Nonperturbative QCD vacuum polarization corrections

Dru Renner

JLab

done in collaboration with **Xu Feng** (KEK),

Marcus Petschlies (Humboldt U.) and **Karl Jansen** (DESY)

Outline

- start with the muon $g-2$ as a concrete example
  - measurements and the standard model differ by $3\,\sigma$
  - illustrates the relevant phenomenology
  - allows me to explain our modified lattice method
- continue to illustrate our method with calculations of
  - $g-2$ for the electron and tau, quite distinct from the muon
  - $\Delta\alpha(Q^2)$, the QCD corrections to the running QED coupling
  - higher-order QCD corrections, using $g_\mu - 2$ as an example
- ask me about: the Alder function $D(Q^2)$, α_s, or muonic hydrogen

Muon $g-2$

Status of muon $g-2$

- anomalous magnetic moment due solely to radiative corrections

$$a_\mu \equiv \frac{g_\mu - 2}{2} = \frac{\alpha}{2\pi} + \mathcal{O}(\alpha^2)$$

- experimental measurement at BNL [Muon G-2, PRD 2006]

$$a_\mu^{\mathrm{ex}} = 1.16592080(63) \times 10^{-3} \quad [0.54\ \mathrm{ppm}]$$

- standard model estimate [Jegerlehner, Nyffeler Phys. Rept. 2009]

$$a_\mu^{\mathrm{th}} = 1.16591790(65) \times 10^{-3} \quad [0.56\ \mathrm{ppm}]$$

- a $3.2\,\sigma$ difference *might* indicate physics beyond the standard model

$$a_\mu^{\mathrm{ex}} - a_\mu^{\mathrm{th}} = 2.90(91) \times 10^{-9}$$

Future experiments

- planned or proposed experiments at Fermilab and J-PARC

$$\sigma(a_\mu^{\text{ex}}) = 6.3 \times 10^{-10} \to 1.6 \times 10^{-10} \quad \text{[using FNAL]}$$

- comparison would be dominated by theory errors ($\sigma(a_\mu^{\text{th}}) = 6.5{\cdot}10^{-10}$)

$$\sigma(a_\mu^{\text{ex}} - a_\mu^{\text{th}}) = 9.1 \cdot 10^{-10} \to 6.7 \cdot 10^{-10}$$

- assuming the measurement remains consistent, i.e. $\pm 2\,\sigma$, gives

$$\sigma(a_\mu^{\text{ex}} - a_\mu^{\text{th}})/(a_\mu^{\text{ex}} - a_\mu^{\text{th}}) = 3.2 \to (2.4 - 6.3)$$

- either way, allowed/excluded BSM physics limited by theory errors
- improvements in the standard model estimate are highly desirable

Theory error budget

- standard model error is dominated by the QCD corrections

Contribution	σ^{th} $[10^{-10}]$
QCD-LO $[\alpha^2]$	5.3
QCD-NLO $[\alpha^3]$	3.9
QED/EW	0.2
Total	6.6

- $\sigma(a_\mu^{\text{ex}}) \to 1.6 \cdot 10^{-10}$ will not probe higher QED/EW corrections

- naively, α^4 QCD correction is not needed at the FNAL precision

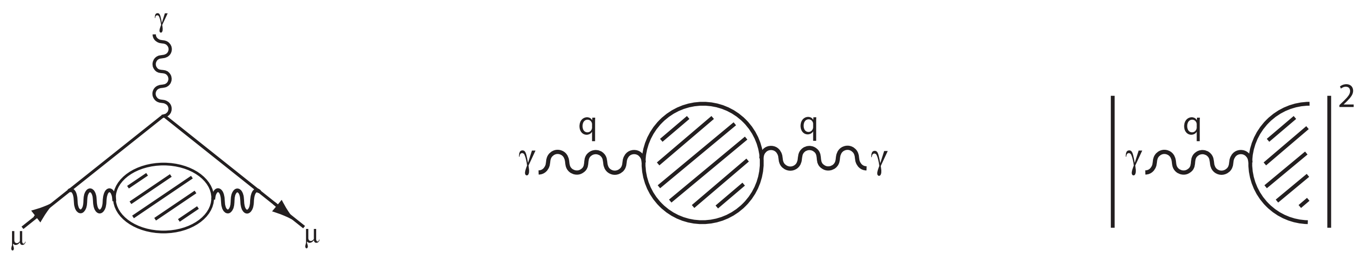
- but the α^2 and α^3 QCD corrections must be improved by factor 4

QCD correction at leading order

- QCD contribution is expanded in α with nonperturbative coefficients

$$a_\mu^{\mathrm{QCD}} = \alpha^2 a_\mu^{\mathrm{hlo}} + \alpha^3 a_\mu^{\mathrm{hnlo}} + \mathcal{O}(\alpha^4)$$

- QCD corrections first occur at $\mathcal{O}(\alpha^2)$, only smaller than QED piece



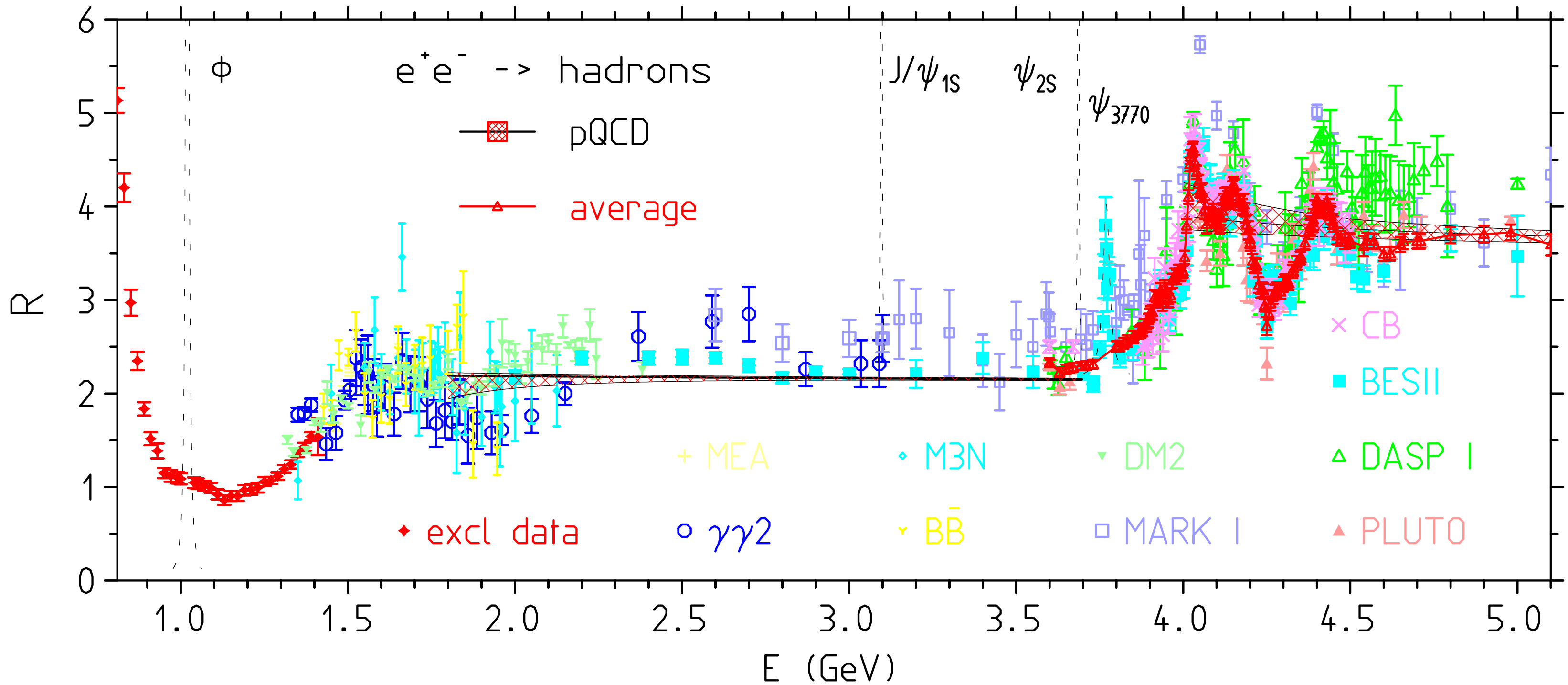
- leading-order hadronic contribution (hlo) is in fact measured

$$a_\mu^{\mathrm{hlo}} = \alpha^2 \int_{4m_\pi^2}^{\infty} \frac{ds}{s} K^{\mathrm{lo}}(s/m_\mu^2) R(s) \qquad R(s) = \frac{\sigma(\gamma^* \to \mathrm{hadrons})}{\sigma(\gamma^* \to e^+ e^-)}$$

