (4+4)	38 (19+19)	8 (1)
4-2ε		11 (7+4)	160 (84+76)	1(4)
4		12 (7+5)	398 (199+199)	14 (2)

Nontrivial dimension

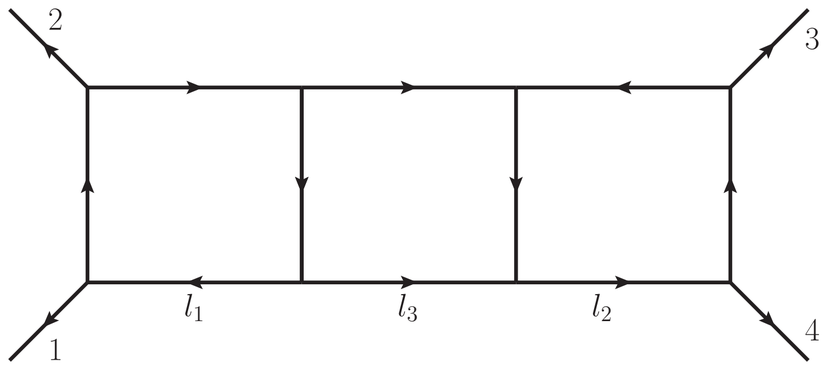
Non-planar

Three-loop!

Even more examples:
arXiv:1209.3747 Bo Feng and Rijun Huang

Triple box results

arXiv:1207.2976, Simon Badger, Hjalte Frellesvig and YZ



Integration-by-parts (IBP) identities

398 terms → 3 master integrals

$$C_1 I_{\text{tribox}}[1] + C_2 I_{\text{tribox}}[l_1 \cdot p_4] + C_3 I_{\text{tribox}}[l_3 \cdot p_4]$$

fit 398 `c` coefficients from products of 8 trees,

from 14 family of cut-solutions

$\mathcal{N}$	n_f	n_s
0	0	0
1	1	0
2	2	1
4	4	3

Yang-Mills with n_f adjoint fermions and n_s adjoint scalars

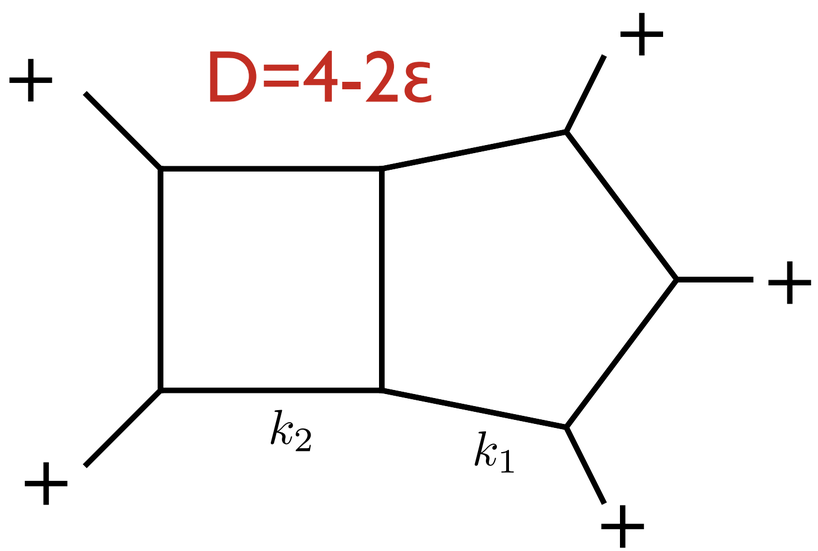
$$\begin{aligned}
C_1^{-+-+}(s,t) = & \\
& -1 + (4-n_f)\frac{st}{u^2} - 2(1+n_s-n_f)\frac{s^2t^2}{u^4} \\
& + \big(2(1-2n_s)+n_f\big)(4-n_f)\frac{s^2t(2t-s)}{4u^4} \\
& - \big(n_f(3-n_s)^2 - 2(4-n_f)^2\big)\frac{st(t^2-4st+s^2)}{8u^4} \\
C_2^{-+-+}(s,t) = & \\
& -(4-n_f)\frac{s}{u^2} + 2(1+n_s-n_f)\frac{s^2t}{u^4} \\
& - \big(2(1-2n_s)+n_f\big)(4-n_f)\frac{s^2(2t-s)}{u^4} \\
& + \big(n_f(3-n_s)^2 - 2(4-n_f)^2\big)\frac{s(t^2-4st+s^2)}{2u^4} \\
C_3^{-+-+}(s,t) = & \\
& + \big(2(1-2n_s)+n_f\big)(4-n_f)\frac{3s^2(2t-s)}{2u^4} \\
& - \big(n_f(3-n_s)^2 - 2(4-n_f)^2\big)\frac{3s(t^2-4st+s^2)}{4u^4}
\end{aligned}$$

