Moments of the 3-Loop Corrections to the Heavy Flavor Contribution to $F_2(x, Q^2)$ for $Q^2 \gg m^2$

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- Introduction and Theory Status
- The Method
- 2 Loop Results
- Asymptotic 3 Loop Results (Fixed Moments) & Anomalous Dimensions
- Towards an all–N Result at 3 Loops.
- Conclusions

References:

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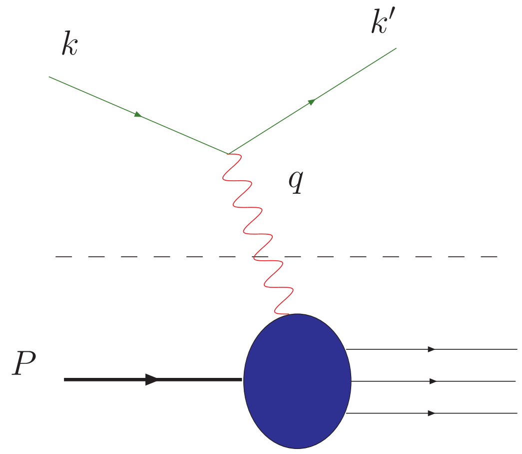
1. Introduction

Deep–Inelastic Scattering:



$\longrightarrow \quad L_{\mu\nu}$

$\longrightarrow \quad W_{\mu\nu}$

$$Q^2 := -q^2, \quad x := \frac{Q^2}{2P\cdot q}\ , \quad \text{Bjorken–x}$$

$$\nu := \frac{P\cdot q}{M}\ ,$$

$$\frac{d\sigma}{dQ^2\, dx} \sim L^{\mu\nu} W_{\mu\nu}$$

The picture of the proton at short distances [Feynman, 1969; Bjorken, Paschos, 1969.]

- The proton mainly consists of light partons.
- There are three valence partons: two up quarks and one down quark.
- The sea–partons are: $u,\ \overline{u},\ d,\ \overline{d},\ s,\ \overline{s}$ and the gluon g.

The hadronic tensor cannot be calculated perturbatively. It can be decomposed into several scalar structure functions. For DIS via single photon exchange, it is given by:

$$W_{\mu\nu}(q,P,s) = \frac{1}{4\pi}\int d^4\xi \exp(iq\xi)\langle P,s \mid [J_\mu^{em}(\xi), J_\nu^{em}(0)] \mid P,s\rangle$$

$$\text{unpol.}\left\{ \quad = \frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_L(x,Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2(x,Q^2)\right.$$

$$\text{pol.}\left\{ \quad -\frac{M}{2Pq}\varepsilon_{\mu\nu\alpha\beta} q^\alpha \left[s^\beta g_1(x,Q^2) + \left(s^\beta - \frac{sq}{Pq}p^\beta\right) g_2(x,Q^2)\right] .\right.$$

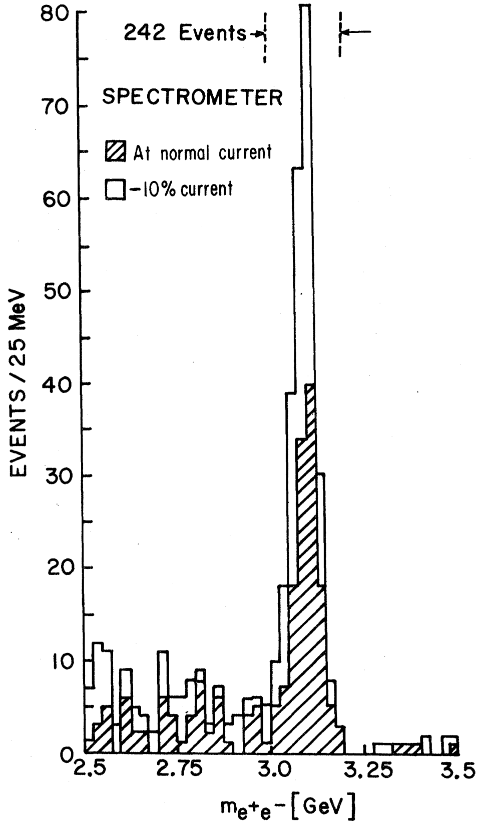
In Bjorken limit, $\{Q^2,\ \nu\} \rightarrow \infty$, x fixed, at twist $\tau = 2$–level:

$$\underbrace{F_i(x,Q^2)}_{\text{structure functions}} = \sum_j \underbrace{\mathcal{C}_{i,j}\left(x, \frac{Q^2}{\mu^2}, \frac{m_k^2}{\mu^2}\right)}_{\text{Wilson coefficients, perturbative}} \otimes \underbrace{f_j(x,\mu^2)}_{\text{parton densities, non-perturbative}} ,$$

$\Longrightarrow$ Wilson coefficients contain both light and heavy flavor contributions:

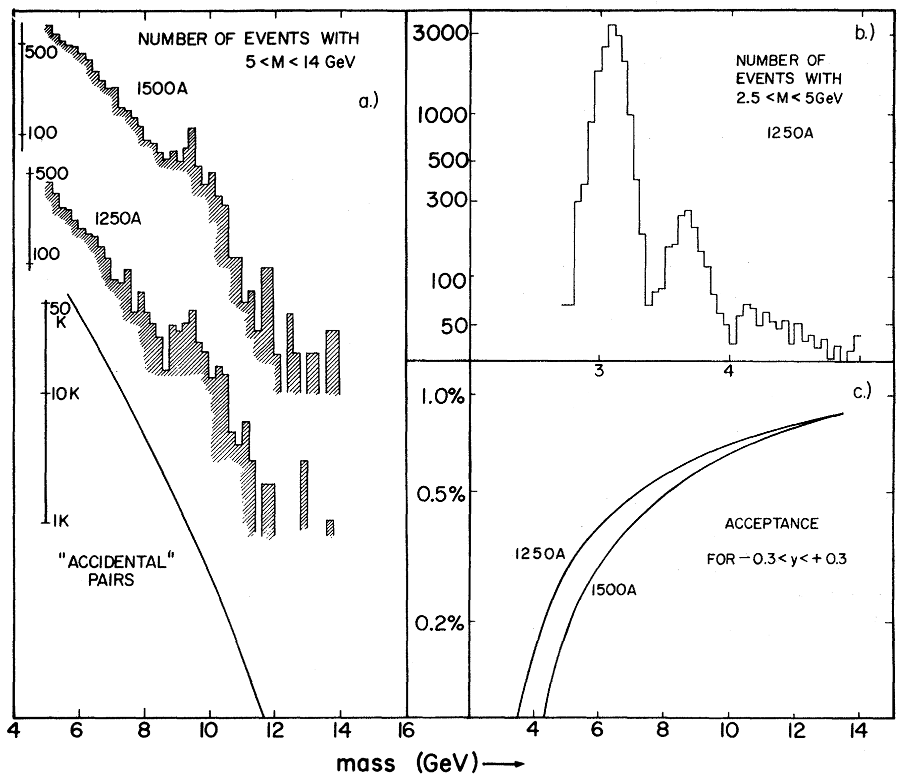
$$\mathcal{C}_{i,j}\left(x, \frac{Q^2}{\mu^2}, \frac{m_k^2}{\mu^2}\right) = C_{i,j}^{\text{light}}\left(x, \frac{Q^2}{\mu^2}\right) + \mathsf{H}_{i,j}\left(x, \frac{Q^2}{\mu^2}, \frac{m_k^2}{\mu^2}\right) , k = c, b .$$

The Discovery of Heavy Quarks



$\mathcal{J}$ [Aubert *et al.*, 1974] @ BNL

Ψ [Augustin *et al.*, 1974] @ SLAC



Υ [Herb *et al.*, 1977] @ FERMILAB

- Masses of charm and bottom [PDG, 2008.]: $m_c \approx 1.3\,\text{GeV}$, $m_b \approx 4.2\,\text{GeV}$

Heavy Quarks in DIS

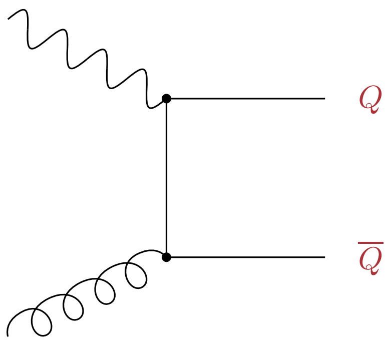
- Assume only light partons in the proton. Light quarks may directly scatter off the exchanged vector boson, the gluon via quark–pair production.

- Heavy quarks (c or b) emerge in final states through hard scattering processes (top outside the HERA region).

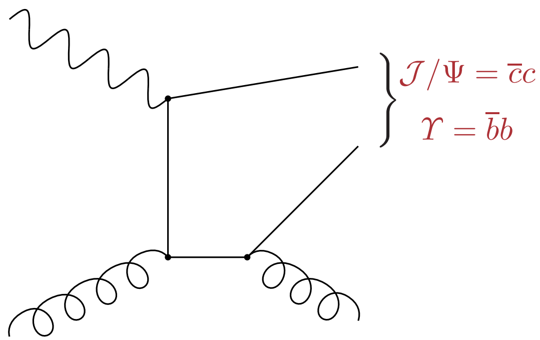
- LO contribution to $F_{(2,L)}$ by heavy quark production: photon-gluon fusion

$$F_{(2,L)}^{Q\overline{Q}}(x,Q^2) = 4e_c^2 a_s \int_{ax}^{1} \frac{dz}{z} H_{(2,L),g}^{(1)}\left(\frac{x}{z}, \frac{m^2}{Q^2}\right) G(z,Q^2) \ , \quad a = 1 + 4m^2/Q^2 \ .$$

- open c(b) production:
  $D_u = \overline{u}c$, ...
  $B_u = \overline{u}b$, ...



- heavy quark resonances:
  $\overline{c}c = \mathcal{J}/\Psi$
  $\overline{b}b = \Upsilon$.



[Witten, 1976; Glück, Reya, 1979, ...]

[Berger, Jones, 1981.]

