INTERPRETATION OF CDMS RESULTS IN A SUPERSYMMETRIC WARPED 5D INELASTIC LITTLE HIGGS MULTIVERSE

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SU(3)/SU(2), THE SIMPLEST WZW TERM

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basic idea: classic result is the closed form for gauged WZW term in QCD-like chiral lagrangian SU(N)xSU(N)/SU(N)

$$\Gamma_{\rm WZW}(U, A_L, A_R) = \Gamma_0(U) + \frac{N}{48\pi^2}\,\mathrm{Tr}\int_{M^4}\Big\{(A_L\alpha^3 + A_R\beta^3) - \frac{i}{2}[(A_L\alpha)^2 - (A_R\beta)^2]$$
$$+ i[(dA_LA_L + A_LdA_L)\alpha + (dA_RA_R + A_RdA_R)\beta] + (A_L^3\alpha + A_R^3\beta)$$
$$+ i(A_LUA_RU^\dagger\alpha^2 - A_RU^\dagger A_LU\beta^2) + i(dA_RdU^\dagger A_LU - dA_LdUA_RU^\dagger)]$$
$$-(dA_LA_L + A_LdA_L)UA_RU^\dagger + (dA_RA_R + A_RdA_R)U^\dagger A_LU - i(A_LUA_RU^\dagger A_L\alpha + A_RU^\dagger A_LUA_R\beta)$$
$$+ i[A_L^3UA_RU^\dagger - A_R^3U^\dagger A_LU - \frac{1}{2}(UA_RU^\dagger A_L)^2\Big\}, \qquad (19$$

Witten 83

Kaymakcalan, Rajeev, Schechter 84

...

derive analog for SU(3)/SU(2) (a special case: doesn't extend to N>3)

$$\Gamma_{WZW}(\Phi, A, A_0) = \Gamma_0(\Phi) + \frac{p}{4\pi^2}\int_{M^4}\sum_{i=1}^{4}\mathcal{L}_i + \mathcal{L}_{G.I.}$$

$$\mathcal{L}_1 = A_0\Phi^\dagger d\Phi d\Phi^\dagger d\Phi - \left(\Phi^\dagger Ad\Phi + d\Phi^\dagger A\Phi\right)d\Phi^\dagger d\Phi\,,$$
$$\mathcal{L}_2 = iA_0dA_0\Phi^\dagger d\Phi - idA_0\Phi^\dagger A\Phi\Phi^\dagger d\Phi - 2iA_0\Phi^\dagger A\Phi d\Phi^\dagger d\Phi + \frac{i}{2}\left[(d\Phi^\dagger A\Phi)^2 - (\Phi^\dagger Ad\Phi)^2\right]$$
$$+\frac{i}{4}\left[\Phi^\dagger(AdA + dAA)d\Phi + d\Phi^\dagger(AdA + dAA)\Phi\right] - \frac{i}{2}\Phi^\dagger(AdA + dAA)\Phi\Phi^\dagger d\Phi - \frac{i}{2}\mathrm{Tr}(AdA)\Phi^\dagger d\Phi$$
$$+\frac{i}{4}\left[\Phi^\dagger dA\Phi(d\Phi^\dagger A\Phi + \Phi^\dagger Ad\Phi) + \Phi^\dagger A\Phi(\Phi^\dagger dAd\Phi - d\Phi^\dagger dA\Phi)\right]\,,$$
$$\mathcal{L}_3 = A_0dA_0\Phi^\dagger A\Phi + A_0\left[-\Phi^\dagger A\Phi d(\Phi^\dagger A\Phi) + \frac{1}{3}\mathrm{Tr}(AdA)\right] - \frac{1}{6}\Phi^\dagger A^2\Phi(\Phi^\dagger Ad\Phi + d\Phi^\dagger A\Phi)$$
$$+\frac{1}{6}\Phi^\dagger A\Phi(\Phi^\dagger A^2d\Phi - d\Phi^\dagger A^2\Phi) + \frac{1}{6}\Phi^\dagger(dAA^2 - A^2dA)\Phi + \frac{1}{3}(\Phi^\dagger A^3d\Phi + d\Phi^\dagger A^3\Phi) - \frac{2}{3}\Phi^\dagger A^3\Phi\Phi^\dagger d\Phi$$
$$-\frac{1}{3}\mathrm{Tr}(A^3)\Phi^\dagger d\Phi - \frac{1}{2}\Phi^\dagger(AdA + dAA)\Phi\Phi^\dagger A\Phi - \frac{1}{3}\mathrm{Tr}(AdA)\Phi^\dagger A\Phi\,,$$
$$\mathcal{L}_4 = -\frac{i}{4}A_0\mathrm{Tr}(A^3) - \frac{3i}{4}\Phi^\dagger A\Phi\Phi^\dagger A^3\Phi - \frac{i}{4}\Phi^\dagger A\Phi\mathrm{Tr}(A^3)\,.$$