New analytic results for non-supersymmetric gauge theory

D-dim integrand reduction

2-loop 5-point QCD

arXiv: 1310.1051: Simon Badger, Hjalte Frellesvig and YZ



$$\mu_{11} = l^2_{[-2\epsilon],1},\ \mu_{22} = l^2_{[-2\epsilon],2} \text{ and } \mu_{12} = 2(l_{[-2\epsilon],1} \cdot l_{[-2\epsilon],2})$$

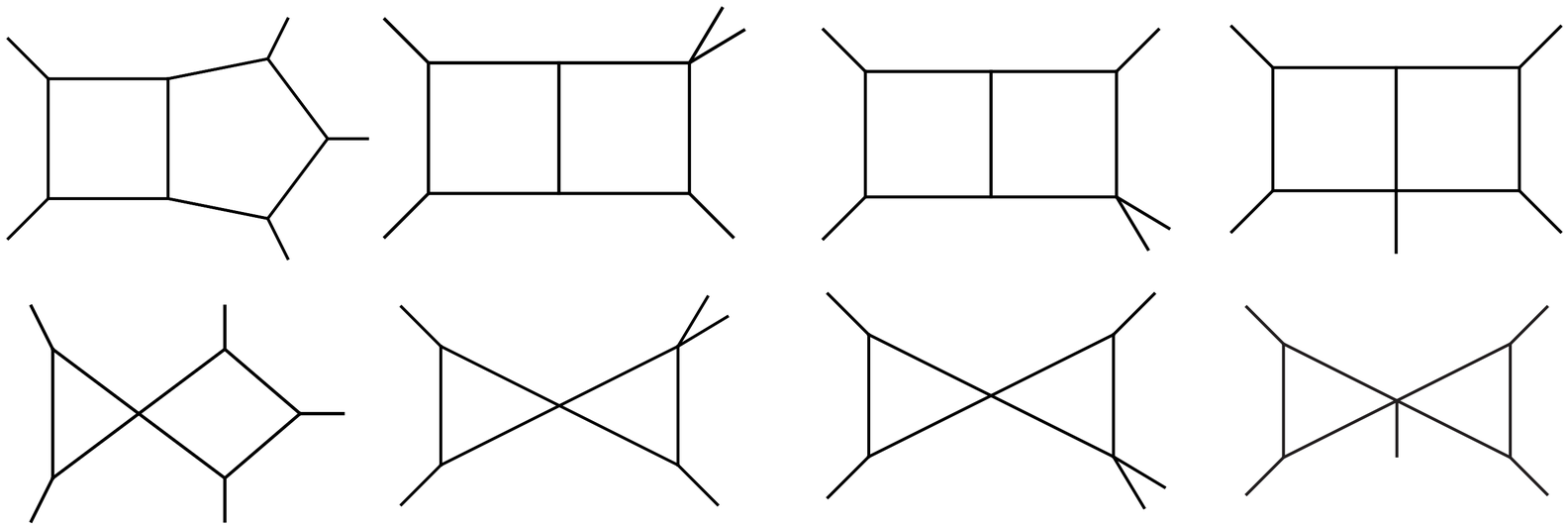
$$\mu_{33} = \mu_{11} + \mu_{22} + \mu_{12}$$

$$\Delta_{431}(1^+, 2^+, 3^+, 4^+, 5^+) = \frac{i\, s_{12} s_{23} s_{45} F_1(D_s, \mu_{11}, \mu_{22}, \mu_{12})}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}(tr_+(1345)(k_1 + p_5)^2 + s_{15}s_{34}s_{45})$$

$$F_1(D_s, \mu_{11}, \mu_{22}, \mu_{12}) = (D_s - 2)(\mu_{11}\mu_{22} + \mu_{11}\mu_{33} + \mu_{22}\mu_{33}) + 4(\mu_{12}^2 - 4\mu_{11}\mu_{22})$$