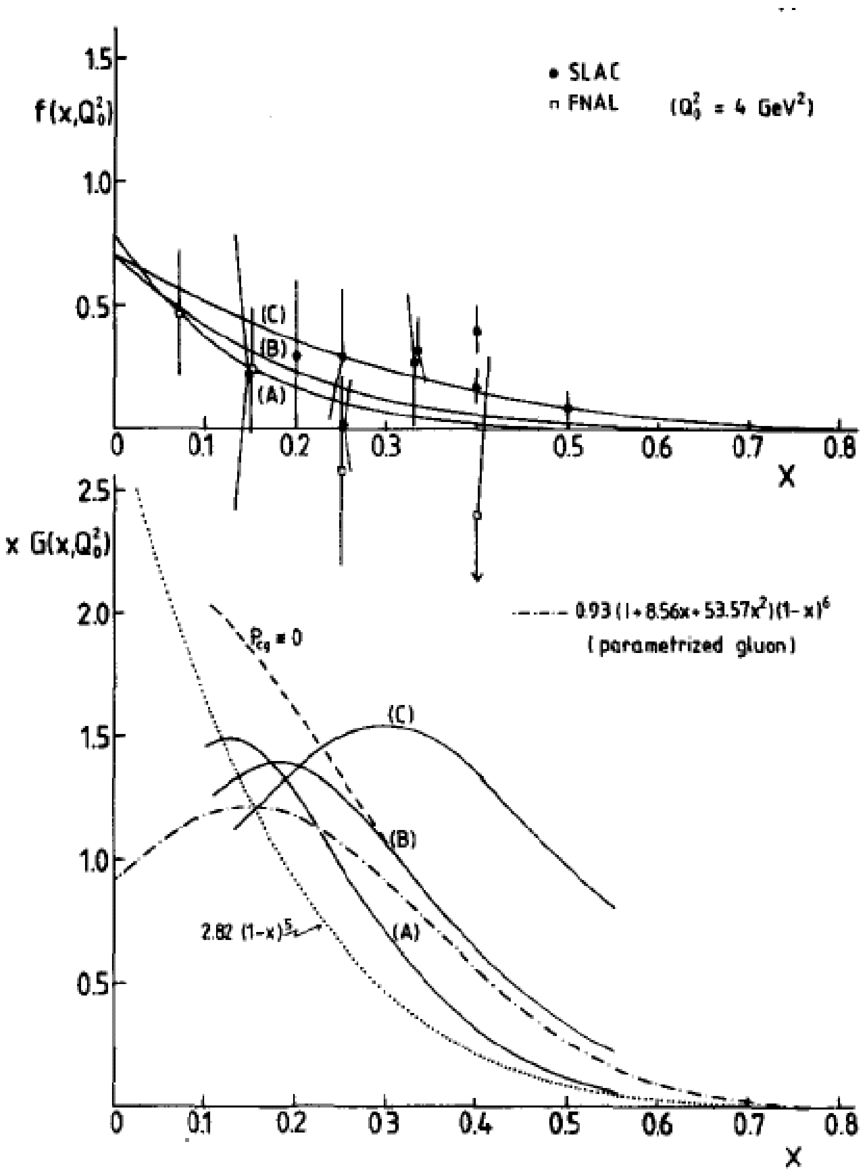
- Observation of charmonium in DIS [Aubert *et al.*, 1983.]

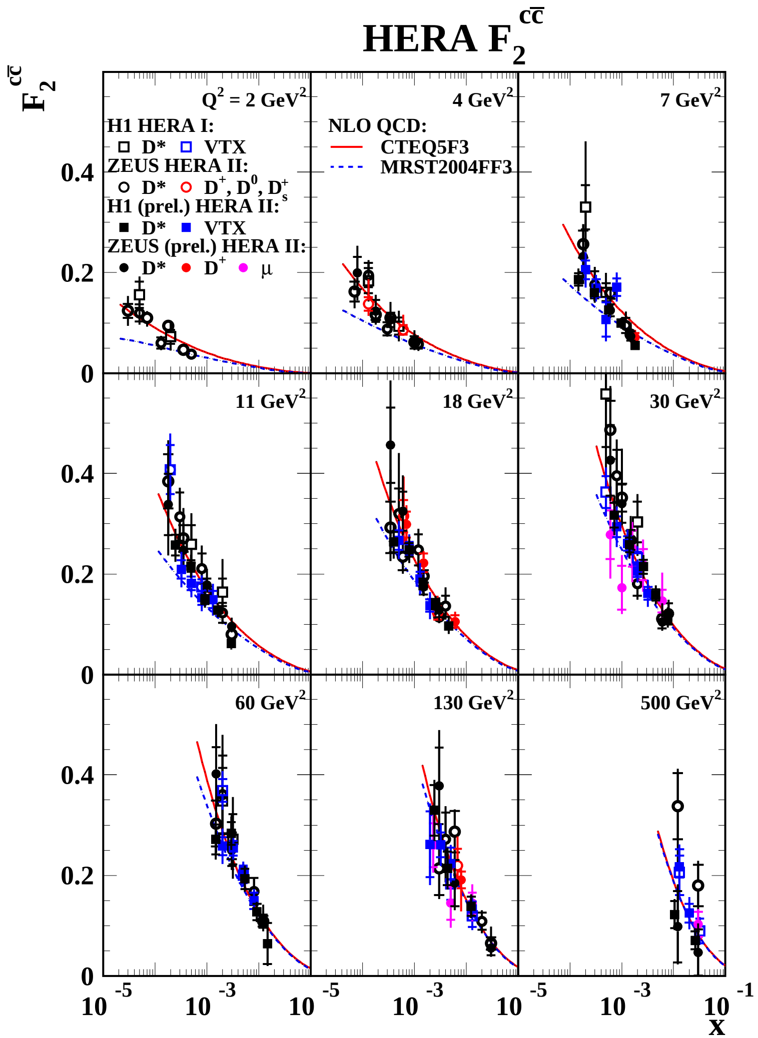
The Gluon Distribution

- Gluon carries roughly 50% of the proton momentum.
- Heavy quark production is an excellent way to extract the gluon density via a measurement of
  - scaling–violations of F_2 ,
  - $F_L^{Q\overline{Q}}$.
- First extraction of the gluon density including heavy quark effects by [Glück, Hoffmann, Reya, 1982.]:

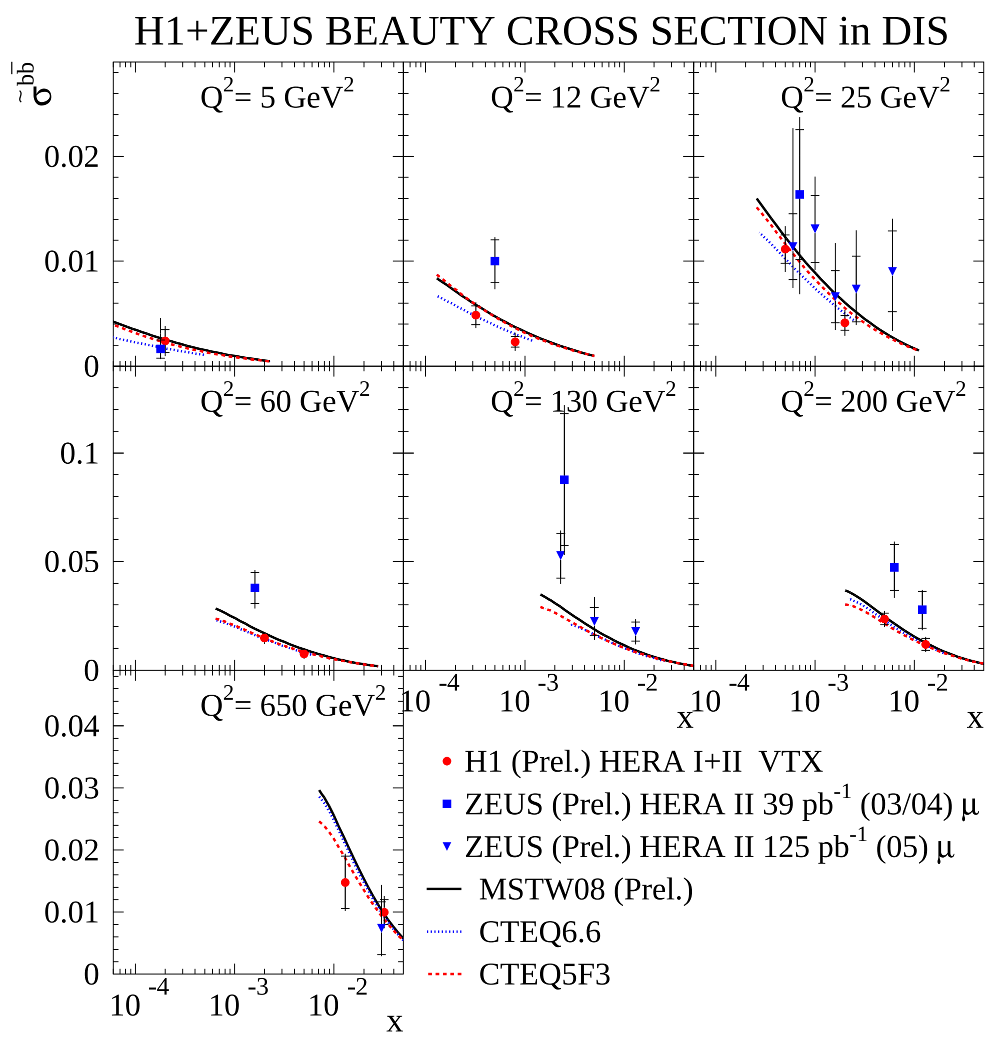
- Unfold the gluon density via

$$G(x,Q^2) = P_{qg}^{-1} \otimes \left[\frac{f(x,Q^2)}{x} - \frac{2}{3} P_{cg} \otimes G(x,Q^2) \right] .$$