$$\mathcal{L}_{G.I.} = c_1\left[\Phi^\dagger(dA - iA^2)\Phi\right]^2$$
$$+ c_2\, i\Phi^\dagger(dA - iA^2)\Phi D\Phi^\dagger D\Phi$$
$$+ c_3\,\Phi^\dagger(dA - iA^2)^2\Phi$$
$$+ c_4\,\Phi^\dagger D\Phi\left[\Phi^\dagger(dA - iA^2)D\Phi - (D\Phi^\dagger)(dA - iA^2)\Phi\right]\,,$$
$$+ c_5\, dA_0\Phi^\dagger(dA - iA^2)\Phi$$

OUTLINE

- intro- anomalies and extensions of electroweak SM
- a 2-d example
- the 4-d action
- equivalence to SU(3)xSU(3)/SU(3)
- reduction to SU(2)xU(1)/U(1)

CAN'T HIDE FERMIONS

Common situation in model building: add some fermion content to a consistent theory → spoil anomaly cancellation.

For consistent (not necessarily complete) theory, need spectator fermions in the linear theory, or extra operator (WZW term) in the nonlinear theory

Appears e.g. in little higgs / technicolor models. consider as example SU(3)/SU(2) little higgs

1) generation-independent gauging

quarks

$$SU(3)_c \times SU(3)_L \times U(1)_X$$

$$Q_L = \begin{pmatrix} u \\ d \\ d' \end{pmatrix}_L \qquad u_R, d_R, d'_R$$

$$Q_L = (3,3)_{\frac{1}{3}}$$
$$u_R = u'_R = (3,1)_{\frac{2}{3}}$$
$$d_R = (3,1)_{-\frac{1}{3}}$$

leptons

$$E_L = \begin{pmatrix} \nu \\ e \\ \nu' \end{pmatrix}_L \qquad \nu_R, e_R, \nu'_R$$

$$E_L = (1,3)_{-\frac{1}{3}}$$
$$\nu_R = \nu'_R = (1,1)_0$$
$$e_R = (1,1)_{-1}$$

can give mass to exotic fermions using triplet higgs fields

$$H_1 = H_2 = (1,3)_{-\frac{1}{3}}$$

Kaplan Schmaltz 04

little higgs idea - fields acquire aligned VEV's

$$H \sim \Lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

scalar fields:

$$12 = 2 + 5 + 4 + 1$$

radial modes

eaten by heavy gauge fields

SM Higgs

singlet

Just this fermion content is inconsistent: $SU(3)_W$, $U(1)_X$ anomalies

(ignoring all questions of vac.alignment, fine-tuning, etc.)

extra fermion content equivalent to e.g. N copies of

$$\Psi_L = (1,3)_{-\frac{1}{2}}\,, \quad q_R = (1,1)_{\frac{3}{2}}$$

$$N = 4N_{\text{generations}} \sim 12$$

extra operator is WZW term for SU(3)/SU(2)

$$N_1\Gamma_{WZW}(\Phi_1) + N_2\Gamma_{WZW}(\Phi_2)$$

$$N_1 + N_2 = 4N_{\text{generations}}$$

$$\mathcal{L} \sim N_1\epsilon_{\mu\nu\rho\sigma}\left[\frac{1}{96\pi^2F^4}H^\dagger D^\mu H H^\dagger F_W^{\nu\rho}D^\sigma H - \frac{2}{8\pi^2\sqrt{3}F}\eta \mathrm{Tr}(F_W^{\mu\nu}F_W^{\rho\sigma}) + \frac{2}{16\pi^2F}D^\mu H^\dagger F_W^{\nu\rho}C^\sigma + \ldots\right]$$
$$+N_2\left[\eta \to -\eta, H \to -H\right]$$

Hill & Hill 07

N=12 an important clue to UV completion

2) generation-independent gauging

generation 1,2

$$Q_L = (3,\bar{3})_0$$
$$u_R = (3,1)_{\frac{2}{3}}$$
$$d_R = d'_R = (3,1)_{-\frac{1}{3}}$$

generation 3

$$Q_L = (3,3)_{\frac{1}{3}}$$
$$u_R = u'_R = (3,1)_{\frac{2}{3}}$$
$$d_R = (3,1)_{-\frac{1}{3}}$$