- thus the "theory" calculation requires significant experimental input

Measurement of $R(s)$

- complicated analysis of $\mathcal{O}(100)$ channels/experiments



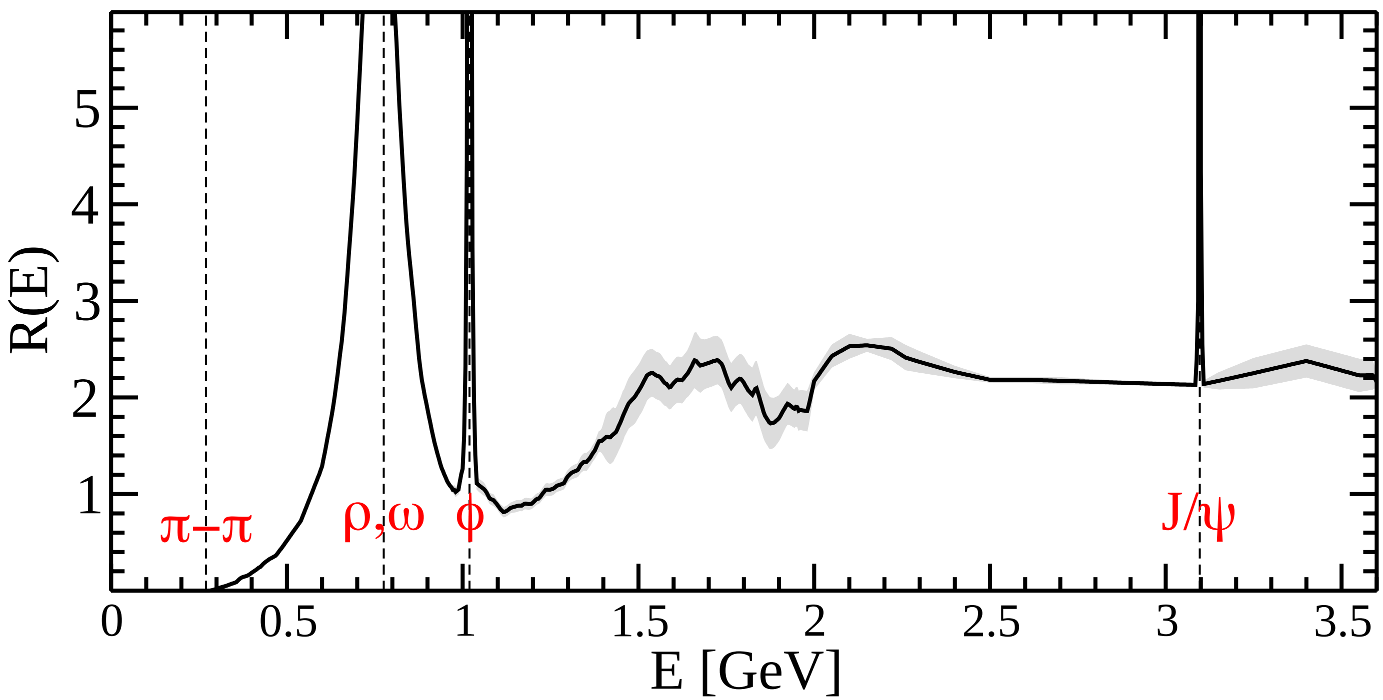
- improvement in $\sigma(e^+e^- \to \text{hadrons})$ coming from many experiments

[Jegerlehner, Nyffeler Phys.Rept.477, 2009]

Phenomenological flavor dependence

- pheno. analysis uses $R_{N_f}(s)$ to extract $N_f = 2$ and 3 contributions

$$R_{N_f}(s) = R(s)(\sum_{N_f} Q_f^2)/(\sum_{N} Q_f^2) \qquad 4m_N^2 \le s \le 4m_{N+1}^2$$

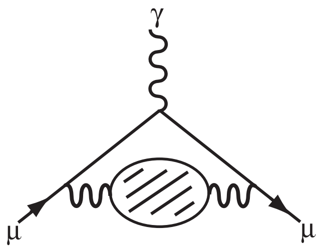


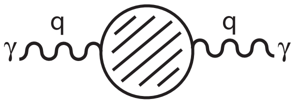
- this is a simple/crude means of estimating importance of strange/charm

[$R(E)$ given by F. Jegerlehner's compilation of $\sigma(e^+e^- \to$ hadrons)]

Lattice calculation of a_μ^{hlo}

- a_μ^{hlo} can also be calculated directly in Euclidean space





- vacuum polarization tensor is a simple two-point function

$$\pi_{\mu\nu}(Q^2) = \int d^4X\, e^{iQ\cdot(X-Y)} \langle J_\mu(X) J_\nu(Y) \rangle = (Q_\mu Q_\nu - Q^2 \delta_{\mu\nu}) \pi(Q^2)$$

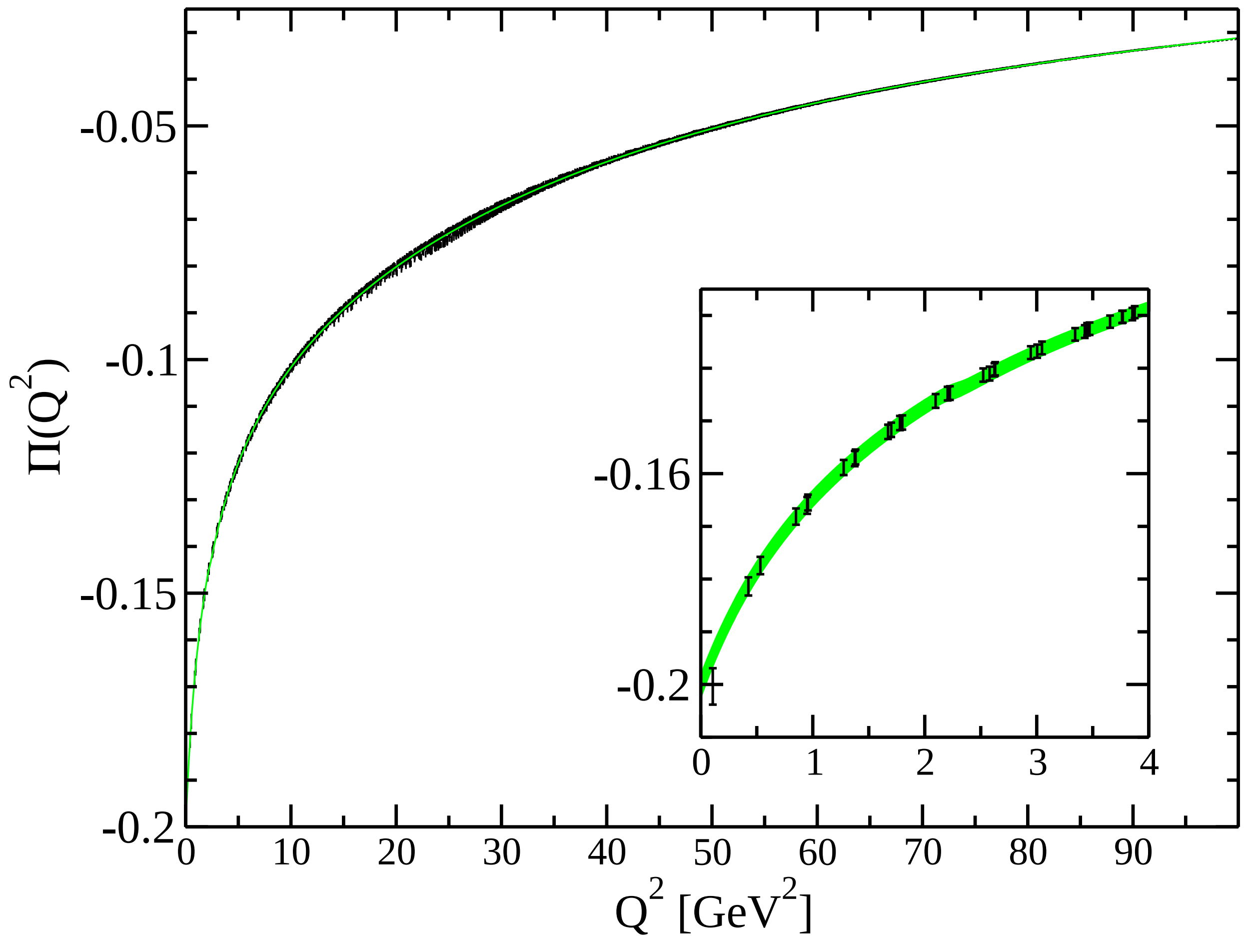
- leading-order QCD contribution [Blum, PRL 2003]

$$a_\mu^{\text{hlo}} = \alpha^2 \int_0^\infty \frac{dQ^2}{Q^2}\, w^{\text{lo}}(Q^2/m_\mu^2)\, \pi_R(Q^2)$$

- $\pi_R(Q^2) = \pi(Q^2) - \pi(0)$ is finite with $R(s) \propto \text{Im}\pi(-s + i\epsilon)$

Advantages of Euclidean space

- no complicated resonance structure, almost boring Q^2 dependence



- straightforward matching to perturbative QCD at large Q^2

Problems with an external scale

- a_l^{hlo} is made dimensionless at the expense of introducing m_l

$$a_l^{\text{hlo}} = \alpha^2 \int_0^\infty \frac{dQ^2}{Q^2}\, w^{\text{lo}}(Q^2/m_l^2)\, \pi_R(Q^2)$$

- the lepton mass is completely unrelated to QCD scales

$$m_e \approx 5.1 \cdot 10^{-4}\ \text{GeV} \qquad m_\mu \approx 0.11\ \text{GeV} \qquad m_\tau \approx 1.8\ \text{GeV}$$

- introduces dependence on lattice spacing in dimensionless quantity

$$\frac{Q^2}{m_l^2} = \frac{1}{a^2}\frac{a^2 Q^2}{m_l^2} = \frac{1}{a^2}\frac{[Q^2]_{\text{latt}}}{[m_l^2]_{\text{GeV}}}$$

- creates strong m_{PS} dep., as seen in leading vector-meson contribution

$$a_{l,V} \propto g_V^2 \frac{m_l^2}{m_V^2}$$

Effective dimension

- d_{eff} captures the dimension of only the QCD scales

$$d_{\text{eff}}[X] = -\frac{a}{X}\frac{\partial X}{\partial a}\bigg|_{g_0=\text{fixed}}$$

- for a standard QCD mass scale M, d_{eff} is the usual mass dimension

$$d_{\text{eff}}[M^n] = n$$

- however, it differs for a composite observable

$$d_{\text{eff}}[\ m_\mu^2/m_V^2\ ] = d_{\text{eff}}[\ 1/m_V^2\ ] = -2$$

- for a_μ^{hlo}, we have a nonperturbative but physical result

$$d_{\text{eff}}[a_\mu^{\text{hlo}}] = -1.887\,(5)$$

Eliminating the external scale

- this understanding leads to a class of modified observables

$$a_{\overline{\mu}}^{\text{hlo}} = \alpha^2 \int_0^\infty \frac{dQ^2}{Q^2} w^{\text{lo}} \left(\frac{Q^2}{H^2} \cdot \frac{H_{\text{phys}}^2}{m_\mu^2} \right) \pi_R(Q^2)$$