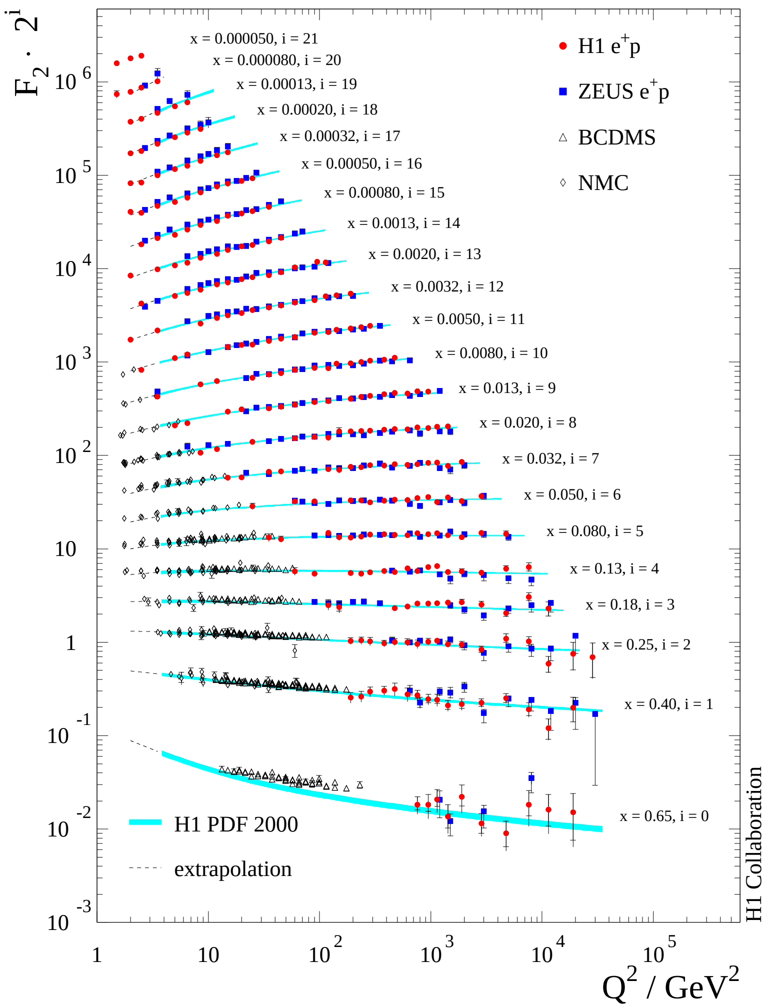


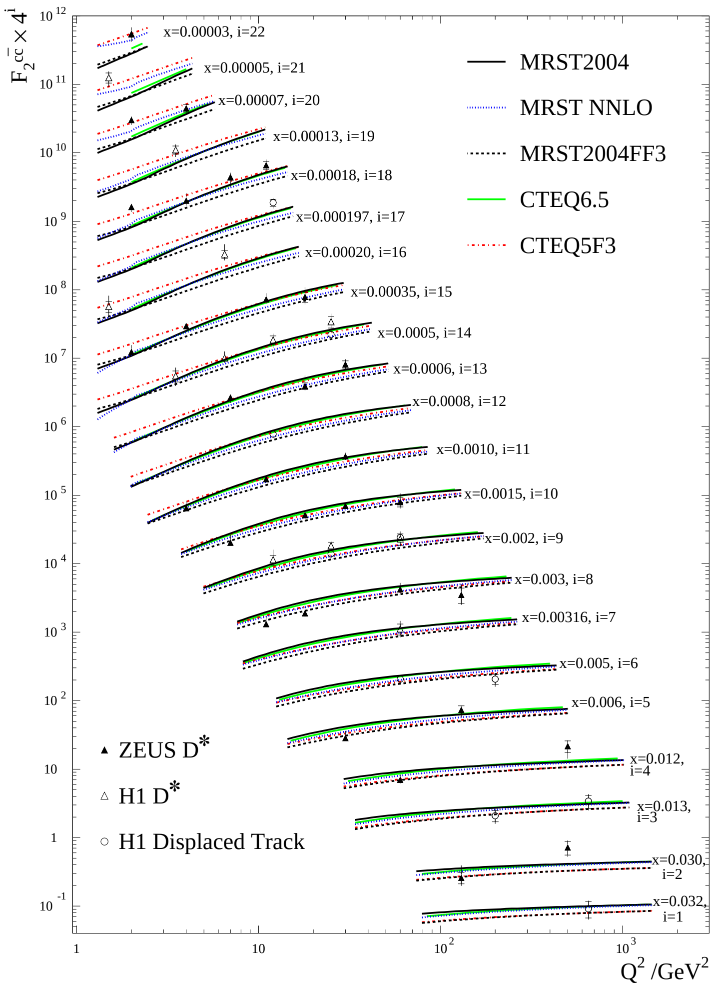


[Krüger (H1 and Z. Coll.), 2008.]



[Krüger (H1 and Z. Coll.), 2008.]





[Thompson, 2007.]

- High statistics for F_2 and $F_2^{c\bar{c}}$. Accuracy will increase in the future.
- $F_2^{c\bar{c}}(x,Q^2) \sim 20-40\,\%$ of $F_2(x,Q^2)$ for small values of x, but different scaling violations.

Splitting Functions

- The scaling violations are described by the splitting functions $P_{ij}(x, a_s)$.
- They describe the probability to find a parton i being radiated from parton j and carrying its momentum fraction x.
- They are related to the anomalous dimensions via a Mellin–Transform:

$$\mathbf{M}[f](N) := \int_0^1 dz z^{N-1} f(z) \ , \ \gamma_{ij}(N, a_s) := -\mathbf{M}[P_{ij}](N, a_s) \ .$$

- The splitting functions govern the scale–evolution of the parton densities.

$$\frac{d}{d \ln Q^2} \begin{pmatrix} \Sigma(N, Q^2) \\ G(N, Q^2) \end{pmatrix} = - \begin{pmatrix} \gamma_{qq} & \gamma_{qg} \\ \gamma_{gq} & \gamma_{gg} \end{pmatrix} \otimes \begin{pmatrix} \Sigma(N, Q^2) \\ G(N, Q^2) \end{pmatrix} \ ,$$

$$\frac{d}{d \ln Q^2} q_{NS}(N, Q^2) = - \gamma_{qq}^{NS} \otimes q_{NS} \ .$$

- The singlet light flavor density is defined by

$$\Sigma(n_f, \mu^2) = \sum_{i=1}^{n_f} (f_i(n_f, \mu^2) + \bar{f}_i(n_f, \mu^2)) \ .$$

- The anomalous dimensions are presently known at NNLO [Moch, Vermaseren, Vogt, 2004.])

Theory Status of Heavy Quark Corrections

Leading Order : $F_{2,L}(x,Q^2)$ [Witten, 1976; Babcock, Sivers, 1978; Shifman, Vainshtein, Zakharov, 1978; Leveille, Weiler, 1979; Glück, Reya, 1979; Glück, Hoffmann, Reya, 1982.]

Leading Order : $g_1(x,Q^2)$ [Watson, 1982; Glück, Reya, Vogelsang, 1991; Vogelsang, 1991]

Leading Order : $g_2(x,Q^2)$ [Blümlein, Ravindran, van Neerven, 2003]
Soft Resummation: $F_{2,L}(x,Q^2)$ [Laenen & Moch, 1998; Alekhin & Moch, 2008]

Next-to-Leading Order : $F_{2,L}(x,Q^2)$ [Laenen, Riemersma, Smith, van Neerven, 1993, 1995]
asymptotic: [Buza, Matiounine, Smith, Migneron, van Neerven, 1996; Bierenbaum, Blümlein, S.K., 2007]
Mellin–space expressions: [Alekhin, Blümlein, 2003.].
Next-to-Leading Order : $g_1(x,Q^2)$ asymptotic: [Buza, Matiounine, Smith, Migneron, van Neerven, 1997; Bierenbaum, Blümlein, S.K., 2009]
Next-to-Next-to-Leading Order : $F_L(x,Q^2)$ asymptotic:
[Blümlein, De Freitas, S.K., van Neerven, 2006.]
$O(\alpha_s^3)$: Light flavor Wilson coefficients: [Moch, Vermaseren, Vogt, 2005.]

$\Longrightarrow$ 3-loop heavy quark corrections needed to reach the same accuracy as for the light flavor contributions.

Need for the Calculation:

- Heavy flavor (charm) contributions to DIS structure functions are rather large [20–40 % at lower values of x] .
- Increase in accuracy of the perturbative description of DIS structure functions.

$\Longleftrightarrow$ QCD analysis and determination of $\Lambda_{\rm QCD}$, resp. $\alpha_s(M_Z^2)$, from DIS data: $\delta\alpha_s/\alpha_s < 1$ %.
(Recent NS N^{3}LO analysis: $\alpha_s(M_Z^2) = 0.1141^{+0.0020}_{-0.0022}$
$\Longrightarrow \delta\alpha_s/\alpha_s \approx 2\%$ [Blümlein, Böttcher, Guffanti, 2007].)

$\Longleftrightarrow$ Precise determination of the gluon and sea quark distributions.

$\Longleftrightarrow$ Derivation of variable flavor number scheme for heavy quark production to $O(a_s^3)$.

Goal:

- Calculation of the heavy flavor Wilson coefficients to higher orders for $Q^2 \geq 25\,\mathrm{GeV}^2$ [sufficient in many applications].
- First recalculation of the fermionic contributions to the NNLO anomalous dimensions.