Fermion anomalies cancel. In this case:

$$N_1 \Gamma_{WZW}(\Phi_1) + N_2 \Gamma_{WZW}(\Phi_2)$$

$$N_1 + N_2 = 0$$

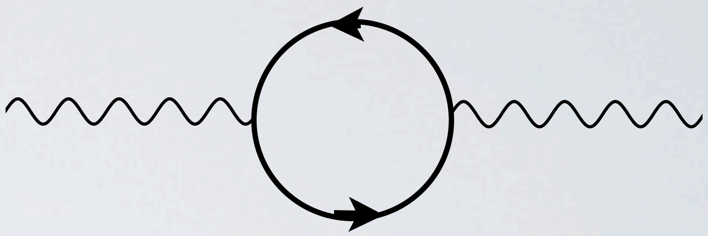
$$\mathcal{L} \sim N_1 \epsilon_{\mu\nu\rho\sigma} \left[\frac{1}{96\pi^2 F^4} H^\dagger D^\mu H H^\dagger F_W^{\nu\rho} D^\sigma H - \frac{2}{8\pi^2\sqrt{3}F} \eta \mathrm{Tr}(F_W^{\mu\nu} F_W^{\rho\sigma}) + \frac{2}{16\pi^2 F} D^\mu H^\dagger F_W^{\nu\rho} C^\sigma + \ldots \right]$$
$$+ N_2 \left[\eta \to -\eta, H \to -H \right]$$

$N_1 = -N_2$ counts "colors" of UV completion, e.g. 2 copies of SU(3)×SU(3)/SU(3), and strongly coupled SU(2). $N_1 \neq 0 \Rightarrow$ breaks potential NGB or T parity

Also more subtle non-decoupling effects

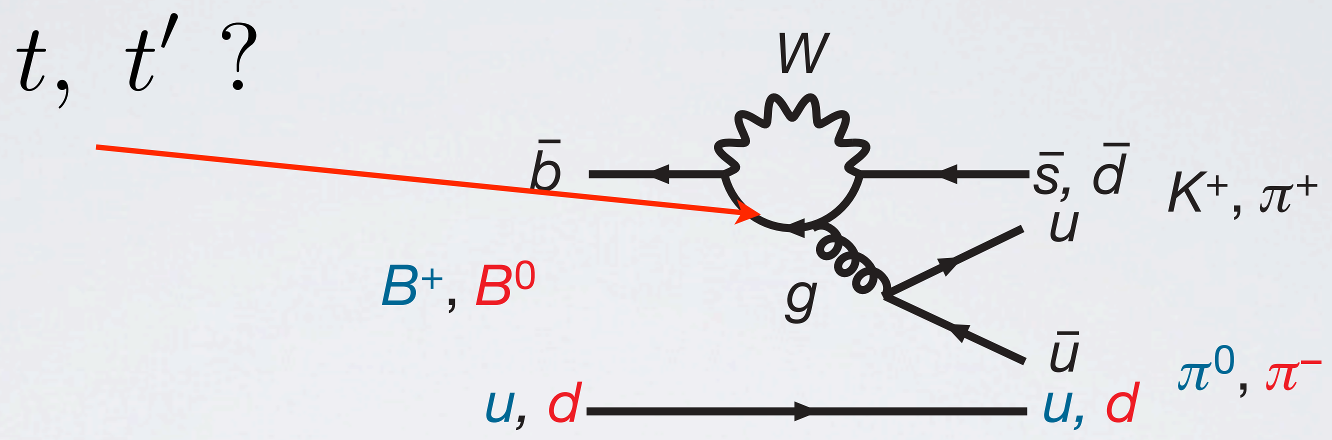
Example: extra generation adds to one-loop S parameter

$$S \sim \sum \frac{1}{6\pi} \sim \frac{2}{3\pi}$$



⇒ Degenerate heavy fermions don't decouple

Can see also in certain flavor transitions



$$O_{\text{eff}} \sim G_F m_b \bar{s} \sigma^{\mu\nu} G_{\mu\nu} b$$

$$C_{\text{eff}} \sim \text{finite} \quad (m_{t'} \to \infty)$$

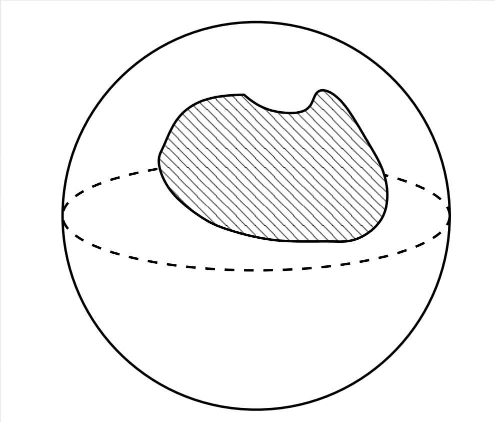
With this motivation, look in detail at the low-energy effective action. In particular, quantized topological action can be treated on its own

- Explicit gauged action for U(3)/U(2) (careful to include U(1) factors)
- By reduction, anomalous action for U(2)/U(1) (e.g. anomalous action for SM higgs NGB's)
- Some simple manipulations that are fun to do at least once

RE: "SIMPLEST"

complexity in defining topological actions come from manipulating five-spheres inside the field space G/H