- H is any hadronic scale and $H(m_{PS} \to m_\pi) = H_{\text{phys}}$, so

$$\lim_{m_{PS} \to m_\pi} a_{\overline{\mu}}^{\text{hlo}} = a_\mu^{\text{hlo}}$$

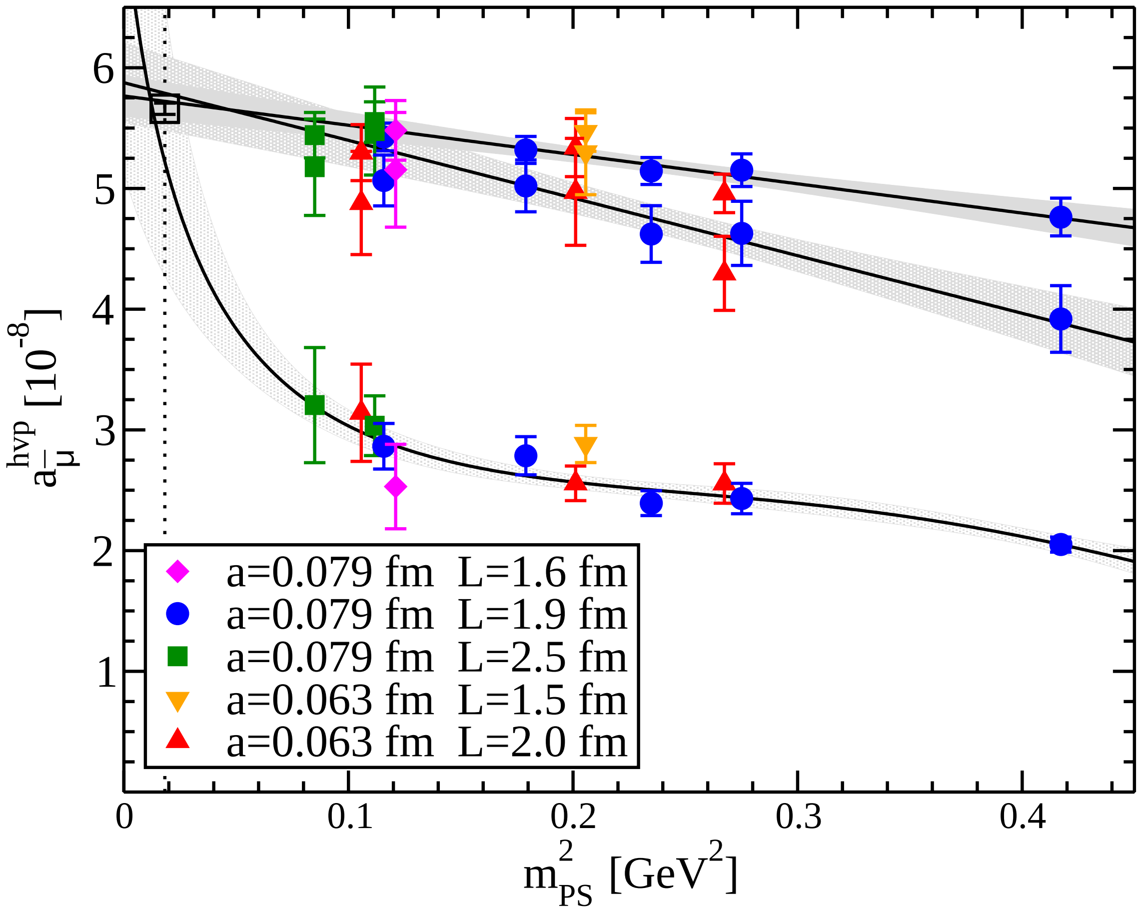
- each $a_{\overline{\mu}}^{\text{hlo}}$ behaves like a proper dimensionless QCD quantity

$$d_{\text{eff}}[a_{\overline{\mu}}^{\text{hlo}}] = 0$$

- each $a_{\overline{\mu}}^{\text{hlo}}$ is composed of hadronic scales only

Modified method for $a_\mu^{\rm hlo}$

- bottom to top: $H = 1$ (std. method), $H = f_V$ and $H = m_V$



- comparing to $N_f = 2$ piece important, full piece is $6.903\,(53) \cdot 10^{-8}$

- our error of 2.8% is in the ballpark of the 0.8% currently used

Electron and tau $g-2$

Electron and tau $g-2$

- high precision measurement of g_e [Harvard, PRL 100:120801 (2008)]

$$g_e/2 = 1.00115965218073\,(28) \quad [0.28\text{ ppt}]$$

- extraction of α from g_e just becoming sensitive to QCD corrections

$$\alpha^{-1} = 137.035999084\,(51) \quad [0.37\text{ ppb}]$$

- g_e provides an very different probe of the QCD vacuum polarization

$$a_e^{\text{hlo}} \approx \frac{4}{3}\alpha^2 m_e^2 \left.\frac{d\pi_R}{dQ^2}\right|_{Q^2=0} \qquad d_{\text{eff}}[a_e^{\text{hlo}}] = -1.999984\,(1)$$

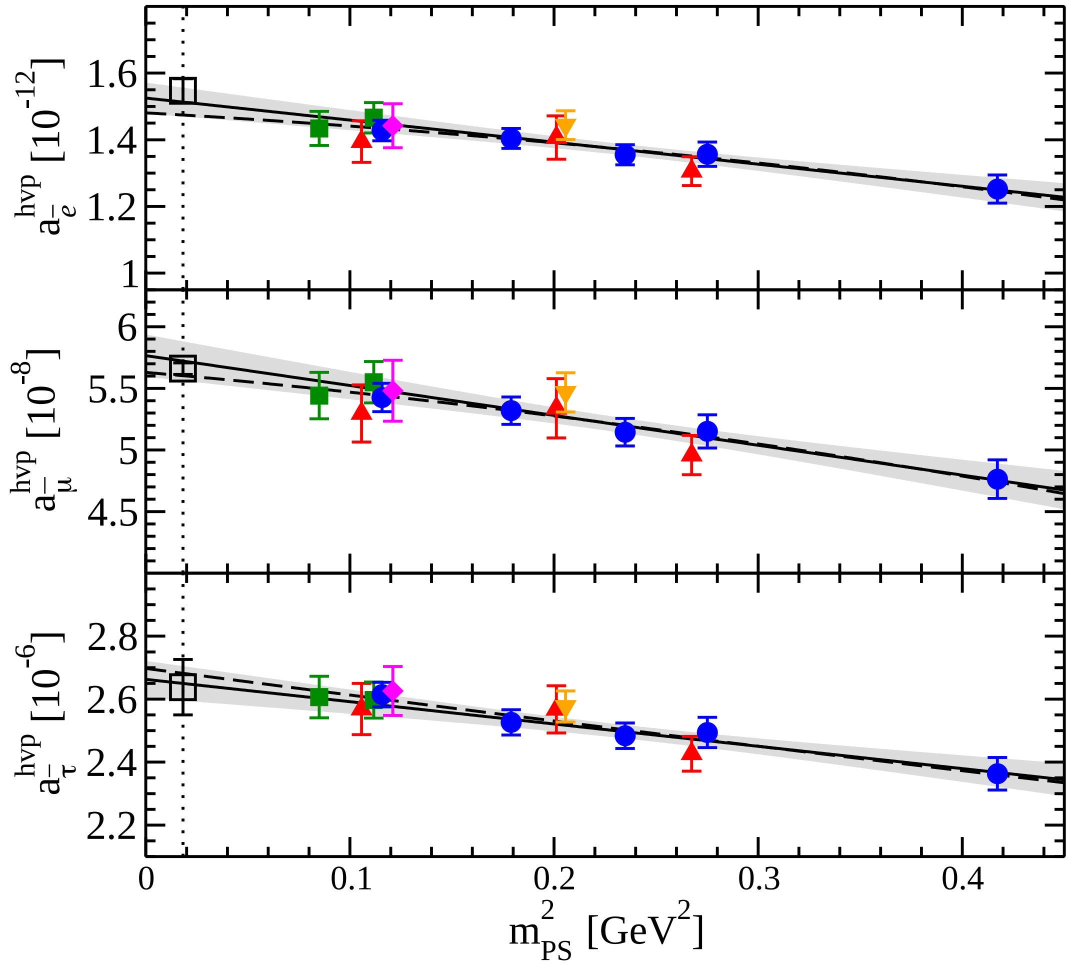
- g_τ is sensitive to larger Q^2 and provides another test of our calculation

$$d_{\text{eff}}[a_\tau^{\text{hlo}}] = -0.936\,(13)$$

- g_τ is much more difficult to measure directly but a_τ^{hlo} is not

Calculation for all three charged leptons

- no QCD perturbation theory, complete nonperturbative calculation



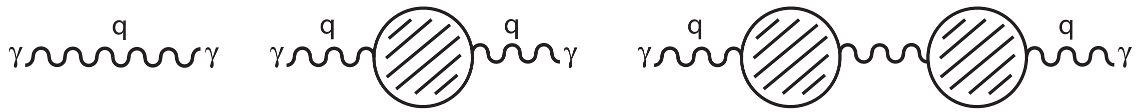
- the e is similar to the μ with our result at 2.8% versus 0.8%
- but for the τ we are doing better with 2.0% versus 3.3%