2. The Method

- Massless RGE and light–cone expansion in Bjorken–limit $\{Q^2,\ \nu\} \rightarrow \infty,\ x$ fixed:

$$\lim_{\xi^2\to 0}\Big[J(\xi), J(0)\Big] \propto \sum_{i,N,\tau} c_{i,\tau}^N(\xi^2,\mu^2)\xi_{\mu_1}...\xi_{\mu_N} O_{i,\tau}^{\mu_1\cdots\mu_m}(0,\mu^2)\ .$$

- Mass factorization of the structure functions into Wilson coefficients and parton densities:

$$F_i(x,Q^2) = \sum_j C_{i,j}\left(x,\frac{Q^2}{\mu^2}\right) \otimes f_j(x,\mu^2)\ ; \qquad \text{Twist } \tau = 2$$

- Light-flavor Wilson coefficients: process dependent ($O(a_s^3)$: [Moch, Vermaseren, Vogt, 2005.])

$$C_{(2,L),i}^{\text{light}}\left(\frac{Q^2}{\mu^2}\right) = \delta_{i,q} + \sum_{l=1}^{\infty} a_s^l C_{(2,L),i}^{\text{light},(l)}\ , \qquad i = q, g$$

- Heavy quark contributions given by heavy quark Wilson coefficients

$$\mathsf{H}_{(2,L),i}^{\mathrm{S}}\left(\frac{Q^2}{\mu^2},\frac{m^2}{Q^2}\right) = \underbrace{H_{(2,L),i}^{\mathrm{S}}\left(\frac{Q^2}{\mu^2},\frac{m^2}{Q^2}\right)}_{\gamma + q_{heavy} \to X} + \underbrace{L_{(2,L),i}^{\mathrm{S,NS}}\left(\frac{Q^2}{\mu^2},\frac{m^2}{Q^2}\right)}_{\gamma + q_{light} \to X}$$

- Consider only one species of heavy quarks

- Factorization for $F_2^{Q\overline{Q}}(x, Q^2)$ at the level of twist $\tau = 2$:

$$
\begin{aligned}
F_2^{Q\overline{Q}}(n_f, x, Q^2, m^2) = \sum_{k=1}^{n_f} e_k^2 \Biggl\{ & \quad L_{2,q}^{\mathsf{NS}}\left(n_f, x, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \otimes \left[f_k(n_f, x, \mu^2) + f_{\overline{k}}(n_f, x, \mu^2)\right] \\
& + \tilde{L}_{2,q}^{\mathsf{PS}}\left(n_f, x, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \otimes \Sigma(n_f, x, \mu^2) \\
& + \tilde{L}_{2,g}^{\mathsf{S}}\left(n_f, x, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \otimes G(n_f, x, \mu^2) \Biggr\} \\
+ e_Q^2 \Biggl\{ & \quad H_{2,q}^{\mathsf{PS}}\left(n_f, x, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \otimes \Sigma(n_f, x, \mu^2) \\
& + H_{2,g}^{\mathsf{S}}\left(n_f, x, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \otimes G(n_f, x, \mu^2) \Biggr\} .
\end{aligned}
$$

- In the limit $Q^2 \gg m_h^2$ [$Q^2 \approx 10\ m^2$ for F_2, g_1]: massive RGE, derivative $m^2\partial/\partial m^2$ acts on Wilson coefficients only: all terms but power corrections calculable through partonic operator matrix elements, $\langle i|A_l|j\rangle$, which are process independent objects!

$$H^{\rm S}_{(2,L),i}\left(\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right) = \underbrace{A^{\rm S}_{ki}\left(\frac{m^2}{\mu^2}\right)}_{\text{massive OMEs}} \otimes \underbrace{C^{\rm S}_{(2,L),k}\left(\frac{Q^2}{\mu^2}\right)}_{\text{light–parton–Wilson coefficients}}.$$

- Similar formula for $L^{S,NS}_{(2,L),i}$. Holds for polarized and unpolarized case.
- OMEs obey expansion

$$A^{\rm S,NS}_{ki}\left(\frac{m^2}{\mu^2}\right) = \langle i|O^{\rm S,NS}_k|i\rangle = \delta_{ki} + \sum_{l=1}^{\infty} a_s^l A^{\rm S,NS,(l)}_{ki}\left(\frac{m^2}{\mu^2}\right), \qquad i = q, g$$

[Buza, Matiounine, Migneron, Smith, van Neerven, 1996; Buza, Matiounine, Smith, van Neerven, 1997.]

- Heavy OMEs also occur as transition functions to define a variable flavor number scheme starting from a fixed flavor number scheme.

[Aivazis, Collins, Olness, Tung, 1994; Buza, Matiounine, Smith, van Neerven, 1998; Chuvakin, Smith, van Neerven, 1998.]

- Expansion up to $O(a_s^3)$ for $F_2^{Q\overline{Q}}(x,Q^2)$ reads

$$
\begin{aligned}
L_{2,q}^{\mathsf{NS}}(n_f) &= a_s^2\left[A_{qq,Q}^{\mathsf{NS},(2)}(n_f)+\hat{C}_{2,q}^{\mathsf{NS},(2)}(n_f)\right]+a_s^3\left[A_{qq,Q}^{\mathsf{NS},(3)}(n_f)+A_{qq,Q}^{\mathsf{NS},(2)}(n_f)C_{2,q}^{\mathsf{NS},(1)}(n_f)+\hat{C}_{2,q}^{\mathsf{NS},(3)}(n_f)\right]\\
\tilde{L}_{2,q}^{\mathsf{PS}}(n_f) &= a_s^3\left[\ \tilde{A}_{qq,Q}^{\mathsf{PS},(3)}(n_f)+A_{gq,Q}^{(2)}(n_f)\ \tilde{C}_{2,g}^{(1)}(n_f+1)+\hat{\tilde{C}}_{2,q}^{\mathsf{PS},(3)}(n_f)\right]\\
\tilde{L}_{2,g}^{\mathsf{S}}(n_f) &= a_s^2\,A_{gg,Q}^{(1)}(n_f)\tilde{C}_{2,g}^{(1)}(n_f+1)+a_s^3\Big[\ \tilde{A}_{qg,Q}^{(3)}(n_f)+A_{gg,Q}^{(1)}(n_f)\ \tilde{C}_{2,g}^{(2)}(n_f+1)\\
&\qquad +A_{gg,Q}^{(2)}(n_f)\ \tilde{C}_{2,g}^{(1)}(n_f+1)+A_{Qg}^{(1)}(n_f)\ \tilde{C}_{2,q}^{\mathsf{PS},(2)}(n_f+1)+\hat{\tilde{C}}_{2,g}^{(3)}(n_f)\Big]\\
H_{2,q}^{\mathsf{PS}}(n_f) &= a_s^2\left[\ A_{Qq}^{\mathsf{PS},(2)}+\ \tilde{C}_{2,q}^{\mathsf{PS},(2)}(n_f+1)\right]+a_s^3\Big[\ A_{Qq}^{\mathsf{PS},(3)}+\ \tilde{C}_{2,q}^{\mathsf{PS},(3)}(n_f+1)+A_{gq,Q}^{(2)}\ \tilde{C}_{2,g}^{(1)}(n_f+1)+A_{Qq}^{\mathsf{PS},(2)}\ C_{2,q}^{\mathsf{NS},(1)}(n_f+1)\Big]\\
H_{2,g}^{\mathsf{S}}(n_f) &= a_s\left[\ A_{Qg}^{(1)}+\ \tilde{C}_{2,g}^{(1)}(n_f+1)\right]+a_s^2\Big[\ A_{Qg}^{(2)}+\ A_{Qg}^{(1)}\ C_{2,q}^{\mathsf{NS},(1)}(n_f+1)+\ A_{gg,Q}^{(1)}\ \tilde{C}_{2,g}^{(1)}(n_f+1)\ +\ \tilde{C}_{2,g}^{(2)}(n_f+1)\Big]\\
&+\ a_s^3\Big[\ A_{Qg}^{(3)}+\ A_{Qg}^{(2)}\ C_{2,q}^{\mathsf{NS},(1)}(n_f+1)+\ A_{gg,Q}^{(2)}\ \tilde{C}_{2,g}^{(1)}(n_f+1)\ +\ A_{gg,Q}^{(1)}\ \tilde{C}_{2,g}^{(2)}(n_f+1)\\
&\qquad +A_{Qg}^{(1)}\left[C_{2,q}^{\mathsf{NS},(2)}(n_f+1)+\ \tilde{C}_{2,q}^{\mathsf{PS},(2)}(n_f+1)\right]+\ \tilde{C}_{2,g}^{(3)}(n_f+1)\Big]\ .
\end{aligned}
$$

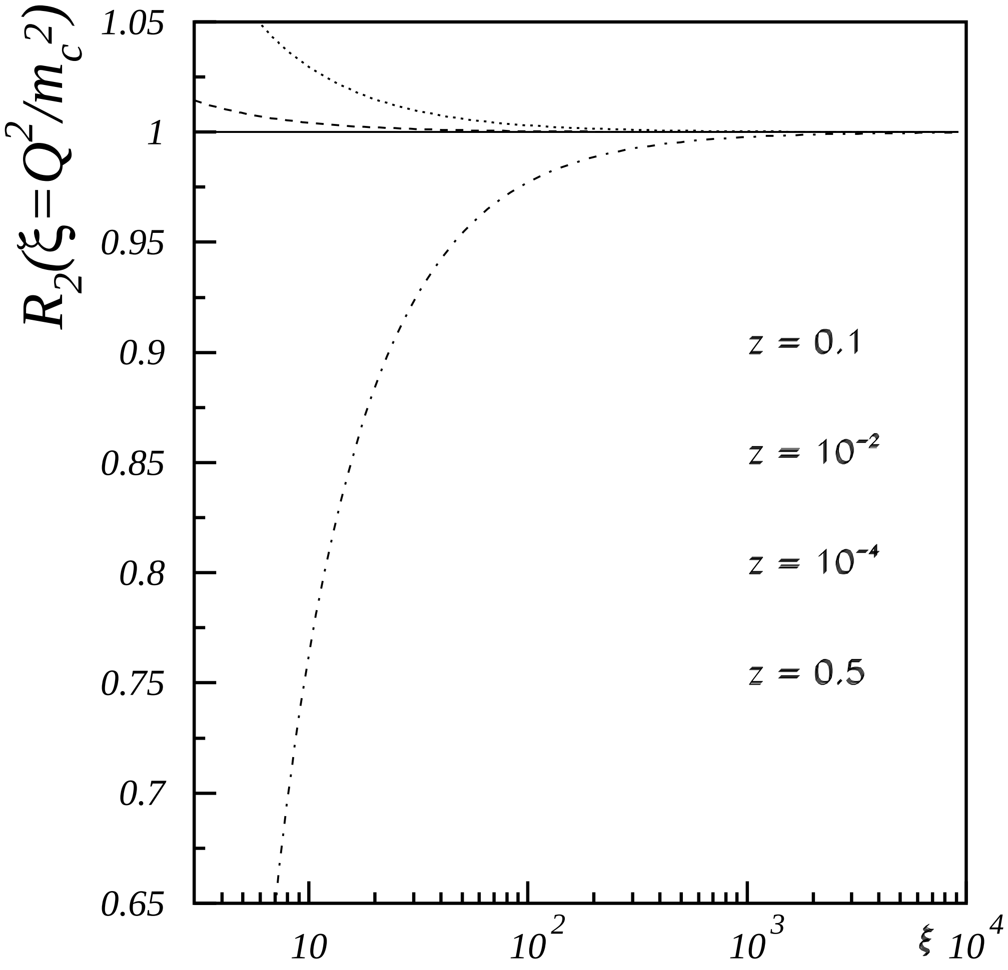
- n_f–dependence non–trivial: $\hat{f}(n_f)\equiv f(n_f+1)-f(n_f)$, $\tilde{f}(n_f)\equiv f(n_f)/n_f$.
- Highlighted terms are (partially) due to heavy quark insertions on external legs and have to be included in the $\overline{\text{MS}}$–scheme $\Longrightarrow$ not considered in previous NLO analyses.
- At NLO, these differences correspond to
  - fully inclusive DIS ($\overline{\text{MS}}$–scheme) as in [Buza, Matiounine, Smith, van Neerven, 1998]
  - DIS with heavy quarks in the final state only [Laenen, Riemersma, Smith, van Neerven, 1993].