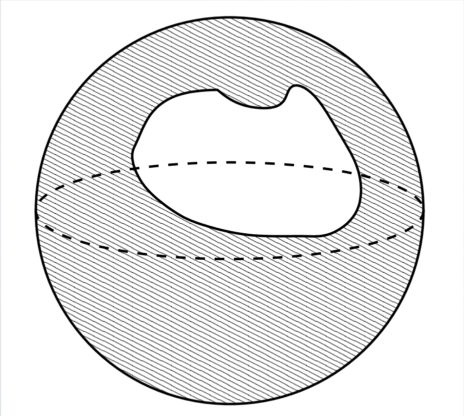
$$\Gamma = C \int_{M^5} d^5x \, \epsilon_{ABCDE} \omega^{ABCDE}$$



5-form that is
-globally invariant (action is G invariant)
-closed (action is four-dimensiona)

$\pi_4(G/H) = 0 \quad \Rightarrow$ given chiral field U(x), i.e., image of spacetime, can form a 5-d surface with image of spacetime as boundary

$\pi_5(G/H) = Z \quad \Rightarrow$ "winding number" assigned to inequivalent mappings



nontrivial action corresponds to existence of closed, but not exact, globally invariant five-form

$$\omega \in H_5(G/H, R)$$
$$d\omega = 0$$
$$\omega \neq d\eta$$

Examples

$$G = SU(N)_L \times SU(N)_R\,, \quad H = SU(N)_V\,, \quad G/H \cong SU(N)$$

$\pi_4(SU(N)) = 0$

$\pi_5(SU(N)) = Z$

$$\Psi_{L,R} \to e^{i\epsilon_{L,R}} \Psi_{L,R}$$

$$\langle \bar{\Psi}_L \Psi_R \rangle \neq 0$$

fields:

$$U(x) \to e^{i\epsilon_L} U(x) e^{-i\epsilon_R}$$

$$U = e^{i\pi^a t^a / f_\pi} \quad U^\dagger U = \mathbb{1}$$

$$G = SU(3)\,, \quad H = SU(2)\,, \quad G/H = S^5$$

$$H \to e^{i\epsilon} H$$

$$\pi_4(S^5) = 0$$

$$\pi_5(S^5) = Z$$

$$\langle H \rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

For SU(3)/SU(2), the field space *is* the five-sphere

fields:

$$\Phi \to e^{i\epsilon}\Phi$$

$$\Phi = e^{i\pi}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \phi^1 + i\phi^2 \\ \phi^3 + i\phi^4 \\ \phi^5 + i\phi^6 \end{pmatrix} \qquad \Phi^\dagger\Phi = 1$$

five forms

Should find exactly one nontrivial, globally invariant five form in each case

Properly normalized form integrates to 2**π** on the simplest nontrivial embedding

sprinkle five derivatives into:

$$\mathrm{Tr}(\ldots U^\dagger \ldots U \ldots U^\dagger \ldots U \ldots)$$

$$(d^2 = 0, \quad d(U^\dagger U) = 0)$$

$$\omega = \frac{i}{240\pi^2}\mathrm{Tr}\left[(U^\dagger dU)^5\right]$$

sprinkle five derivatives into:

$$\ldots \Phi^\dagger \ldots \Phi \ldots \Phi^\dagger \ldots \Phi$$

$$(d^2 = 0, \quad d(\Phi^\dagger \Phi) = 0)$$

$$\omega = 2\pi \frac{1}{\pi^2}\left[-\frac{i}{8}\Phi^\dagger d\Phi (d\Phi^\dagger d\Phi)^2\right]$$

$$\frac{1}{\sqrt{1 - \sum_{i=1}^5 (\phi^i)^2}} d\phi^1 d\phi^2 d\phi^3 d\phi^4 d\phi^5$$

volume of 5-sphere

notes

- 3 is special: $SU(N)/SU(N-1) = S^{2N-1}\,, \quad \pi_5(S^{2N-1}) = 0 (N > 3)$

- will see that anomalous variation of gauged action corresponds to *even* number of fermions

- will see that gauged action is equivalent to (a limit of) the SU(3)xSU(3)/SU(3) action (eating and decoupling)