Running of α

QCD corrections to the QED coupling

- an effective QED coupling is normally defined by

$$\alpha(Q^2) = \frac{\alpha}{1 - \Delta\alpha(Q^2)}$$



- the hadronic piece is again related to $\pi_R(Q^2)$

$$\Delta\alpha_{\mathrm{had}}(Q^2) = 4\pi\alpha\pi_R(Q^2)$$

- precision of α is eroded by QCD corrections

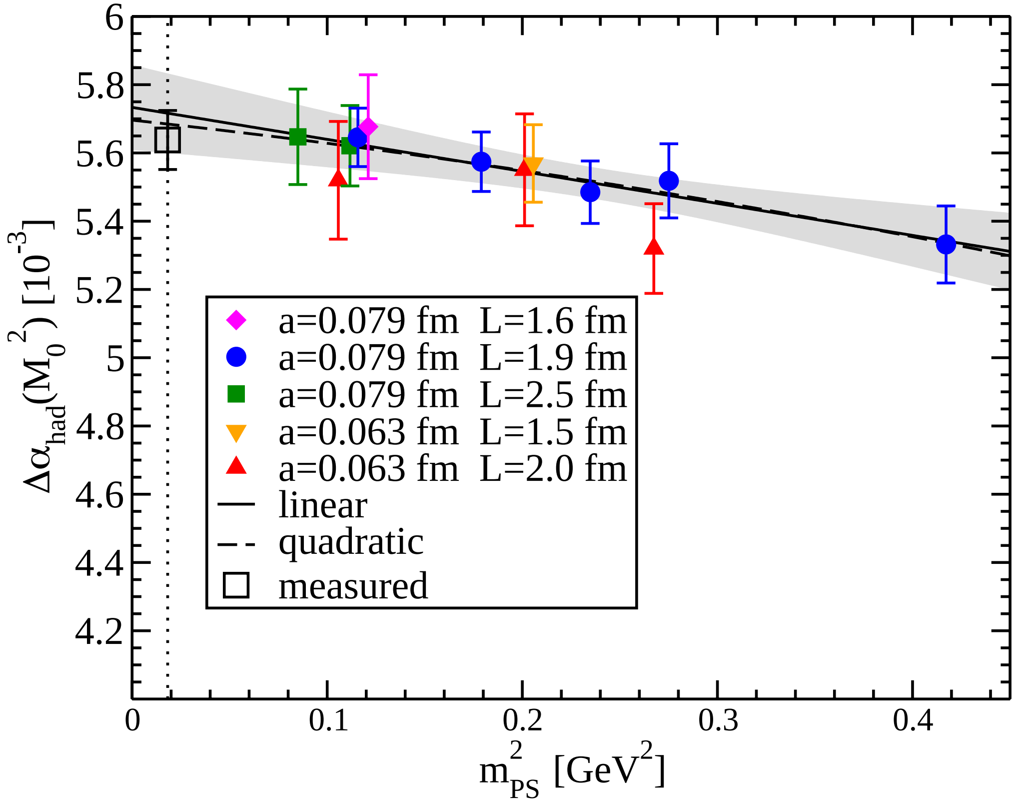
$$\frac{\sigma_\alpha}{\alpha} \approx 4 \cdot 10^{-10} \qquad \rightarrow \qquad \frac{\sigma_{\alpha(M_Z^2)}}{\alpha(M_Z^2)} \approx 3 \cdot 10^{-4}$$

- this impacts many SM predictions, for example the Gfitter fit for m_H

$$\begin{aligned} m_H &= 44^{+62}_{-43}\,\mathrm{GeV} \quad \text{without} \quad \Delta\alpha(M_Z^2) \\ m_H &= 96^{+31}_{-24}\,\mathrm{GeV} \quad \text{with} \quad \Delta\alpha(M_Z^2) \end{aligned}$$

Modified definition of $\Delta\alpha_{\mathrm{had}}(Q^2)$

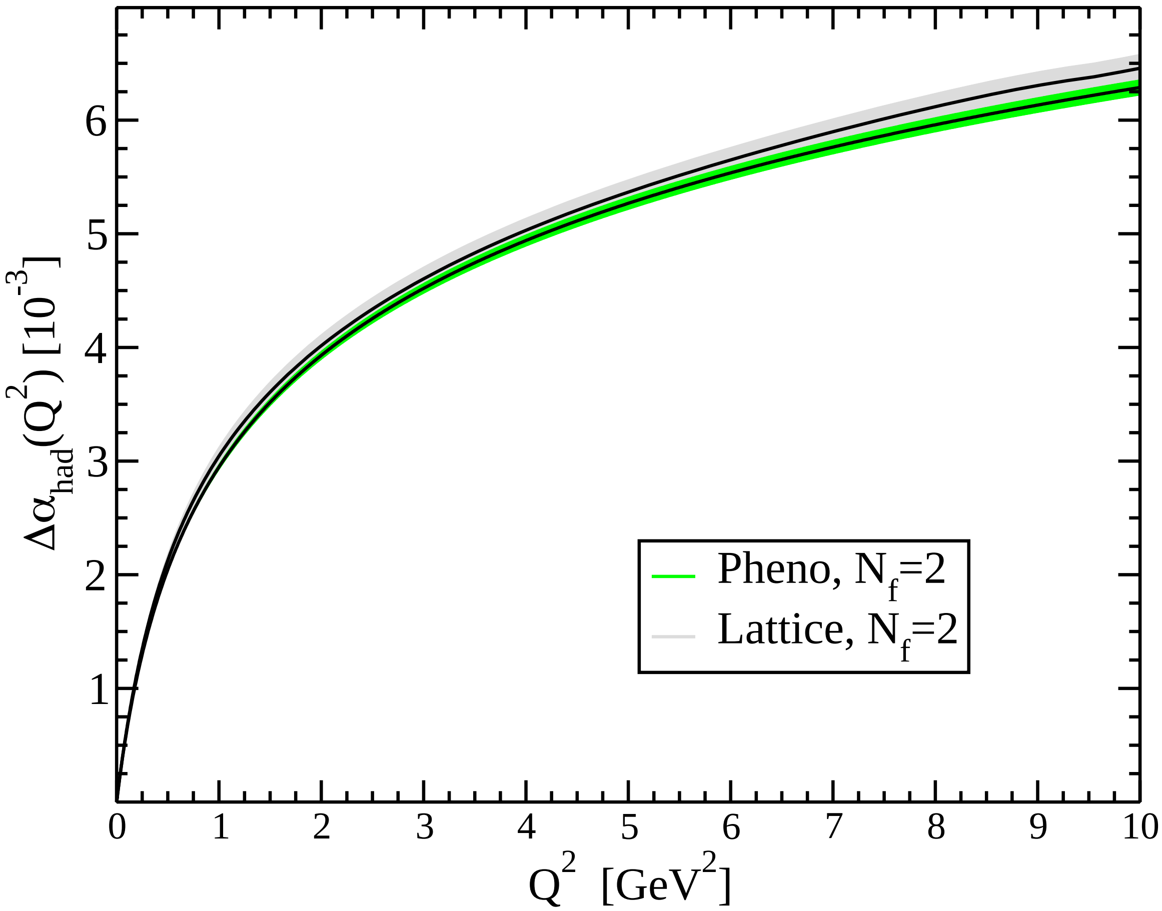
- treat Q^2 as an external scale and similarly define a new observable
- $M_0 = 2.5$ GeV is a common matching point in pheno. work



- our 2.1% accuracy is nearly competetive with the currently used 1.1%

Hadronic running of the QED coupling

- lattice artifacts only show up slowly for $Q^2 \gtrsim 7\ \mathrm{GeV}^2$



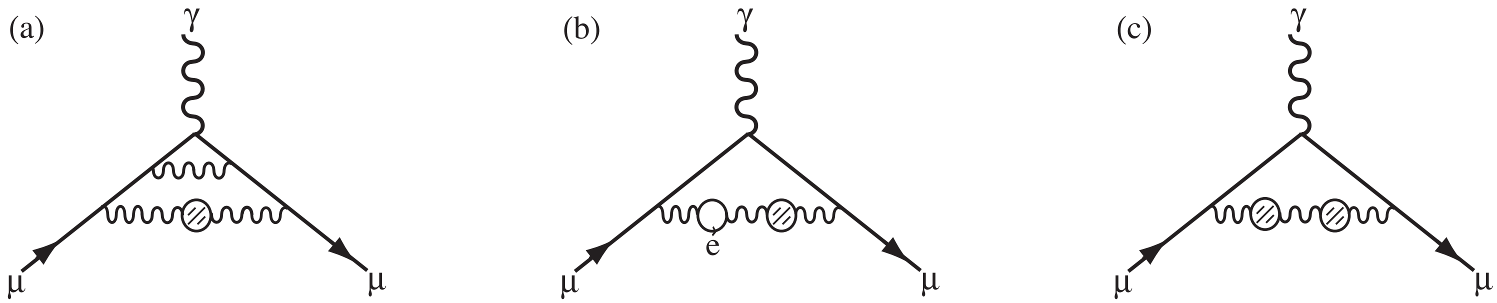
- α_s from $\pi(Q^2)$ used to determine $\Delta\alpha(M_Z^2) - \Delta\alpha(M_0^2)$ at 5 loops

$$\Delta\alpha(M_Z^2) = \Delta\alpha(M_0^2) + \Delta\alpha(M_Z^2) - \Delta\alpha(M_0^2) = 0.01715\,(42)$$

Higher order corrections

NLO QCD correction to $g_\mu - 2$

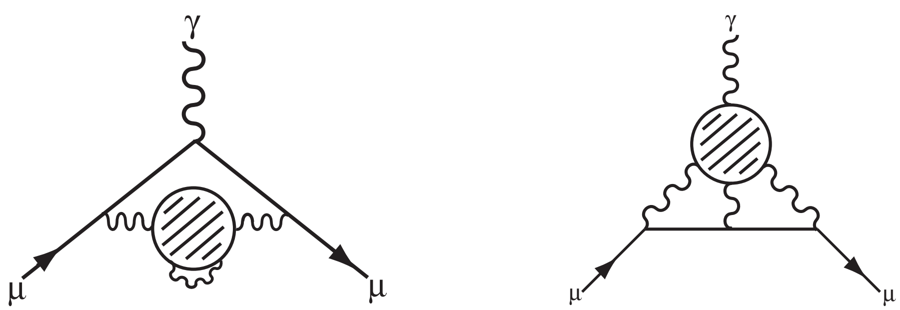
- calculated all three classes of 17 NLO diagrams involving $\pi_R(Q^2)$



- complete non-pert. NLO (α^3) correction, excluding light-by-light

$$a_\mu^{\text{nlo,hvp}} = -7.99\,(20)\cdot 10^{-10} \qquad \text{Lattice, } N_f = 2$$
$$a_\mu^{\text{nlo,hvp}} = -7.78\,(16)\cdot 10^{-10} \qquad \text{Pheno, } N_f = 2$$

- light-by-light corrections require a different technology



- ongoing work by Blum et. al, QCDSF, JLQCD

$$a_\mu^{\text{nlo,lbl}} = 8\,(4)\cdot 10^{-10} \leftrightarrow 12\,(4)\cdot 10^{-10} \qquad \text{Pheno}$$

Outlook for muon $g-2$

- a precision of 3% (2%) currently achieved for a_μ^{lo} ($\Delta\alpha$) for $N_f = 2$
- our $N_f = 4$ calculation, aiming at 3% is starting now
- 1% with $N_f = 4$ may be feasible for a_μ^{lo}, would match BNL precision
- FNAL/JPARC precisions would require another factor of 3 for a_μ^{lo}
- $a_\mu^{\text{nlo,vp}}$ with $N_f = 4$ should be possible at FNAL/JPARC precisions
- $a_\mu^{\text{nlo,lbl}}$ is an active research program, more ideas are still coming
- there are now 6 lattice groups working on the muon $g-2$