- Comparison for LO:

$$R_2\left(\xi \equiv \frac{Q^2}{m^2}\right) \quad \equiv \quad \frac{H_{2,g}^{(1)}}{H_{2,g,(asym)}^{(1)}} \, .$$

- Comparison to exact order $O(a_s^2)$ result: asymptotic formulas valid for $Q^2 \geq 20$ $(\,\mathrm{GeV}/c)^2$ in case of $F_2^{c\bar{c}}(x,Q^2)$ and $Q^2 \geq 1000\ (\,\mathrm{GeV}/c)^2$ for $F_L^{c\bar{c}}(x,Q^2)$

- Drawbacks:
  - Power corrections $(m^2/Q^2)^k$ can not be calculated using this method.
  - Two heavy quark masses are still too complicated $\Longrightarrow$ 2 scale problem to be treated analytically.
  - Only inclusive quantities can be calculated $\Longrightarrow$ structure functions.



FFNS:

- Fixed order perturbation theory and Fixed number of light partons in the proton.
- The heavy quarks are produced extrinsically only.
- The large logarithmic terms in the heavy quark coefficient functions entirely determine the charm component of the structure function for large values of Q^2.

VFNS:

- Define threshold above which the heavy quark is treated as light, thereby obtaining a parton density.
- Remove the mass singular terms from the asymptotic heavy quark coefficient functions and absorb them into parton densities.
- Heavy Flavor initial state parton densities for the LHC. E.g. for $c\,\bar{s} \rightarrow W^+$.

The VFNS is derived from the FFNS directly. New parton density appears corresponding to the heavy quark, which is now treated as light (massless). $\Longrightarrow$ Relations between parton densities for n_f and $n_f + 1$ flavors.

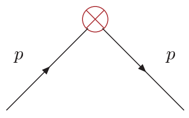
$$
\begin{aligned}
f_{Q+\bar{Q}}(n_f+1,\mu^2) &= A_{Qq}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right)\otimes\Sigma(n_f,\mu^2)+A_{Qg}\left(n_f,\frac{\mu^2}{m^2}\right)\otimes G(n_f,\mu^2)\ . \\
G(n_f+1,\mu^2) &= A_{gq,Q}\left(n_f,\frac{\mu^2}{m^2}\right)\otimes\Sigma(n_f,\mu^2)+A_{gg,Q}\left(n_f,\frac{\mu^2}{m^2}\right)\otimes G(n_f,\mu^2)\ .
\end{aligned}
$$

Only possible in regions of phase space where the condition for the validity of the parton model $\boldsymbol{\tau_{\rm int}/\tau_{\rm life} \ll 1}$ is strictly observed.

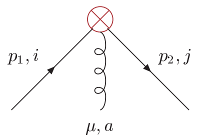
Operator Insertions in Light–Cone Expansion

E.g. singlet heavy quark operator:

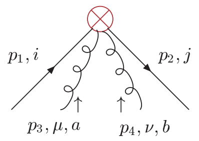
$$O_Q^{\mu_1,\dots,\mu_N}(z) = \frac{1}{2} i^{N-1} S[\overline{q}(z)\gamma^{\mu_1} D^{\mu_2} \dots D^{\mu_N} q(z)] - \text{Trace Terms}\ .$$



$$\not\Delta \gamma_{\pm} (\Delta \cdot p)^{N-1}\ ,$$



$$g t^a_{ji} \Delta^\mu \not\Delta \gamma_{\pm} \sum_{j=0}^{N-2} (\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-j-2}\ ,$$



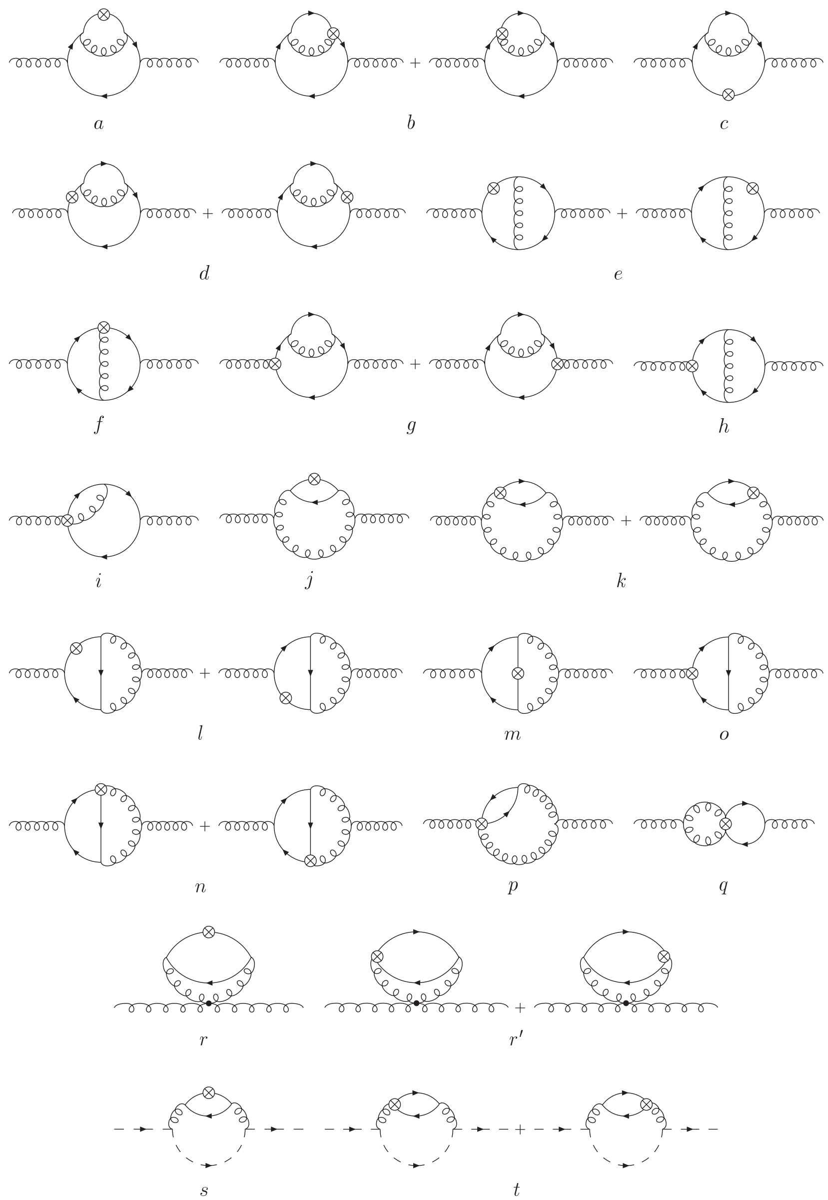
$$g^2 \Delta^\mu \Delta^\nu \not\Delta \gamma_{\pm} \sum_{0 \le j < l}^{N-2} \Big[(\Delta p_1)^{N-l-2} (\Delta p_1 + \Delta p_4)^{l-j-1} (\Delta p_2)^j (t^a t^b)_{ji} + (\Delta p_1)^{N-l-2} (\Delta p_1 + \Delta p_3)^{l-j-1} (\Delta p_2)^j (t^b t^a)_{ji} \Big]\ ,$$

$$\gamma_+ = 1\ , \quad \gamma_- = \gamma_5\ .$$

Δ: light–like momentum, $\Delta^2 = 0$.

$\Longrightarrow$ Additional vertices with 2 and more gluons at higher orders.

- Diagrams contain two scales: the mass m and the Mellin–parameter N.
- 2–point functions with on–shell external momentum, $p^2 = 0$.
  $\longrightarrow$ reduce to massive tadpoles for $N = 0$.
- Graphs shown here contribute to $\hat{A}_{Qg}^{(2)}$.



Renormalization

- Mass renormalization (on–mass shell scheme)
- Charge renormalization: MOM scheme for the gluon propagator.
  MOM scheme → $\overline{\mathrm{MS}}$ scheme:

$$a_s^{\mathrm{MOM}} = a_s^{\overline{\mathrm{MS}}} - \beta_{0,Q} \ln\left(\frac{m^2}{\mu^2}\right) a_s^{\overline{\mathrm{MS}}^2} + \left[\beta_{0,Q}^2 \ln^2\left(\frac{m^2}{\mu^2}\right) - \beta_{1,Q} \ln\left(\frac{m^2}{\mu^2}\right) - \beta_{1,Q}^{(1)}\right] a_s^{\overline{\mathrm{MS}}^3} .$$

  $\Longrightarrow$ Accounts at NLO for difference due to heavy quark insertions on external legs.

- Renormalization of ultraviolet singularities
  $\Longrightarrow$ are absorbed into Z–factors given in terms of anomalous dimensions γ_{ij}.
- Factorization of collinear singularities into Γ–factors Γ_{NS}, $\Gamma_{ij,S}$ and $\Gamma_{qq,PS}$.

Generic formula for operator renormalization and mass factorization:

$$A_{ij} = Z_{il}^{-1} \hat{A}_{lm} \Gamma_{mj}^{-1}$$

$\Longrightarrow$ $O(\varepsilon)$–terms of the 2–loop OMEs are needed for renormalization at 3–loops.