2-D EXAMPLE

To get a handle on the algebra, consider one dimension lower

$$G = SU(2)\,, \quad H = \text{identity}\,, \quad G/H \cong S^3$$

$$H \to e^{i\epsilon} H$$

$$\pi_2(S^3) = 0$$

$$\pi_3(S^3) = Z$$

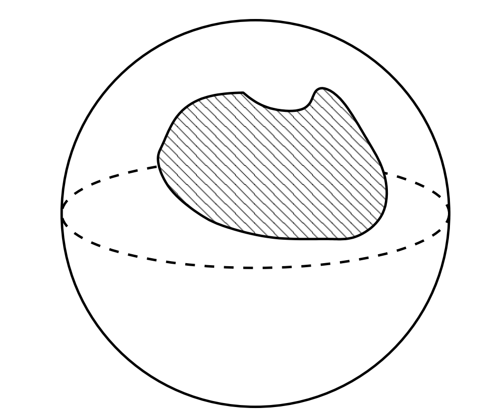
$$\langle H \rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Phi \to e^{i\epsilon}\Phi \qquad \Phi = e^{i\pi}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \phi^1 + i\phi^2 \\ \phi^3 + i\phi^4 \end{pmatrix}$$

$$\omega = 2\pi \frac{1}{2\pi^2}\left[\frac{1}{2}\Phi^\dagger d\Phi d\Phi^\dagger d\Phi\right] \qquad \Gamma(\phi) = \int \omega = \text{“area”}$$

volume of 3-sphere

$$\frac{1}{\sqrt{1 - \sum_{i=1}^3 (\phi^i)^2}} d\phi^1 d\phi^2 d\phi^3$$



gauging the 3-sphere

Let's try to gauge by brute force

$$\Phi \longrightarrow e^{i\epsilon}\Phi$$

Should find something 2-dimensional by stokes theorem, but have a left-over piece:

$$\begin{aligned}\delta\Gamma_0 &= \frac{ip}{2\pi}\int_{M^3} \Phi^\dagger d\epsilon\Phi d\Phi^\dagger d\Phi - \Phi^\dagger d\Phi\Phi^\dagger d\epsilon d\Phi + \Phi^\dagger d\Phi d\Phi^\dagger d\epsilon\Phi \\ &= \frac{ip}{2\pi}\int_{M^3} d\left[\Phi^\dagger\epsilon\Phi d\Phi^\dagger d\Phi + \Phi^\dagger d\Phi\Phi^\dagger\epsilon d\Phi + \Phi^\dagger d\Phi d\Phi^\dagger\epsilon\Phi\right] \\ &\quad -2\epsilon^A\left[d\Phi^\dagger d\Phi(\Phi^\dagger\sigma^A d\Phi + d\Phi^\dagger\sigma^A\Phi)\right]\end{aligned}$$

But use a magic identity involving Pauli matrices and fields on three-sphere:

$$d(\Phi^\dagger\sigma^A\Phi)d\Phi^\dagger d\Phi = 0$$

Should find something globally invariant (vanishes for const. **ε**):

$$\delta\Gamma_0 = \frac{ip}{2\pi}\int_{M^2} \Phi^\dagger\epsilon\Phi d\Phi^\dagger d\Phi + \Phi^\dagger d\Phi\Phi^\dagger\epsilon d\Phi + \Phi^\dagger d\Phi d\Phi^\dagger\epsilon\Phi$$

Again, a magic identity saves the day

$$d\Phi^\dagger\sigma^A d\Phi = d(\Phi^\dagger\sigma^A\Phi)\Phi^\dagger d\Phi - \Phi^\dagger\sigma^A\Phi d\Phi^\dagger d\Phi$$

Finally, integrate by parts,

$$\delta\Gamma_0 = \frac{ip}{4\pi}\int_{M^2} \Phi^\dagger d\epsilon d\Phi + d\Phi^\dagger d\epsilon\Phi$$

Variation is 2-d and local, can be cancelled by a gauge field with

$$A \to e^{i\epsilon}(A + id)e^{-i\epsilon}$$

$$\Gamma_1 = \frac{ip}{4\pi}\int_{M^2} -\Phi^\dagger A d\Phi - d\Phi^\dagger A\Phi$$

Result is not gauge invariant, but variation is independent of pions:

$$\delta(\Gamma_0 + \Gamma_1) = \frac{p}{2\pi}\int_{M^2} \frac{1}{2}\mathrm{Tr}(Ad\epsilon)$$

Gauged action:

$$\Gamma_{WZW} = \Gamma_0 + \Gamma_1 = \int_{M^3}\omega + \frac{ip}{4\pi}\int_{M^2} -\Phi^\dagger A d\Phi - d\Phi^\dagger A\Phi$$

Non-uniqueness due to gauge-invariant operator:

$$\Gamma_{G.I.} = c\int_{M^2}\Phi^\dagger(dA - iA^2)\Phi$$

equivalence to SU(2)xSU(2)/SU(2)