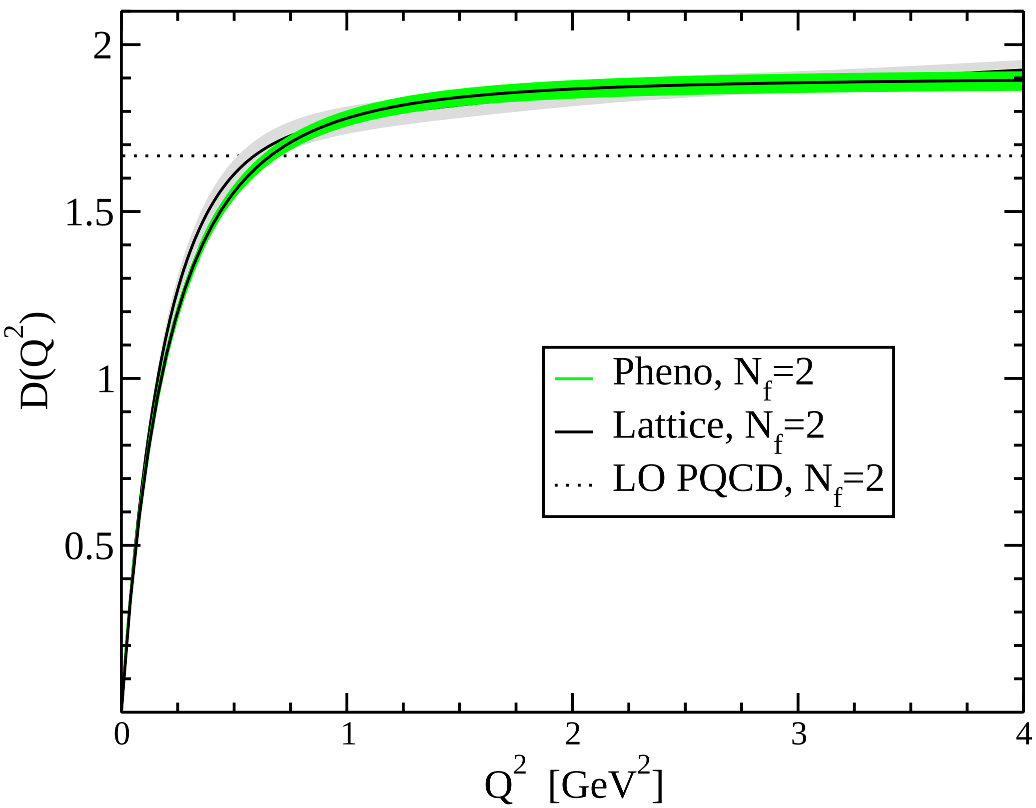
Extra slides

Adler function

- the Adler function eliminates the UV divergence by a derivative

$$D(Q^2) = 12\pi^2 Q^2 \frac{d\pi_R}{dQ^2} \quad \rightarrow \quad \overline{D}(Q^2) = D(Q^2/H^2_{\text{phys}} \cdot H^2)$$

- this makes $D(Q^2)$ much more sensitive to cut-off effects



$$\alpha_s^{(2)}(2\ \text{GeV}^2) = 0.263\,(16)$$

$$\Lambda^{(2)} = 222\,(27)\ \text{MeV}$$

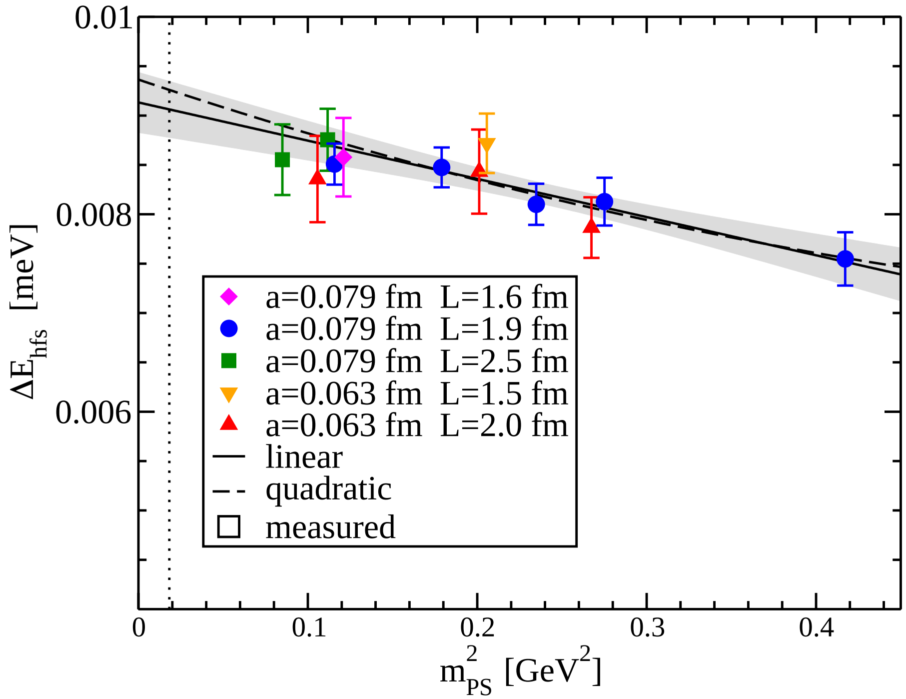
- can determine α_s and Λ at each Q^2 (2 GeV^2 used) without OPE

Muonic hydrogen

- the LO QCD corrections to the 2P/2S splitting in $\mu^- p$

$$\Delta E_{\rm hfs}^{\rm hlo} = 2\pi\alpha^5\mu^3 \left.\frac{d\pi_R}{dQ^2}\right|_{Q^2=0}$$

- this is closely related to $a_e^{\rm hlo}$ and similarly tests the low Q^2 region



Lattice, $N_f = 2$

$$\Delta E_{\rm hfs}^{\rm hlo} = 9.06\,(29)\ \mu\text{eV}$$

Pheno, $N_f = 2$

$$\Delta E_{\rm hfs}^{\rm hlo} = 9.17\,(07)\ \mu\text{eV}$$

- small compared to current $5\,\sigma$ discrepancy, only rough checks needed

$$E_{\rm ex} - E_{\rm th} = 0.316\,(63)\ \text{meV}$$

Definition of a_μ^{hlo} for $a > 0$

- the large Q^2 behavior is parameterized by fitting to

$$\pi_R(Q^2) = c + \ln Q^2 \cdot \sum_n a_n Q^{2n}$$

- to be precise, we fix the definition at non-zero lattice spacing with

$$\int_0^\infty dQ^2 \to \int_0^{Q^2_{\mathrm{uv}}} dQ^2 \qquad\qquad Q^2_{\mathrm{uv}} = 16/a^2$$

- the integral is convergent, so this is just a choice of cut-off effects
- this choice does not require QCD perturbation theory
- this definition does not force us to introduce a lattice spacing
- this last point is important given that $d_{\mathrm{eff}}[a_\mu] \approx -2$

Definition of $a_\mu^{\rm hlo}$ for $L < \infty$

- define π_R for low Q^2 by including the lowest meson and fitting the a_n

$$\pi_R(Q^2) = \frac{5}{9} g_V^2 \frac{Q^2}{Q^2 + m_V^2} + \sum_n a_n Q^{2n}$$

- fit ensures that $\pi_R(Q^2)$ matches lattice calculation for accessible Q^2
- extrapolation provides a well-defined finite-volume definition
- explicit vector-meson term is systematically reabsorbed as L increases

$$\frac{5}{9} g_V^2 \frac{Q^2}{Q^2 + m_V^2} = \sum_n b_n Q^2 \quad \text{for} \quad Q^2 < m_V^2$$

- this is not a systematic error but a proper finite-volume definition
- a practical matter of explicitly verifying controlled finite-size effects

Details on the effective dimension

- d_{eff} attempts to capture the dimensionality of only the QCD scales

$$d_{\text{eff}}[X] = -\frac{a}{X}\,\frac{\partial X}{\partial a}\bigg|_{g_0=\text{fixed}}$$

- for a standard mass scale M, definition is the usual mass dimension

$$d_{\text{eff}}[M^n] = -\frac{a}{M^n}\frac{\partial}{\partial a}\left(\frac{1}{a^n}\widehat{M}^n(g_0)\right) = -\frac{a}{M^n}\widehat{M}^n(g_0)\frac{\partial}{\partial a}\left(\frac{1}{a^n}\right) = n$$

- however, it differs for a composite observable

$$d_{\text{eff}}\left[\frac{m_\mu^2}{m_V^2}\right] = d_{\text{eff}}\left[\frac{1}{m_V^2}\right] = -2$$

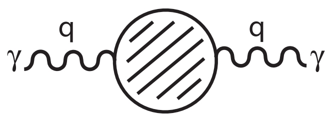
- for a_μ, we have an expression that must be evaluated on the lattice

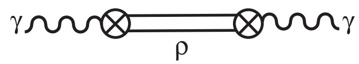
$$d_{\text{eff}}[a_\mu] = -2\left(\int\frac{dQ^2}{Q^2}w(Q^2/m_\mu^2)Q^2\frac{d\pi_R}{dQ^2}\right)\Big/\left(\int\frac{dQ^2}{Q^2}w(Q^2/m_\mu^2)\pi_R\right) < 0$$

- you can easily prove that $d_{\text{eff}}[a_\mu] \to -2$ (0) for $m_\mu \to 0$ (∞)