3. 2–Loop Results

- Single scale problem, depending only on one variable, z.
  $\Longrightarrow$ Calculation in Mellin-space for space–like q^2, $Q^2 = -q^2$: $\quad 0 \leq z \leq 1$

$$\mathbf{M}[f](N) := \int_0^1 dz z^{N-1} f(z) \ .$$

- Analytic results for general value of Mellin N are obtained in terms of harmonic sums [Blümlein, Kurth, 1999; Vermaseren, 1999.]

$$S_{a_1,\ldots,a_m}(N) = \sum_{n_1=1}^{N} \sum_{n_2=1}^{n_1} \cdots \sum_{n_m=1}^{n_{m-1}} \frac{(\mathrm{sign}(a_1))^{n_1}}{n_1^{|a_1|}} \frac{(\mathrm{sign}(a_2))^{n_2}}{n_2^{|a_2|}} \cdots \frac{(\mathrm{sign}(a_m))^{n_m}}{n_m^{|a_m|}} \ ,$$
$$N \in \mathbb{N}, \ \forall \ l, \ a_l \in \mathbb{Z} \setminus 0 \ ,$$
$$S_{-2,1}(N) = \sum_{i=1}^{N} \frac{(-1)^i}{i^2} \sum_{j=1}^{i} \frac{1}{j} \ .$$

- Algebraic and structural simplification of the harmonic sums [J. Blümlein, 2003, 2007.].

- Analytic continuation to complex N via analytic relations or integral representations, e.g.

$$\mathbf{M}\left[\frac{\mathrm{Li}_2(x)}{1+x}\right](N+1) - \zeta_2 \beta(N+1) = (-1)^{N+1}\left[S_{-2,1}(N) + \frac{5}{8}\zeta_3\right] \ .$$

- Use of generalized hypergeometric functions for general analytic results

$$
{}_3F_2\left[\begin{matrix} a_0, a_1, a_2 \\ b_1, b_2 \end{matrix} ; z\right] = \sum_{i=0}^{\infty} \frac{(a_0)_i (a_1)_i (a_2)_i}{(b_1)_i (b_2)_i} \frac{z^i}{\Gamma(i+1)} .
$$

$$
= \frac{1}{B(a_1, b_1) B(a_2, b_2)} \int_0^1 dx_1 \int_0^1 dx_2 \frac{x_1^{a_1-1}(1-x_1)^{b_1-a_1-1} x_2^{a_2-1}(1-x_2)^{b_2-a_2-1}}{(1-zx_1x_2)^{a_0}}
$$

- Use of Mellin-Barnes integrals for numerical checks for fixed values of N (MB [Czakon, 2006.])

- Summation of a lot of new infinite one-parameter sums into harmonic sums. E.g.:

$$
N \sum_{i,j=1}^{\infty} \frac{S_1(i) S_1(i+j+N)}{i(i+j)(j+N)} = 4S_{2,1,1} - 2S_{3,1} + S_1\Big(-3S_{2,1} + \frac{4S_3}{3}\Big) - \frac{S_4}{2}
$$

$$
- S_2^2 + S_1^2 S_2 + \frac{S_1^4}{6} + 6S_1\zeta_3 + \zeta_2\Big(2S_1^2 + S_2\Big) .
$$

  Use of integral techniques and the Mathematica package SIGMA [Schneider, 2007.], [Bierenbaum, Blümlein, S. K., Schneider, 2007, 2008.]

- Partial checks for fixed values of N using SUMMER, [Vermaseren, 1999.]

We calculated all 2–loop $O(\varepsilon)$–terms in the unpolarized case

and several 2–loop $O(\varepsilon)$–terms in the polarized case:

$$\overline{a}_{Qg}^{(2)}, \quad \overline{a}_{Qq}^{(2),\mathbf{PS}}, \quad \overline{a}_{gg,Q}^{(2)}, \quad \overline{a}_{gq,Q}^{(2)}, \quad \overline{a}_{qq,Q}^{(2),\mathbf{NS}}.$$

$$\Delta\overline{a}_{Qg}^{(2)}, \quad \Delta\overline{a}_{Qq}^{(2),\mathbf{PS}}, \quad \Delta\overline{a}_{qq,Q}^{(2),\mathbf{NS}}.$$

We verified all corresponding 2–loop $O(\varepsilon^0)$–results by van Neerven et. al.

- A remark on the appearing functions:

  van Neerven et al. to O(1): unpolarized: 48 basic functions; polarized: 24 basic functions.

  O(1): $\{S_1, \ S_2, \ S_3, \ S_{-2}, S_{-3}\}$, $\quad S_{-2,1} \Longrightarrow$ 2 basic objects.

  O(ε): $\{S_1, \ S_2, \ S_3, \ S_4, \ S_{-2}, S_{-3}, S_{-4}\}$, $\quad S_{2,1}, \quad S_{-2,1}, \quad S_{-3,1}, \quad S_{2,1,1}, \quad S_{-2,1,1}$
  $\Longrightarrow$ 6 basic objects

- harmonic sums with index $\{-1\}$ cancel (holds even for each diagram)

  [Blümlein, 2004; Blümlein, Ravindran, 2005,2006; Blümlein, S. K., 2007; Blümlein, Moch in preparation.]

- Expectation for 3–loops: weight 5 (6) harmonic sums

Example: Unpolarized case, Singlet, O(ε)

$$
\begin{aligned}
\overline{a}_{Qg}^{(2)} = T_F C_F \Biggl\{ & \frac{2}{3}\frac{(N^2+N+2)(3N^2+3N+2)}{N^2(N+1)^2(N+2)}\zeta_3 + \frac{P_1}{N^3(N+1)^3(N+2)}S_2 + \frac{N^4-5N^3-32N^2-18N-4}{N^2(N+1)^2(N+2)}S_1^2 \\
& + \frac{N^2+N+2}{N(N+1)(N+2)}\Bigl(16S_{2,1,1} - 8S_{3,1} - 8S_{2,1}S_1 + 3S_4 - \frac{4}{3}S_3S_1 - \frac{1}{2}S_2^2 - S_2S_1^2 - \frac{1}{6}S_1^4 + 2\zeta_2S_2 - 2\zeta_2S_1^2 - \frac{8}{3}\zeta_3S_1\Bigr) \\
& - 8\frac{N^2-3N-2}{N^2(N+1)(N+2)}S_{2,1} + \frac{2}{3}\frac{3N+2}{N^2(N+2)}S_1^3 + \frac{2}{3}\frac{3N^4+48N^3+43N^2-22N-8}{N^2(N+1)^2(N+2)}S_3 + 2\frac{3N+2}{N^2(N+2)}S_2S_1 + 4\frac{S_1}{N^2}\zeta_2 \\
& + \frac{N^5+N^4-8N^3-5N^2-3N-2}{N^3(N+1)^3}\zeta_2 - 2\frac{2N^5-2N^4-11N^3-19N^2-44N-12}{N^2(N+1)^3(N+2)}S_1 + \frac{P_2}{N^5(N+1)^5(N+2)}\Biggr\} \\
+ T_F C_A \Biggl\{ & \frac{N^2+N+2}{N(N+1)(N+2)}\Bigl(16S_{-2,1,1} - 4S_{2,1,1} - 8S_{-3,1} - 8S_{-2,2} - 4S_{3,1} - \frac{2}{3}\beta''' + 9S_4 - 16S_{-2,1}S_1 \\
& + \frac{40}{3}S_1S_3 + 4\beta''S_1 - 8\beta'S_2 + \frac{1}{2}S_2^2 - 8\beta'S_1^2 + 5S_1^2S_2 + \frac{1}{6}S_1^4 - \frac{10}{3}S_1\zeta_3 - 2S_2\zeta_2 - 2S_1^2\zeta_2 - 4\beta'\zeta_2 - \frac{17}{5}\zeta_2^2\Bigr) \\
& + \frac{4(N^2-N-4)}{(N+1)^2(N+2)^2}\Bigl(-4S_{-2,1} + \beta'' - 4\beta'S_1\Bigr) - \frac{2}{3}\frac{N^3+8N^2+11N+2}{N(N+1)^2(N+2)^2}S_1^3 + 8\frac{N^4+2N^3+7N^2+22N+20}{(N+1)^3(N+2)^3}\beta' \\
& + 2\frac{3N^3-12N^2-27N-2}{N(N+1)^2(N+2)^2}S_2S_1 - \frac{16}{3}\frac{N^5+10N^4+9N^3+3N^2+7N+6}{(N-1)N^2(N+1)^2(N+2)^2}S_3 - 8\frac{N^2+N-1}{(N+1)^2(N+2)^2}\zeta_2S_1 \\
& - \frac{2}{3}\frac{9N^5-10N^4-11N^3+68N^2+24N+16}{(N-1)N^2(N+1)^2(N+2)^2}\zeta_3 - \frac{P_3}{(N-1)N^3(N+1)^3(N+2)^3}S_2 - \frac{2P_4}{(N-1)N^3(N+1)^3(N+2)^2}\zeta_2 \\
& - \frac{P_5}{N(N+1)^3(N+2)^3}S_1^2 + \frac{2P_6}{N(N+1)^4(N+2)^4}S_1 - \frac{2P_7}{(N-1)N^5(N+1)^5(N+2)^5}\Biggr\}\,.
\end{aligned}
$$

4. Fixed Moments at 3–Loops

Contributing OMEs:

Singlet	A_{Qg}	A_{Qg}	$A_{gg,Q}$	$A_{gq,Q}$	} mixing
Pure–Singlet		$A_{Qq}^{\rm PS}$	$A_{qq,Q}^{\rm PS}$		
Non–Singlet	$A_{qq,Q}^{\rm NS,+}$	$A_{qq,Q}^{\rm NS,-}$	$A_{qq,Q}^{\rm NS,v}$		

- Unpolarized anomalous dimensions are known up to $O(a_s^3)$ [Moch, Vermaseren, Vogt, 2004.]
  $\Longrightarrow$ All terms needed for the renormalization of unpolarized 3–loop heavy OMEs are present.
  $\Longrightarrow$ The calculation provides first independent checks on $\gamma_{qg}^{(2)}$, $\gamma_{qq}^{(2),\rm PS}$ and on respective color projections of $\gamma_{qq}^{(2),\rm NS\pm}$, $\gamma_{gg}^{(2)}$ and $\gamma_{gq}^{(2)}$.
- The calculation proceeds in the same way in the polarized case.
- Calculation in Mellin–space:
  For fixed N: three–loop "self-energy" type diagrams with an operator insertion
  $\Longrightarrow$ Calculation using MATAD [Steinhauser, 2001] and FORM [Vermaseren, 2000].