There is an exact equivalence of the preceding action to the $SU(2)_L \times SU(2)_R/SU(2)$ action in two-dimensions

$$\Gamma_{WZW} = -\frac{N}{12\pi}\int_{M^3} \mathrm{Tr}(U^\dagger dU U^\dagger dU U^\dagger dU) + \frac{N}{4\pi}\int_{M^2} \mathrm{Tr}\left[-iA_L\alpha - iA_R\beta + A_L U A_R U^\dagger\right]$$

$$N = p\,, \quad A_L = A\,, \quad A_R = 0\,, \quad \Phi = U\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Gamma_{WZW}(N = p, A_L = A, A_R = 0, U) = \Gamma_{WZW}(p, A, \Phi)$$

proof:

$$\begin{aligned} \mathrm{Tr}(P\alpha P\alpha^2) &= \frac{1}{6}\mathrm{Tr}(\alpha^3) \\ \mathrm{Tr}[P(\alpha\beta - \beta\alpha)] &= \mathrm{Tr}(\alpha\beta) \end{aligned}$$

$$P^2 = P\,, \quad \mathrm{e.g.}: P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow$$

$$\begin{aligned} \frac{-p}{12\pi}\mathrm{Tr}(\alpha^3) &= \frac{p}{2\pi}\Phi^\dagger d\Phi d\Phi^\dagger d\Phi \\ \frac{-ip}{4\pi}\mathrm{Tr}(A\alpha) &= \frac{-ip}{4\pi}(\Phi^\dagger A d\Phi + d\Phi^\dagger A\Phi) \end{aligned}$$

notes

- U(1) factors straightforward

$$\Phi \longrightarrow e^{i(\epsilon+\epsilon_0)}\Phi$$

For equivalence:

$$A_L = A - A_0\,, \quad A_R = -2A_0 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

- equivalence is less trivial in four-dimensional example: 5 NGB's of SU(3)/SU(2) ↔ 8 NGB's of SU(3)xSU(3)/SU(3) ⇒ "eating and decoupling"

- can show that it is always possible to add terms with gauge fields such that total variation is independent of pions, by either of two methods:
  - differential geometry

$$\delta\Gamma \sim \int \xi_{\{ab\}} d\epsilon_a A_b\,, \quad d\xi_{\{ab\}} \sim i_{\{a}i_{b\}}\omega = 0$$

  - equivalence to anomaly-integration expression a la Wess-Zumino

4-D EXAMPLE

We've already found the ungauged topological action (area on five-sphere), and by general arguments know that we can brute-force gauge.

Lots of magical identities between Gellmann matrices and complex triplets

Convenient to organize with diff. geom. (becomes slightly less magical)

Will spare you the algebra,

$$\Gamma_{WZW}(\Phi, A, A_0) = \Gamma_0(\Phi) + \frac{p}{4\pi^2} \int_{M^4} \sum_{i=1}^{4} \mathcal{L}_i + \mathcal{L}_{G.I.}$$

$$\begin{aligned}
\mathcal{L}_1 &= A_0 \Phi^\dagger d\Phi d\Phi^\dagger d\Phi - \left(\Phi^\dagger A d\Phi + d\Phi^\dagger A \Phi\right) d\Phi^\dagger d\Phi\,, \\
\mathcal{L}_2 &= i A_0 dA_0 \Phi^\dagger d\Phi - i dA_0 \Phi^\dagger A \Phi \Phi^\dagger d\Phi - 2i A_0 \Phi^\dagger A \Phi d\Phi^\dagger d\Phi + \frac{i}{2}\left[(d\Phi^\dagger A\Phi)^2 - (\Phi^\dagger A d\Phi)^2\right] \\
&\quad + \frac{i}{4}\left[\Phi^\dagger (AdA + dAA) d\Phi + d\Phi^\dagger (AdA + dAA)\Phi\right] - \frac{i}{2}\Phi^\dagger (AdA + dAA)\Phi\Phi^\dagger d\Phi - \frac{i}{2}\mathrm{Tr}(AdA)\Phi^\dagger d\Phi \\
&\quad + \frac{i}{4}\left[\Phi^\dagger dA \Phi (d\Phi^\dagger A\Phi + \Phi^\dagger A d\Phi) + \Phi^\dagger A \Phi (\Phi^\dagger dA d\Phi - d\Phi^\dagger dA \Phi)\right]\,, \\
\mathcal{L}_3 &= A_0 dA_0 \Phi^\dagger A \Phi + A_0 \left[-\Phi^\dagger A \Phi d(\Phi^\dagger A \Phi) + \frac{1}{3}\mathrm{Tr}(AdA)\right] - \frac{1}{6}\Phi^\dagger A^2 \Phi (\Phi^\dagger A d\Phi + d\Phi^\dagger A \Phi) \\
&\quad + \frac{1}{6}\Phi^\dagger A \Phi (\Phi^\dagger A^2 d\Phi - d\Phi^\dagger A^2 \Phi) + \frac{1}{6}\Phi^\dagger (dAA^2 - A^2 dA)\Phi + \frac{1}{3}(\Phi^\dagger A^3 d\Phi + d\Phi^\dagger A^3 \Phi) - \frac{2}{3}\Phi^\dagger A^3 \Phi \Phi^\dagger d\Phi \\
&\quad - \frac{1}{3}\mathrm{Tr}(A^3)\Phi^\dagger d\Phi - \frac{1}{2}\Phi^\dagger (AdA + dAA)\Phi\Phi^\dagger A\Phi - \frac{1}{3}\mathrm{Tr}(AdA)\Phi^\dagger A \Phi\,, \\
\mathcal{L}_4 &= -\frac{i}{4}A_0 \mathrm{Tr}(A^3) - \frac{3i}{4}\Phi^\dagger A \Phi \Phi^\dagger A^3 \Phi - \frac{i}{4}\Phi^\dagger A \Phi \mathrm{Tr}(A^3)\,.
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_{G.I.} &= c_1 \left[\Phi^\dagger (dA - iA^2)\Phi\right]^2 \\
&\quad + c_2\, i\Phi^\dagger (dA - iA^2)\Phi D\Phi^\dagger D\Phi \\
&\quad + c_3\, \Phi^\dagger (dA - iA^2)^2 \Phi \\
&\quad + c_4\, \Phi^\dagger D\Phi \left[\Phi^\dagger (dA - iA^2) D\Phi - (D\Phi^\dagger)(dA - iA^2)\Phi\right]\,, \\
&\quad + c_5\, dA_0 \Phi^\dagger (dA - iA^2)\Phi
\end{aligned}$$