Vector meson contribution to a_μ

- the vector-mesons dominate the hadronic contribution to a_μ





- on general grounds we expect any model to give

$$a_{\mu,V} \approx c\frac{m_\mu^2}{m_V^2}$$

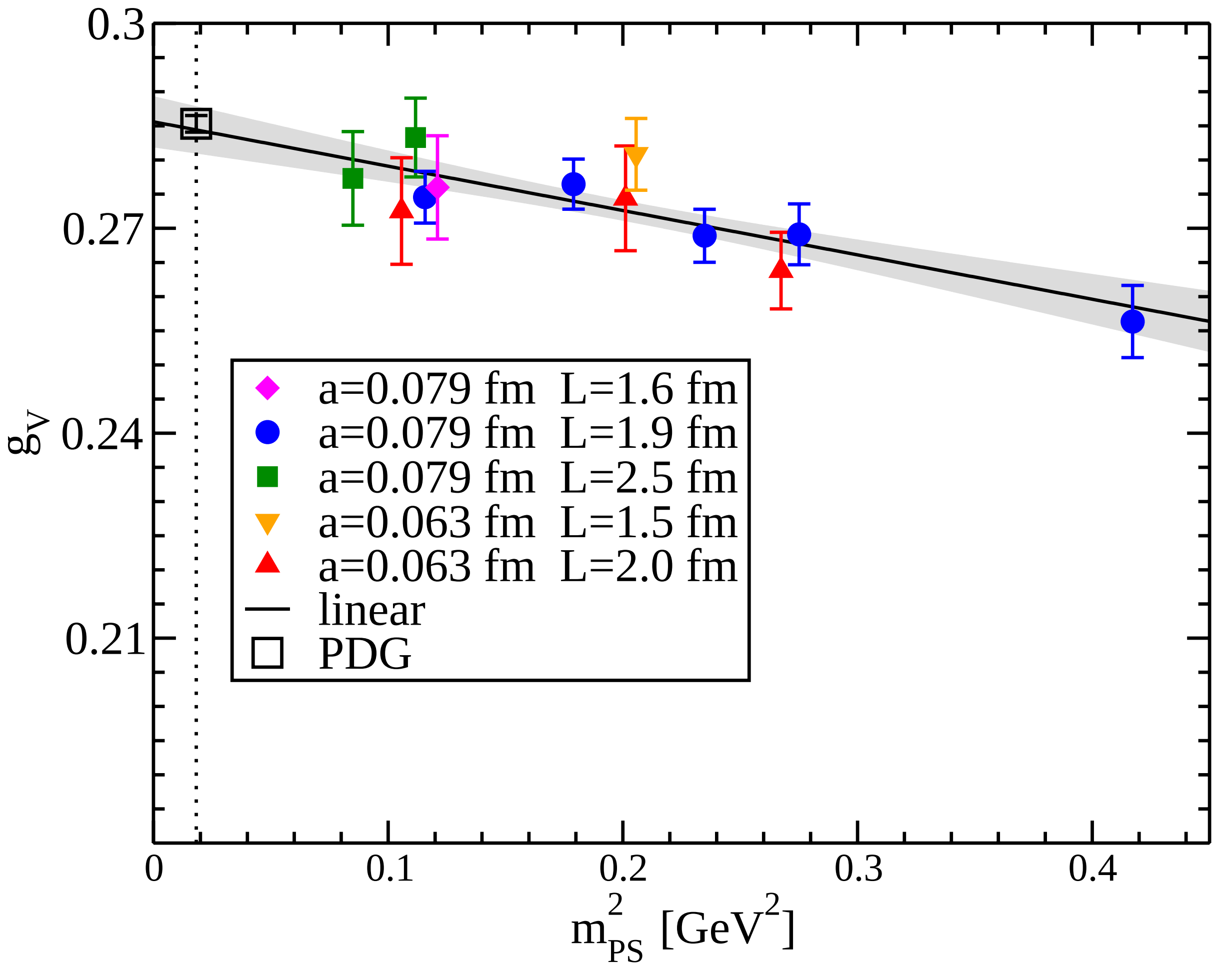
- tree-level chiral perturbation theory gives

$$a_{\mu,V} = \alpha^2 g_V^2 f(m_\mu^2/m_V^2) = \frac{2}{3}\alpha^2 g_V^2 \frac{m_\mu^2}{m_V^2} + \mathcal{O}(m_\mu^4/m_V^4)$$

- this allows us to model the vector meson contribution to a_μ^{hlo}

Electromagnetic coupling of vector-meson

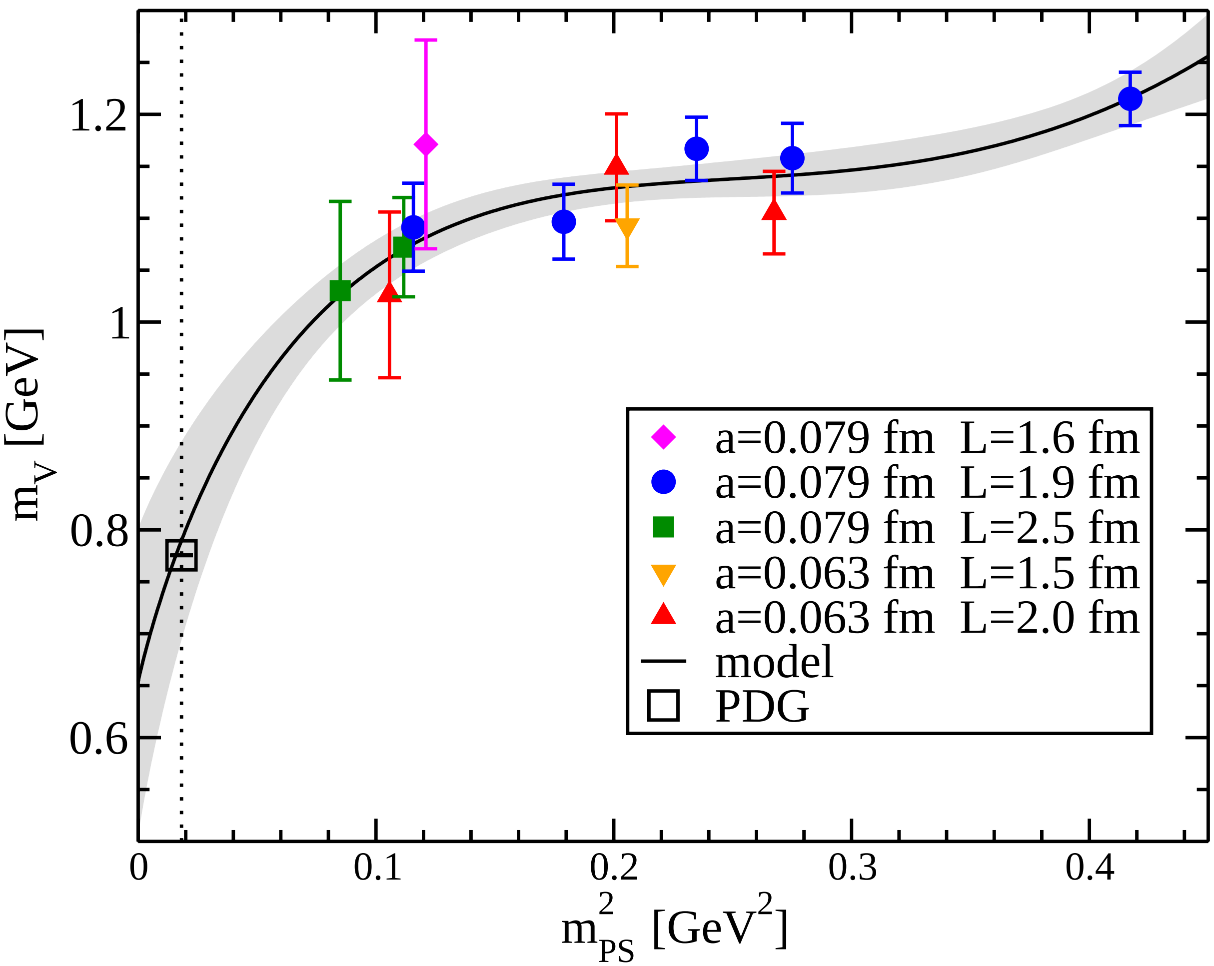
- dimensionless quantities are typically better calculated



- result for g_V represents quantitative success for our calculation

Mass of vector-meson

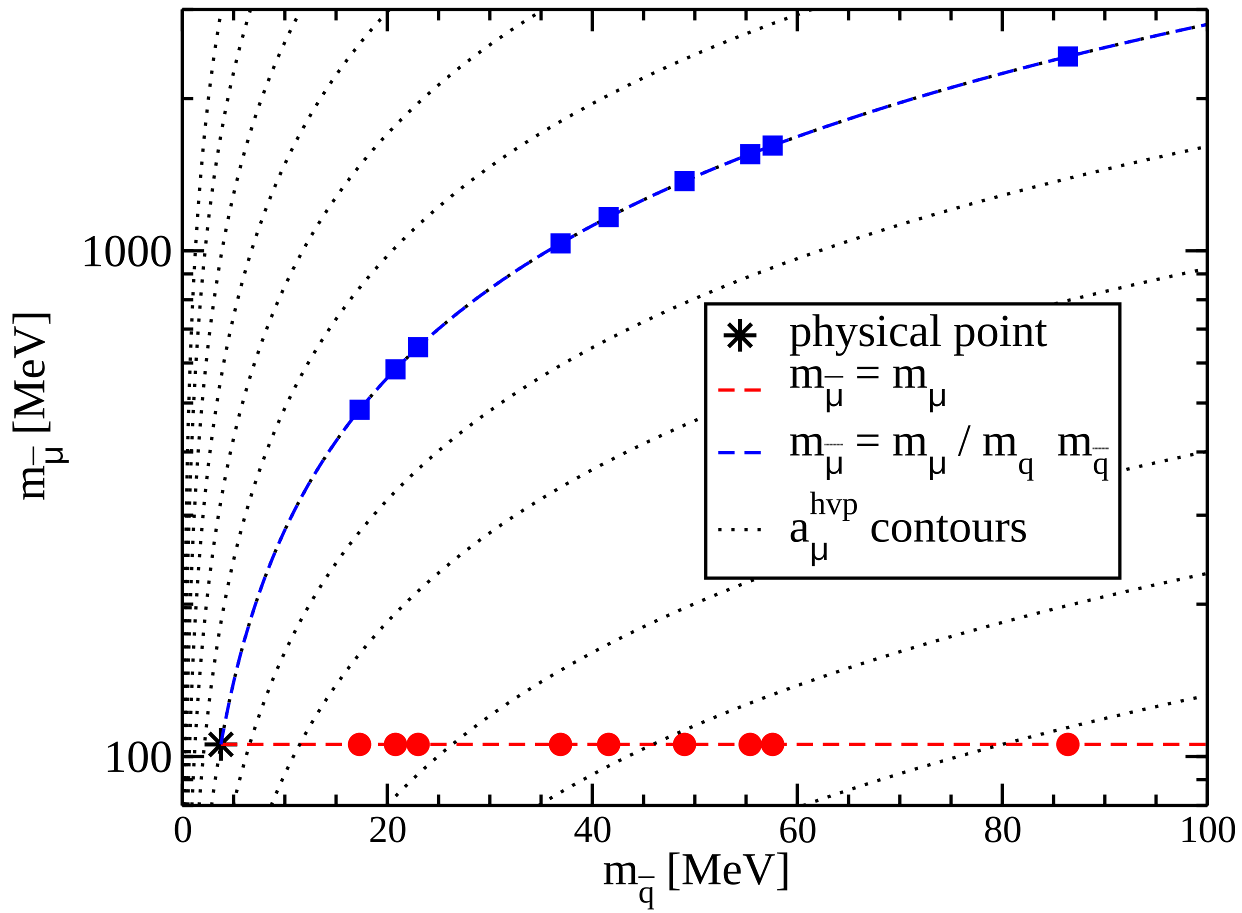
- dimensionful quantities are sensitive to the overall scale setting