- Feynman rules + cut solution
- 6D spinor helicity formalism

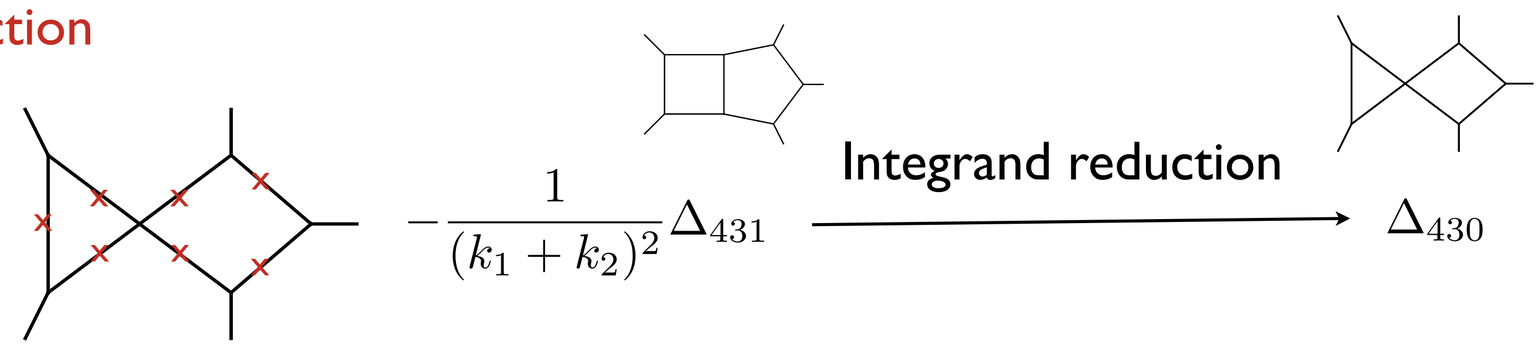
2-loop 5-gluon amplitude



arXiv: 1310.1051

first result on 2-loop 5-gluon helicity amplitude in QCD

subtraction



all coefficients are analytically found
IR structure: consistent with Catani's factorization

non-planar part: under progress, same methods

Momentum-twistor parametrization

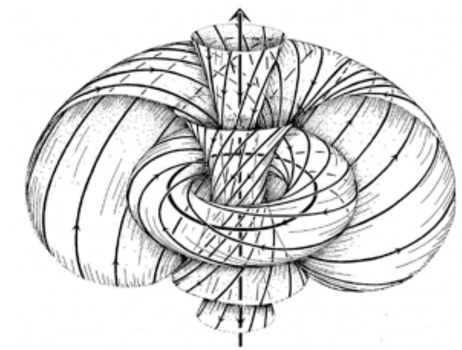
Analytic computation

Andrew Hodges

Spinor helicity formalism $(\lambda, \tilde{\lambda})$ ⟶ Momentum-twistor parametrization (λ, μ)

- momentum conservation
- Schouten identity
- Fierz identity
- ...

all constraints resolved



$$\boxed{\tilde{\lambda}_i = \frac{\langle i,\, i+1\rangle \mu_{i-1} + \langle i+1,\, i-1\rangle \mu_i + \langle i-1,\, i\rangle \mu_{i+1}}{\langle i,\, i+1\rangle \langle i-1,\, i\rangle}}$$

5-point

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \frac{1}{x_1} & \frac{1}{x_1}+\frac{1}{x_2} & \frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3} \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & x_4 & 1 \\ 0 & 0 & 1 & 1 & \frac{x_5}{x_4} \end{pmatrix}$$

In the final result, it is easy to convert $\{x_1, x_2, x_3, x_4, x_5\}$ to s_{ij}, tr_5...

n-point, under progress

Maximal unitarity method

master integrals

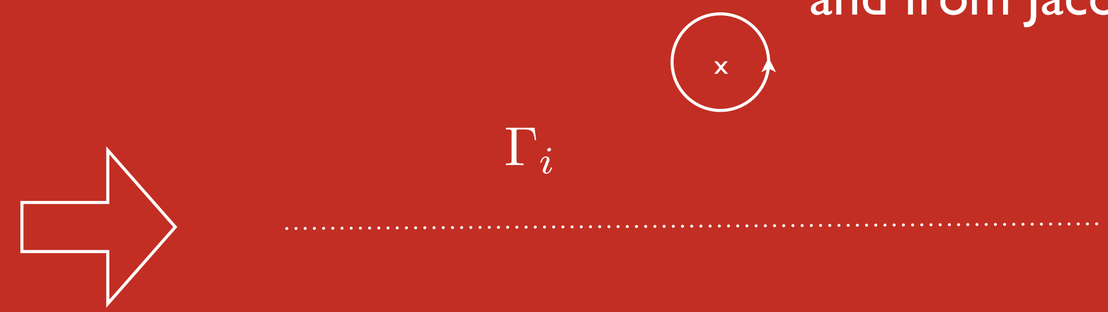
$$A = \sum_i C_i I_i + \text{Rational terms}$$

extract coefficients by contour integrals

Ruth Britto
Freddy Cachazo
Bo Feng
David Kosower
Kasper Larsen
....