Left-over gauge variation (=anomaly):

$$\delta\Gamma = -\frac{p}{12\pi^2}\int_{M^4}\mathrm{Tr}\Big\{\epsilon\left[(dA)^2 - \frac{i}{2}d(A^3)\right] - \frac{1}{2}\epsilon_0\left[(dA)^2 - \frac{i}{2}d(A^3)\right] - \frac{1}{2}\epsilon\left[2dA\,dA_0 - \frac{i}{2}d(A_0A^2)\right]\Big\} + 3\epsilon_0(dA_0)^2$$

$$= -\frac{2p}{24\pi^2}\int_{M^4}\mathrm{Tr}\Big\{\left(\epsilon - \frac{1}{2}\epsilon_0 1\!\!1_3\right)\left[\left(dA - \frac{1}{2}dA_0 1\!\!1_3\right)^2 - \frac{i}{2}d\left[\left(A - \frac{1}{2}A_0 1\!\!1_3\right)^3\right]\right]\Big\} - \left(-\frac{3}{2}\epsilon_0\right)\left(-\frac{3}{2}dA_0\right)^2$$

= anomaly of triplet L, singlet R fermion

$$\Psi_L = \begin{pmatrix} \psi_{1L} \\ \psi_{2L} \\ \psi_{3L} \end{pmatrix}, \quad q_R$$

$$\Psi_L \to e^{i(\epsilon - \epsilon_0/2)}\Psi_L\,, \quad q_R \to e^{-3i\epsilon_0/2}q_R$$

equivalence to SU(3)xSU(3)/SU(3)

SU(3)/SU(2) ~ SU(3)xSU(3)/SU(3), but kaons, eta without the pions.
Make this explicit?

In fact, exact equivalence:

$$\Gamma_{SU(3)/SU(2)}(p, \Phi, A) = \Gamma_{SU(3)\times SU(3)/SU(3)}(2p, \tilde{U}, \tilde{A}_L, \tilde{A}_R)$$

In terms of nonlinear realization of SU(3) on SU(3)/SU(2):

$$\xi \rightarrow e^{i\epsilon}\xi e^{-i\epsilon'(\epsilon,\xi)}$$

Dictionary:

projects onto SU(2)

$$\tilde{U} = \xi \,,$$

$$\tilde{A}_L = A \,,$$

$$\tilde{A}_R = \sum_{A=1}^{3} \frac{\lambda_A}{2} \mathrm{Tr}(\lambda_A[\xi^\dagger (A + id)\xi]) \,,$$

To prove equivalence, first note that gauge variations are identical (ignore U(1) factor to start)

$$\tilde{U} = \xi\,, \qquad \xi \to e^{i\epsilon}\xi e^{-i\epsilon'(\epsilon,\xi)}$$

$$\tilde{A}_L = A\,,$$

$$\tilde{A}_R = \sum_{A=1}^{3} \frac{\lambda_A}{2}\mathrm{Tr}(\lambda_A[\xi^\dagger(A+id)\xi])\,,$$

A_L, A_R transform with **ε**, **ε'**, respectively

$$\delta\Gamma \sim \int \mathrm{Tr}\left[\epsilon(d\tilde{A}_L)^2 - \epsilon'(d\tilde{A}_R)^2\right] \sim \int \mathrm{Tr}\left[\epsilon(dA)^2\right]$$

no continuous anomaly in SU(2)

Can extend to include U(1) using

$$A_L = \tilde{A}_L - \frac{1}{2}A_0\,, \quad A_R = \tilde{A}_R - \frac{3}{2}A_0 \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}$$

Motivation for this limit is "eating and decoupling"