- phenomenological fit includes the PDG value of m_ρ

Renormalization of QCD + QED

- introduce a variable muon mass $m_{\overline{\mu}}$ and quark mass $m_{\overline{q}}$

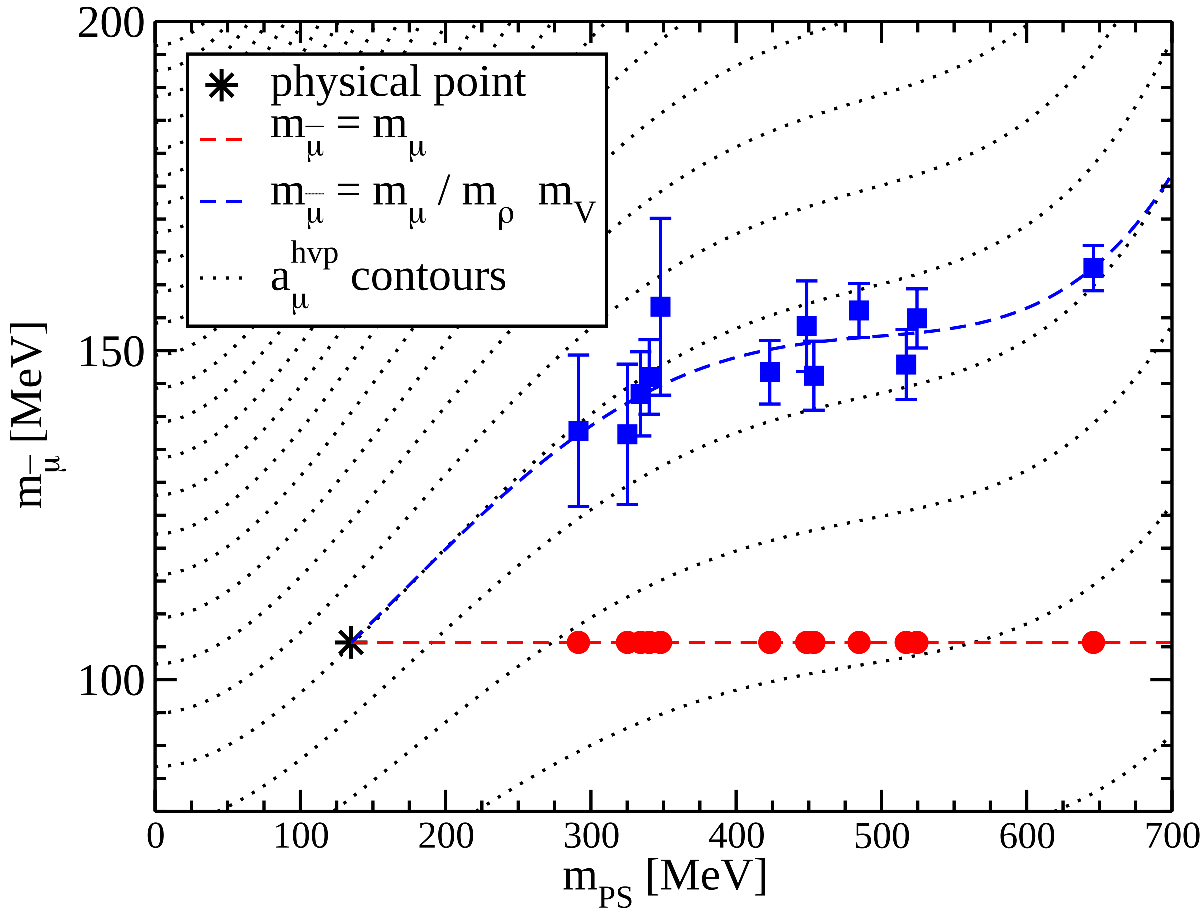


- both paths, with $m_{\overline{\mu}}$ or $m_{\overline{\mu}}/m_{\overline{q}}$ fixed, define valid physical limits

- but $m_{\overline{\mu}} = (m_\mu/m_q)\, m_{\overline{q}}$ follows a contour of a_μ^{hlo} in pQCD

Hadronic scheme

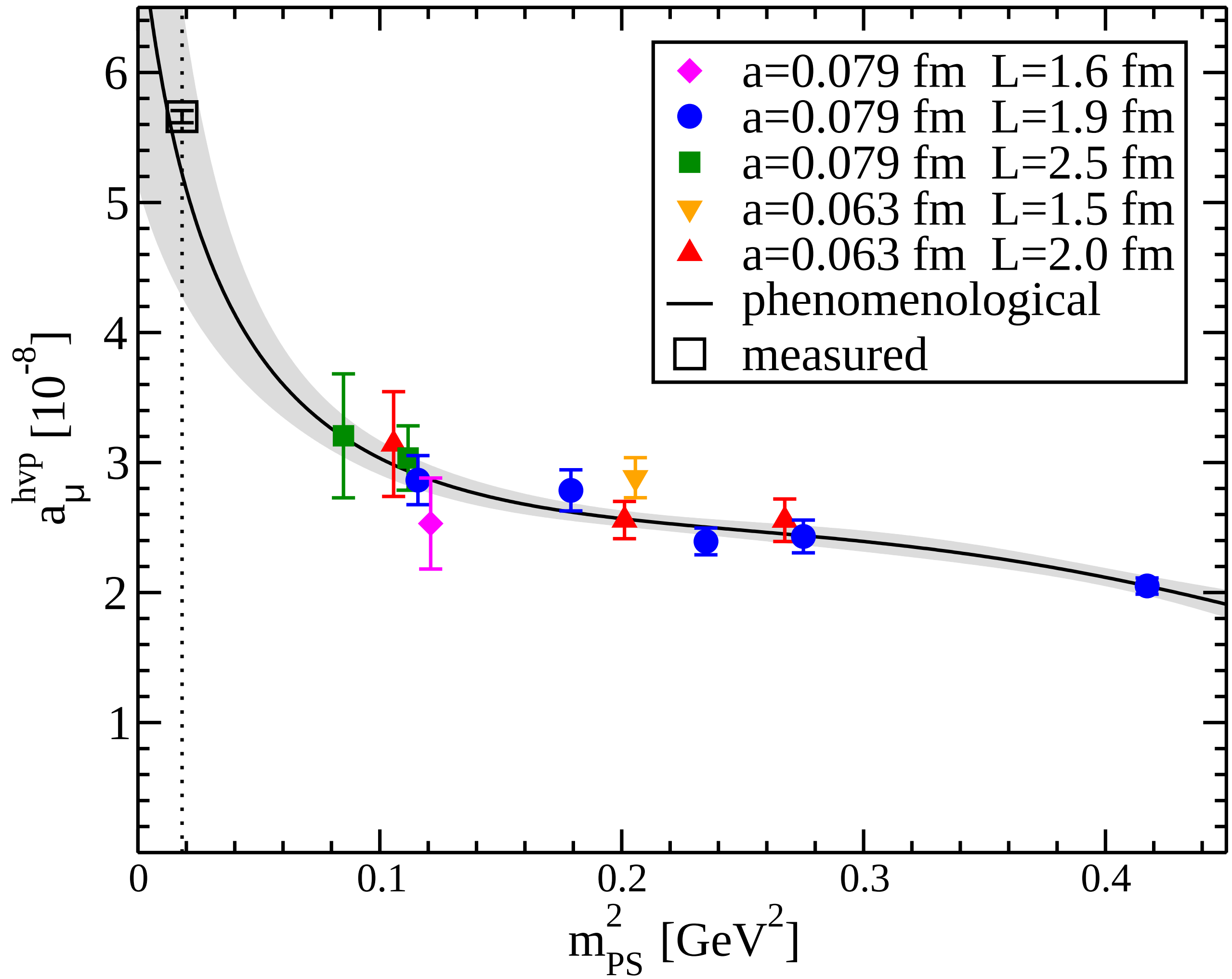
- introduce variable muon mass $m_{\overline{\mu}}$ and pseudo-scalar mass m_{PS}



- curve $m_{\overline{\mu}} = (m_\mu/m_\rho)\, m_V$ is implicitly defined so that $m_{\overline{\mu}} \to m_\mu$
- contours from VMD model (ask me) matched to the lattice calc.