locus of poles $D_1 = \ldots = D_n = 0$

and from Jacobian



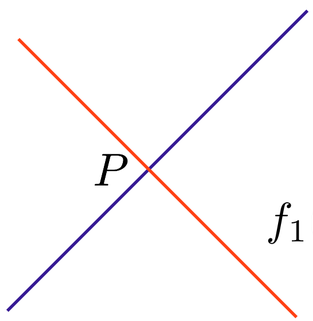
$$A = \int \frac{d^D k_1}{(2\pi)^D} \cdots \frac{d^D k_L}{(2\pi)^D} \frac{N}{D_1 \ldots D_n}$$

$$C_i = \oint_{\Gamma_i} \frac{d^D k_1}{(2\pi)^D} \cdots \frac{d^D k_L}{(2\pi)^D} \frac{N}{D_1 \ldots D_n}$$

multidimensional contours

multivariate residues

(z_1, z_2)



$f_2(z_1, z_2) = 0$

$f_1(z_1, z_2) = 0$

$$Res_P\left(\frac{hdz_1 \wedge dz_2}{f_1 f_2}\right) \equiv \frac{1}{(2\pi i)^2} \oint_{|f_1|=\epsilon} \oint_{|f_2|=\epsilon} \frac{hdz_1 \wedge dz_2}{f_1 f_2}$$

Trivial! if $f_1(z_1, z_2) = f_1(z_1)$, $f_2(z_1, z_2) = f_2(z_2)$..., it is reduced to univariate residues.

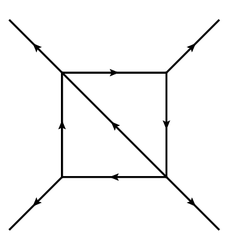
Non-degenerate, if $J = \det\left(\frac{\partial f_i}{\partial z_j}\right)_P \neq 0$, then

$$Res_P\left(\frac{hdz_1 \wedge dz_2}{f_1 f_2}\right) = \frac{h(P)}{J(P)}$$

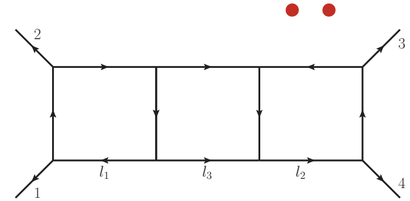
one-loop diagrams, two-loop double box...

Degenerate, if $J = \det\left(\frac{\partial f_i}{\partial z_j}\right)_P = 0$.

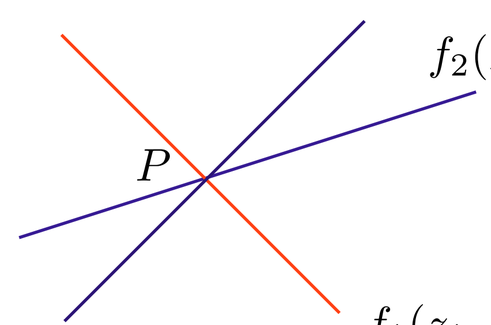
For example, $\{f_1, f_2\} = \{z_1, z_1^2 - z_2^2\}$ and $h = z_2$



??



(z_1, z_2)