Fixed Moments using MATAD

- three–loop "self-energy" type diagrams with an operator insertion
- Extension: additional scale compared to massive propagators: Mellin variable N
- Genuine tensor integrals due to

$$\Delta^{\mu_1}...\Delta^{\mu_n} \langle p|O_{\mu_1...\mu_n}|p\rangle = \Delta^{\mu_1}...\Delta^{\mu_n} \langle p|S\,\bar{\Psi}\gamma_{\mu_1}D_{\mu_2}...D_{\mu_n}\Psi|p\rangle = A(N)\cdot(\Delta p)^N$$

$$D_\mu = \partial_\mu - igt_a A_\mu^a \quad , \qquad \Delta^2 = 0.$$

- Construction of a projector to obtain the desired moment in N [undo Δ-contraction]
- 3–loop OMEs are generated with QGRAF [Nogueira, 1993.]
- Color factors are calculated with [van Ritbergen, Schellekens, Vermaseren, 1998.]
- Translation to suitable input for MATAD [Steinhauser, 2001.]

Tests performed:

- Various 2–loop calculations for $N = 2, 4, 6, ...$ were repeated
  $\longrightarrow$ agreement with our previous calculation.
- Several non–trivial scalar 3–loop diagrams were calculated using Feynman–parameters for all N
  $\longrightarrow$ agreement with MATAD.

General Structure of the Result: the PS –case

$$A_{Qq}^{(3),\mathsf{PS},\overline{\mathrm{MS}}} = \frac{\hat{\gamma}_{qg}^{(0)}\gamma_{gq}^{(0)}}{48}\Bigg\{\gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 6\beta_0 + 16\beta_{0,Q}\Bigg\}\ln^3\left(\frac{m^2}{\mu^2}\right)$$

$$+\frac{1}{8}\Bigg\{-4\hat{\gamma}_{qq}^{(1),\mathsf{PS}}\Big(\beta_0 + \beta_{0,Q}\Big) + \hat{\gamma}_{qg}^{(0)}\Big(\hat{\gamma}_{gq}^{(1)} - \gamma_{gq}^{(1)}\Big) - \gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(1)}\Bigg\}\ln^2\left(\frac{m^2}{\mu^2}\right)$$

$$+\frac{1}{16}\Bigg\{8\,\hat{\gamma}_{qq}^{(2),\mathsf{PS}} - 8n_f\hat{\tilde{\gamma}}_{qq}^{(2),\mathsf{PS}} - 32a_{Qq}^{(2),\mathsf{PS}}(\beta_0 + \beta_{0,Q}) + 8\hat{\gamma}_{qg}^{(0)}a_{gq,Q}^{(2)} - 8\gamma_{gq}^{(0)}a_{Qg}^{(2)}$$

$$-\zeta_2\hat{\gamma}_{qg}^{(0)}\gamma_{gq}^{(0)}\Big(\gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 6\beta_0 + 8\beta_{0,Q}\Big)\Bigg\}\ln\left(\frac{m^2}{\mu^2}\right)$$

$$+4(\beta_0 + \beta_{0,Q})\overline{a}_{Qq}^{(2),\mathsf{PS}} + \gamma_{gq}^{(0)}\overline{a}_{Qg}^{(2)} - \hat{\gamma}_{qg}^{(0)}\overline{a}_{gq,Q}^{(2)} + \zeta_3\frac{\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)}}{48}\Big(\gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 6\beta_0\Big)$$

$$+\frac{\hat{\gamma}_{qg}^{(0)}\gamma_{gq}^{(1)}\zeta_2}{16} + C_F\Big(-(4 + \frac{3}{4}\zeta_2)\hat{\gamma}_{qg}^{(0)}\gamma_{gq}^{(0)} - 4\hat{\gamma}_{qq}^{(1),\mathsf{PS}} + 12a_{Qq}^{(2),\mathsf{PS}}\Big) + a_{Qq}^{(3),\mathsf{PS}}\,.$$

All terms but $a_{Qq}^{(3),\mathsf{PS}}$ known <u>for all N</u>.

- There are similar formulas for the other OMEs.

5. Results

- Using MATAD, we calculated the OMEs ($\approx$ 250 days of computer time/ 2700 diagrams)

$$A_{Qq}^{(3),\mathrm{PS}}: \quad (2,4,..,12); \quad A_{qq,Q}^{(3),\mathrm{PS}}, A_{gq,Q}^{(3)}: \quad (2,4,..,14);$$
$$A_{qq,Q}^{(3),\mathrm{NS}\pm}: \quad (2,3,..,14); \quad A_{Q(q)g}^{(3)}, A_{gg,Q}^{(3)}: \quad (2,4,..,10);$$

  and find agreement with the predictions obtained from renormalization.

- Additional checks are provided by sums rules for $N = 2$, which are fullfilled by our result.

- All terms proportional to ζ_2 cancel in the renormalized result in the $\overline{\mathrm{MS}}$–scheme.

- We observe the number

$$\mathrm{B4} = -4\zeta_2 \ln^2 2 + \frac{2}{3}\ln^4 2 - \frac{13}{2}\zeta_4 + 16\mathrm{Li}_4\left(\frac{1}{2}\right) = -8\sigma_{-3,-1} + \frac{11}{2}\zeta_4$$

  which does not appear in massless calculations and is due to genuine massive effects.

Example: non–logarithmic term of $A_{Qg}^{(3)}$ for $N = 2$

$$
\begin{aligned}
A_{Qg}^{(3),\overline{\rm MS}}(\mu^2 = m^2, N = 2) = T_F {C_A}^2 \Biggl(& \frac{174055}{4374} - \frac{88}{9}{\sf B}_4 + 72\zeta_4 - \frac{29431}{324}\zeta_3 \Biggr) \\
& + T_F C_F C_A \Biggl(-\frac{18002}{729} + \frac{208}{9}{\sf B}_4 - 104\zeta_4 + \frac{2186}{9}\zeta_3 - \frac{64}{3}\zeta_2 + 64\zeta_2 \ln(2) \Biggr) \\
& + T_F {C_F}^2 \Biggl(-\frac{8879}{729} - \frac{64}{9}{\sf B}_4 + 32\zeta_4 - \frac{701}{81}\zeta_3 + 80\zeta_2 - 128\zeta_2 \ln(2) \Biggr) + {T_F}^2 C_A \Biggl(-\frac{21586}{2187} + \frac{3605}{162}\zeta_3 \Biggr) \\
& + {T_F}^2 C_F \Biggl(-\frac{55672}{729} + \frac{889}{81}\zeta_3 + \frac{128}{3}\zeta_2 \Biggr) + n_f {T_F}^2 C_A \Biggl(-\frac{7054}{2187} - \frac{704}{81}\zeta_3 \Biggr) + n_f {T_F}^2 C_F \Biggl(-\frac{22526}{729} + \frac{1024}{81}\zeta_3 - \frac{64}{3}\zeta_2 \Biggr) .
\end{aligned}
$$

The constant terms: N = 10 $a_{Qg}^{(3)} + a_{qg,Q}^{(3)}$:

$$
\begin{aligned}
a_{Qg}^{(3)}\Big|_{N=10} = \; & T_F \left(\frac{m^2}{\mu^2}\right)^{3\varepsilon/2} \Biggl\{ n_f T_F \Biggl(C_A \left[-\frac{1505896}{245025}\zeta_3 + \frac{189965849}{188669250}\zeta_2 + \frac{297277185134077151}{15532837481700000} \right] \\
& + C_F \left[\frac{62292104}{13476375}\zeta_3 - \frac{49652772817}{93391278750}\zeta_2 - \frac{1178560772273339822317}{1076425637481810000000} \right] \Biggr) + C_A^2 \left[-\frac{563692}{81675} \mathsf{B}_4 \right. \\
& + \frac{483988}{9075}\zeta_4 - \frac{103652031822049723}{415451499724800}\zeta_3 - \frac{20114890664357}{581101290000}\zeta_2 \\
& \left. + \frac{6830363463566924692253659}{685850575063965696000000} \right] + C_A C_F \left[\frac{1286792}{81675}\mathsf{B}_4 - \frac{643396}{9075}\zeta_4 \right. \\
& \left. - \frac{76189716747743790 7}{33236119977984000}\zeta_3 + \frac{15455008277}{660342375}\zeta_2 + \frac{872201479486471797889957487}{29928025093700321280000000} \right] \\
& + C_F^2 \left[-\frac{11808}{3025}\mathsf{B}_4 + \frac{53136}{3025}\zeta_4 + \frac{9636017147214304991}{7122025709568000}\zeta_3 + \frac{14699237127551}{15689734830000}\zeta_2 \right. \\
& \left. - \frac{247930147349635960148869654541}{148143724213816590336000000} \right] + T_F C_A \left[\frac{4206955789}{377338500}\zeta_2 + \frac{123553074914173}{5755172290560}\zeta_3 \right. \\
& \left. + \frac{2323118975810619964 5229}{633397356480430080000} \right] + T_F C_F \left[-\frac{502987059528463}{113048027136000}\zeta_3 + \frac{24683221051}{46695639375}\zeta_2 \right. \\
& \left. - \frac{1831993118263044461191 2149}{14108926115601580032000000} \right] - \frac{896}{1485} T_F^2 \zeta_3 \Biggr\} .
\end{aligned}
$$

$$
\begin{aligned}
a_{qg,Q}^{(3)}\Big|_{N=10} = \; & n_f T_F^2 \left(\frac{m^2}{\mu^2}\right)^{3\varepsilon/2} \Biggl\{ C_A \left[-\frac{1505896}{245025}\zeta_3 + \frac{1109186999}{377338500}\zeta_2 + \frac{6542127929072987}{191763425700000} \right] \\
& + C_F \left[\frac{62292104}{13476375}\zeta_3 - \frac{83961181063}{93391278750}\zeta_2 - \frac{353813854966442889041}{215285127496362000000} \right] \Biggr) \Biggr\}
\end{aligned}
$$