Phenomenological description of a_μ^{hlo}

- can combine model expectations with our calc. of g_V and m_V



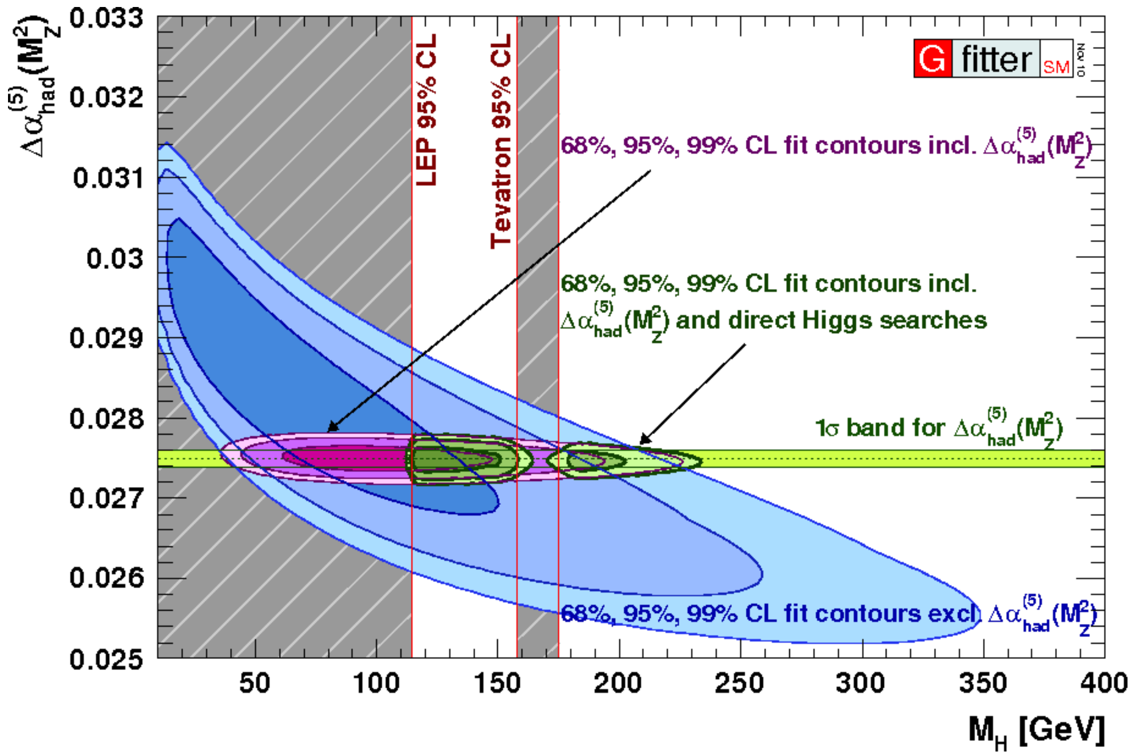
- apparently strong m_{PS} dependence of m_V is reflected in a_μ^{hlo}

Standard model predictions and $\Delta\alpha_{\text{had}}$

- precision of α ($\sigma_\alpha/\alpha \approx 4 \cdot 10^{-10}$) is eroded by QCD corrections

$$\frac{\sigma_{\alpha(M_Z)}}{\alpha(M_Z)} \approx 3 \cdot 10^{-4} \qquad \frac{\sigma_{G_F}}{G_F} \approx 9 \cdot 10^{-6} \qquad \frac{\sigma_{M_Z}}{M_Z} \approx 2 \cdot 10^{-5}$$

- this impacts many SM predictions, for example m_H



Modified definition of $\Delta\alpha(Q^2)$

- a change of variables gives $a_{\overline{\mu}}^{\text{hvp}}$ as

$$a_{\overline{\mu}}^{\text{hvp}} = \alpha^2 \int_0^\infty \frac{dQ^2}{Q^2}\, w(Q^2/m_\mu^2)\, \pi_R\left(Q^2/H_{\text{phys}}^2 \cdot H^2\right)$$

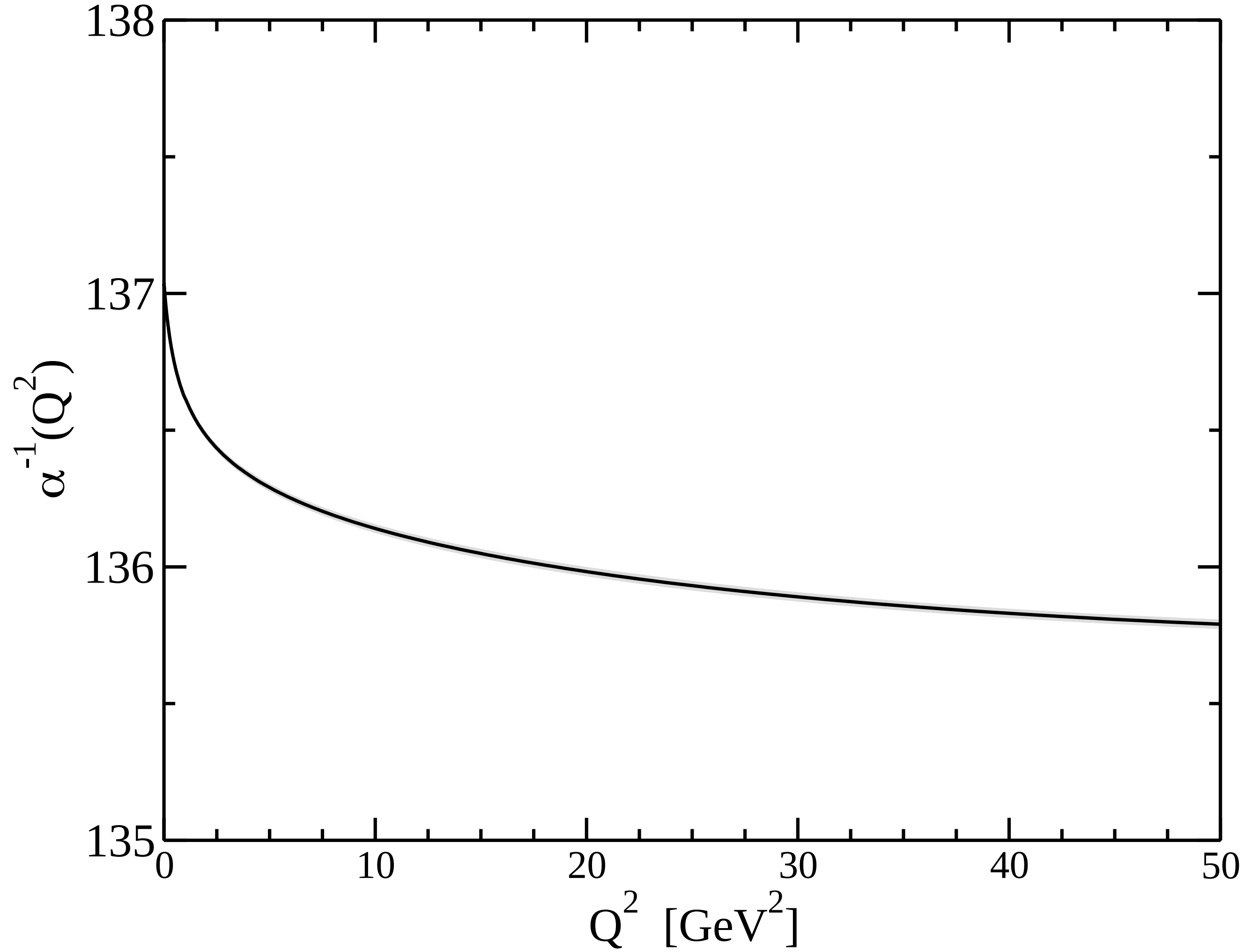
- this suggests treating Q^2 as an external scale like m_μ^2 and defining

$$\Delta\overline{\alpha}_{\text{had}}(Q^2) = 4\pi\alpha\pi_R\left(Q^2/H_{\text{phys}}^2 \cdot H^2\right)$$

- this choice for $\pi_R(Q^2)$ then defines all other observables consistently

Running of α

- includes only the QCD corrections, remember full $\alpha^{-1}(M_Z) \approx 129$



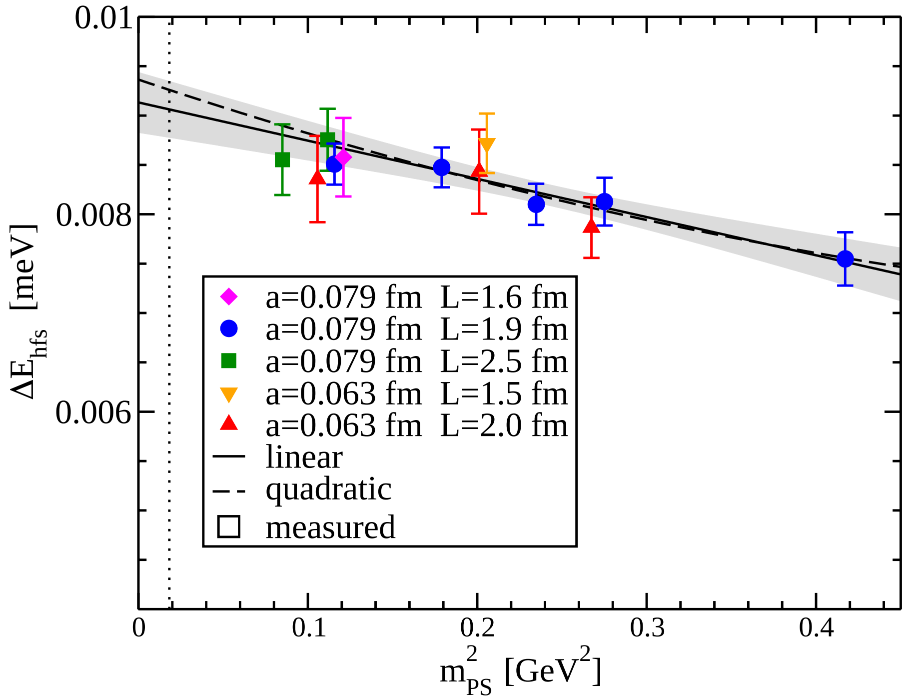
- future work will need matching to pQCD and/or larger Q^2

Muonic hydrogen

- the LO QCD corrections to the 2P/2S splitting in $\mu^- p$

$$\Delta E_{\text{hfs}}^{\text{hlo}} = 2\pi\alpha^5\mu^3 \left.\frac{d\pi_R}{dQ^2}\right|_{Q^2=0}$$

- this is closely related to a_e^{hlo} and similarly tests the low Q^2 region



Lattice, $N_f = 2$

$$\Delta E_{\text{hfs}}^{\text{hlo}} = 9.06\,(29)\ \mu\text{eV}$$

Pheno, $N_f = 2$

$$\Delta E_{\text{hfs}}^{\text{hlo}} = 9.17\,(07)\ \mu\text{eV}$$

- small compared to current $5\,\sigma$ discrepancy, only rough checks needed

$$E_{\text{ex}} - E_{\text{th}} = 0.316\,(63)\ \text{meV}$$

Isospin violating corrections

- by varying from m_π^0 to m_π^+, the standard method changes by

$$\Delta_{m_u \neq m_d} = 9.0 \cdot 10^{-11}$$

- by taking the maximum variation under m_π^0 to m_π^+ and ρ^0 to ρ^+

$$\Delta_{m_u \neq m_d} = 8.0 \cdot 10^{-11}$$

- this suggests isospin violating effects are potentially $\mathcal{O}(10^{-10})$