$f_2(z_1, z_2) = 0$

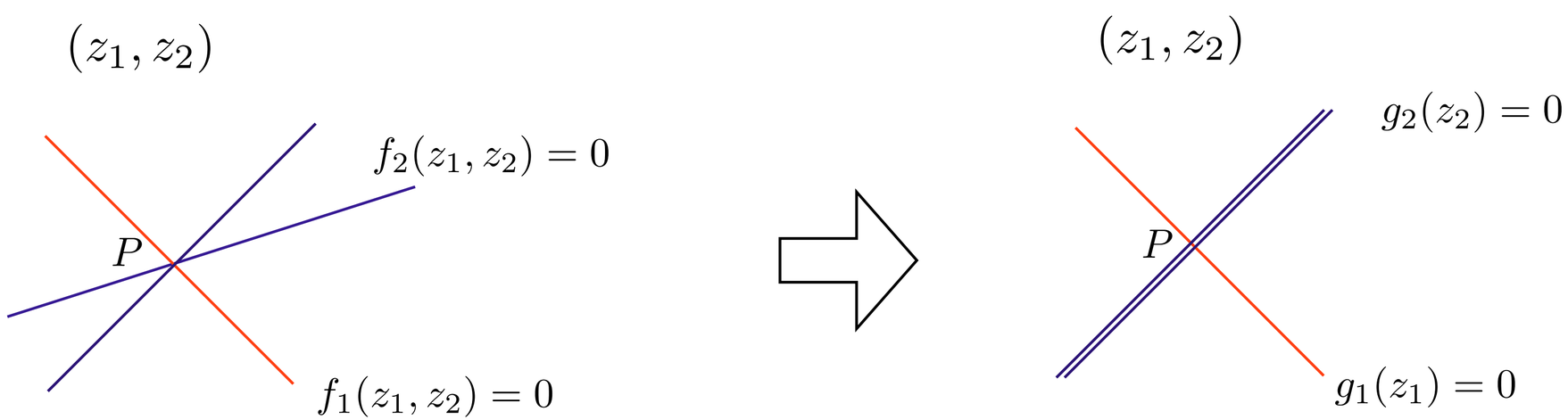
$f_1(z_1, z_2) = 0$

algebraic geometry approach

Transformation law

$I = \langle f_1, \ldots, f_n \rangle$ be the zero-dimensional ideal and $J = \langle g_1, \ldots, g_n \rangle$ be a zero-dimensional ideal such that $J \subset I$. So $g_i = a_{ij} f_j$, where the a_{ij}'s are polynomials. Let A be the matrix of a_{ij}'s,

$$Res_{\{f_1,\ldots,f_n\},\xi}\left(\frac{h(z)dz_1 \wedge \cdots \wedge dz_n}{f_1(z)\cdots f_n(z)}\right) = Res_{\{g_1,\ldots,g_n\},\xi}\left(\frac{h(z)dz_1 \wedge \cdots \wedge dz_n}{g_1(z)\cdots g_n(z)} \det A\right).$$

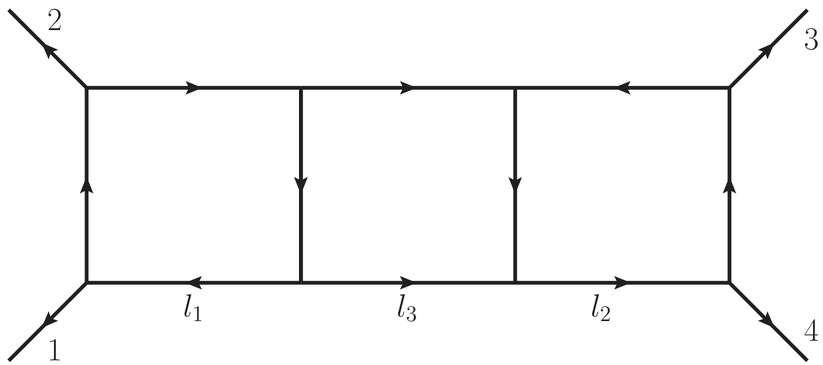


$$\begin{pmatrix} z_1 \\ z_2^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z_1 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_1^2 - z_2^2 \end{pmatrix}.$$

All multivariate residues can be calculated in this way

triple box example

arXiv:1310.6006, Mads Sogaard and YZ



- 14 branches
- 64 residues
- 23 independent residues

degenerate residues exist

using Integration-by-parts relations to find weights of contours...

$$\Omega_1 = \frac{1}{8}\chi^3 s_{12}^{10}\{-1,0,-2,0,1,1,0,0,1,-1,-1,0,1,0,1,0,2,0,-1,1,-1,-1,0\}\ ,$$

$$\Omega_2 = \frac{1}{4}\chi^2 s_{12}^{9}\{0,1,2,-1,-2,-1,0,-1,-1,1,0,0,-1,1,0,-1,-2,1,2,-1,1,0,0\}\ ,$$

$$\Omega_3 = \frac{1}{4}\chi^2 s_{12}^{9}\{1,-1,-2,3,3,0,-2,1,0,0,1,2,0,-1,-1,1,2,-3,-3,0,0,1,0\}\ .$$

$$C_1 I_{\text{tribox}}[1] + C_2 I_{\text{tribox}}[l_1 \cdot p_4] + C_3 I_{\text{tribox}}[l_3 \cdot p_4]$$

consistent with the result from integrand reduction...

generalize to n-loop, any diagram

Conclusion

- Algebraic geometry approach to high-loop amplitudes
  - Gröbner Basis → Integrand basis
  - Primary decomposition → Global unitarity cut structure
  - Multivariate residues → maximal cut method
- First steps towards automating high-loop amplitudes
- Promising for NNLO 2 →3, 4 processes