- We obtain e.g. for the moments of the $\hat{\gamma}_{qg}^{(2)}$ anomalous dimension

N	$\hat{\gamma}_{qg}^{(2)}/T_F$
2	$(1+2n_f)T_F\left(\frac{8464}{243}C_A - \frac{1384}{243}C_F\right) + \frac{\zeta_3}{3}\left(-416C_AC_F + 288C_A^2 + 128C_F^2\right) - \frac{7178}{81}C_A^2 + \frac{556}{9}C_AC_F - \frac{8620}{243}C_F^2$
4	$(1+2n_f)T_F\left(\frac{4481539}{303750}C_A + \frac{9613841}{3037500}C_F\right) + \frac{\zeta_3}{25}\left(2832C_A^2 - 3876C_AC_F + 1044C_F^2\right)$
	$-\frac{295110931}{3037500}C_A^2 + \frac{278546497}{2025000}C_AC_F - \frac{757117001}{12150000}C_F^2$
6	$(1+2n_f)T_F\left(\frac{86617163}{11668860}C_A + \frac{1539874183}{340341750}C_F\right) + \frac{\zeta_3}{735}\left(69864C_A^2 - 94664C_AC_F + 24800C_F^2\right)$
	$-\frac{58595443051}{653456160}C_A^2 + \frac{1199181909343}{8168202000}C_AC_F - \frac{2933980223981}{40841010000}C_F^2$
8	$(1+2n_f)T_F\left(\frac{10379424541}{2755620000}C_A + \frac{7903297846481}{1620304560000}C_F\right) + \zeta_3\left(\frac{128042}{1575}C_A^2 - \frac{515201}{4725}C_AC_F + \frac{749}{27}C_F^2\right)$
	$-\frac{24648658224523}{289340100000}C_A^2 + \frac{4896295442015177}{32406091200000}C_AC_F - \frac{4374484944665803}{56710659600000}C_F^2$
10	$(1+2n_f)T_F\left(\frac{1669885489}{988267500}C_A + \frac{1584713325754369}{323600780868750}C_F\right) + \zeta_3\left(\frac{1935952}{27225}C_A^2 - \frac{2573584}{27225}C_AC_F + \frac{70848}{3025}C_F^2\right)$
	$-\frac{21025430857658971}{255684567600000}C_A^2 + \frac{926990216580622991}{6040547909550000}C_AC_F - \frac{1091980048536213833}{13591232796487500}C_F^2$

- Agreement for the terms $\propto T_F$ of the anomalous dimensions $\gamma_{ij}^{(2),\mathsf{NS}^{\pm},\ \mathsf{S},\ \mathsf{PS}}$ with
  [Larin, Nogueira, Ritbergen, Vermaseren, 1997; Moch, Vermaseren, Vogt, 2004.]

- How far can we go ? $N = 14$ in some cases; generally: $N = 10 \Longrightarrow$ Phenomenology

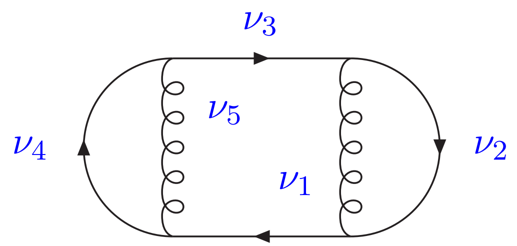
- Unfortunately not enough to perform the automatic
  fixed moments $\rightarrow$ all moments turn. [Blümlein, Kauers, S.K., Schneider, 2009].

- Recently with B. Tödtli: Calculation of moments $N = 1, ..., 13$ of the
  transversity heavy OMEs $A_{qq,Q}^{h,(2,3)}$
  $\Longrightarrow$ Agreement with anomalous dimensions $\gamma_{qq}^{h,(1,2)}$ from
  [Kumano, 1997; 2–Loop: Hayashigaki, Kanazawa, Koike, 1997; Vogelsang, 1998; 3–Loop, N$\leq$ 8: Gracey, 2006]

6. Towards an all–N Result

Representations in terms of Feynman parameters

Consider e.g the 3–loop tadpole diagram



Using Feynman–parameters, one obtains a representation in terms of a double sum

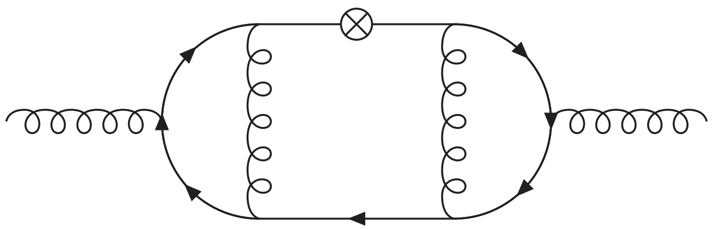
$$I = C\Gamma\left[\begin{matrix} 2+\varepsilon/2-\nu_1, 2+\varepsilon/2-\nu_5, \nu_{12}-2-\varepsilon/2, \nu_{45}-2-\varepsilon/2, \nu_{1345}-4-\varepsilon, \nu_{12345}-6-3/2\varepsilon \\ \nu_1, \nu_2, \nu_4, 2+\varepsilon/2, \nu_{345}-2-\varepsilon/2, \nu_{12345}-4-\varepsilon \end{matrix}\right]$$

$$\sum_{m,n=0}^{\infty} \frac{(\nu_{345}-2-\varepsilon/2)_{n+m}(\nu_{12345}-6-3/2\varepsilon)_m(2+\varepsilon/2-\nu_1)_m(2+\varepsilon/2-\nu_5)_n(\nu_{45}-2-\varepsilon/2)_n}{m!n!(\nu_{12345}-4-\varepsilon)_{n+m}(\nu_{345}-2-\varepsilon/2)_m(\nu_{345}-2-\varepsilon/2)_n},$$

which derives from an Appell–function of the first kind, F_1 .

$$F_1\left[a;b,b';c;x,y\right] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_n(b')_m}{(1)_m(1)_n(c)_{m+n}} x^n y^m \ .$$

For any diagram deriving from the tadpole–ladder topology, one obtains for fixed values of N a finite sum over double sums of the same type. Consider e.g. the scalar diagram



For the above diagram, we obtained an result for arbitrary N using similar summation techniques as in the 2–loop case and the package SIGMA.

$$L_3 = -\frac{4(N+1)S_1+4}{(N+1)^2(N+2)}\zeta_3 + \frac{2S_{2,1,1}}{(N+2)(N+3)} + \frac{1}{(N+1)(N+2)(N+3)}\Bigg\{-2(3N+5)S_{3,1} - \frac{{S_1}^4}{4}$$
$$+\frac{4(N+1)S_1-4N}{N+1}S_{2,1} + 2\Big((2N+3)S_1 + \frac{5N+6}{N+1}\Big)S_3 + \frac{9+4N}{4}{S_2}^2 + \Big(2\frac{7N+11}{(N+1)(N+2)} + \frac{5N}{N+1}S_1$$
$$-\frac{5}{2}{S_1}^2\Big)S_2 + \frac{N}{N+1}{S_1}^3 + \frac{2(3N+5){S_1}^2}{(N+1)(N+2)} + \frac{4(2N+3)S_1}{(N+1)^2(N+2)} - \frac{(2N+3)S_4}{2} + 8\frac{2N+3}{(N+1)^3(N+2)}\Bigg\}.$$

$\Longrightarrow$ Complete solution for the 3–loop case might be found by studying generalized hypergeometric functions and their relations to Feynman–integrals combined with advanced summation techniques.

Single Scale Feynman Integrals as Recurrent Quantities

- A large number of single scale 2– and 3–loop processes can be expressed in terms of nested harmonic sums. This holds for anomalous dimensions, Wilson coefficients, space- and time-like, polarized/unpolarized, Drell-Yan process, hadronic Higgs Boson production in the heavy mass limit, HO QED corrections in e^+e^- annihilation, soft+virtual corrections to Bhabha scattering, Heavy Flavor Wilson Coefficients at $Q^2 \gg m^2$.
[Blümlein and Ravindran, 2004/05; Blümlein and Moch 2005; Blümlein and S.K. 2007]

- Polynomials in N and Nested Harmonic Sums or linear combinations thereof obey recurrence relations, e.g.:

$$F(N+1) - F(N) = \frac{\text{sign}(a)^{N+1}}{(N+1)^{|a|}} \Longrightarrow F(N) = S_a(N) = \sum_{i=1}^{N} \frac{\text{sign(a)}^i}{i^{|a|}} .$$

- It is very likely that single scale Feynman diagrams always obey difference equations

$$\sum_{k=0}^{l} \left[\sum_{i=0}^{d} c_{i,k} N^i \right] F(N+k) = 0 .$$

$\Longrightarrow$ seek for solutions in terms of harmonic sums [Blümlein, Kauers, S.K. and Schneider, 2009]

7. Conclusions

- The heavy flavor contributions to F_2 are rather large in the region of lower values of x.
- QCD precision analyses require the description of the heavy quark contributions to 3–loops.
- Complete analytic results are known in the region $Q^2 \gg m^2$ at NLO for $F_{2,L}^{Q\overline{Q}}(x,Q^2), g_{1,2}^{Q\overline{Q}}(x,Q^2)$. They are expressed in terms of massive operator matrix elements and the corresponding massless Wilson coefficients.
- $F_L^{Q\overline{Q}}(x,Q^2)$ is known to NNLO for $Q^2 \gg m^2$.
- The calculation of fixed moments of the massive operator matrix elements at $O(a_s^3)$ has been finished for $N = 10, 12, 14$

  $\Longrightarrow F_2^{Q\overline{Q}}(x,Q^2)$ to NNLO for $Q^2 \gg m^2$.

  $\Longrightarrow$ Logarithmic terms are known for all N.
- We also calculate the matrix elements necessary to transform from the FFNS to the VFNS.
- First phenomenological parametrization to come up soon.
- Moments of the fermionic contributions to the 3-loop anomalous dimensions have been confirmed for the first time by an independent calculation.