Consider strong coupling of $SU(2)_R$ gauge field

$$\frac{-1}{4g^2} F_{\mu\nu}^2 \to 0$$

Fields become nondynamical, enforcing the locking condition

$$\frac{\delta}{\delta A_R} \mathrm{Tr}\Big[(\partial_\mu U - iA_{L\mu} U + iU A_{R\mu})(\partial_\mu U^\dagger + iU A_{L\mu} - iA_{R\mu} U) \Big] = 0$$

$$\Rightarrow \quad A_R \quad = \sum_{A=1}^{3} \frac{\lambda_A}{2} \mathrm{Tr}(\lambda_A [U^\dagger (A + id) U])$$

Work in gauge where A_R eats the pions and decouples:

$$U = \xi \sim \exp\left[i \begin{pmatrix} \eta & K \\ K^\dagger & -2\eta \end{pmatrix} \right]$$

- even quantization reflects $\pi_4(SU(2)) = Z_2 \neq 0$
- inability to extend to SU(4)/SU(3), etc: can't gauge $SU(N)_L$ (N>2)

Two actions have same gauge transformation: all that remains is to fix the coefficients of the gauge-invariant operators, e.g. by considering actions at

$$\xi = \mathbb{1} \qquad \Phi = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Straightforward but tedious calculation:

$$c_1 = c_2 = c_3 = c_5 = 0\,, \quad c_4 = \frac{7}{12}$$

Note: on SU(3)xSU(3)/SU(3) side, an ambiguity due to gauge-invariant operator:

$$\mathcal{L} \sim \epsilon_{\mu\nu\rho\sigma} \mathrm{Tr}(U F_R^{\mu\nu} U^\dagger F_L^{\rho\sigma})$$

Fix this by imposing parity (e.g. in QCD)

Reduction to SU(2)xU(1)/U(1)

A classic construction of the anomalous action of n_f=2 QCD employs a reduction of the WZW term for n_f=3

$$\pi_5(SU(2)) = 0 \qquad\qquad \pi_5(SU(3)) = Z$$

A similar construction in the present case gives a topological construction of anomalous action for SM-like Higgs field

$$\Phi = \begin{pmatrix} 0 \\ H \end{pmatrix}, \quad A = \begin{pmatrix} 0 & \\ & W \end{pmatrix}, \quad A_0 = \frac{1}{2}B$$

E.g., consider integrating out a generation of quarks (or leptons), consisting of L and R doublet of fermions with

$$A_L = W + yB\,, \quad A_R = \left(y + \frac{\sigma^3}{2}\right) B$$

Anomaly of fermions:

$$\begin{aligned} \delta\Gamma &= -\frac{y}{24\pi^2}\int_{M^4} \mathrm{Tr}\Big\{\epsilon_W\left[2dW dB - \frac{i}{2}d(BW^2)\right] \\ &\quad +\epsilon_B\left[(dW)^2 - \frac{i}{2}d(W^3)\right]\Big\} - \frac{3}{2}\epsilon_B(dB)^2 \end{aligned}$$

Matched by previous expression provided

$$p = -2y$$

⇒ integer p sufficient for y=-1/2, y= 3 x (1/6)

Gauge invariant operators can be fixed by enforcing custodial symmetry (e.g. degenerate limit for mass of heavy quarks, or leptons):

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$$\begin{aligned}\mathcal{L}_{G.I.} = c_1 \, & \left[\Phi^\dagger(dA - iA^2)\Phi\right]^2 \\ & + c_2 \, i\Phi^\dagger(dA - iA^2)\Phi D\Phi^\dagger D\Phi \\ & + c_3 \, \Phi^\dagger(dA - iA^2)^2\Phi \\ & + c_4 \, \Phi^\dagger D\Phi \left[\Phi^\dagger(dA - iA^2)D\Phi - (D\Phi^\dagger)(dA - iA^2)\Phi\right] \, , \\ & + c_5 \, dA_0\Phi^\dagger(dA - iA^2)\Phi\end{aligned}$$

$$c_1 - \frac{1}{2}c_2 = c_3 = c_5 = 0 \, , \quad c_4 = \frac{1}{2}$$

E.g., integrate a complete heavy SM generation (and heavy higgs boson): these operators summarize what remains of terms with epsilon tensor (remaining actions cancel)

(No) Skyrmion

Recall in QCD, conserved baryon current follows from the existence of a globally invariant 3-form

$$J \sim \mathrm{Tr}\big[(U^\dagger dU)^3\big]$$

$$J_\mu \sim \epsilon_{\mu\nu\rho\sigma} \mathrm{Tr}\big[(U^\dagger \partial_\nu U)(U^\dagger \partial_\rho U)(U^\dagger \partial_\sigma U)\big]$$

$$\partial_\mu J^\mu \sim d\left(\mathrm{Tr}\big[(U^\dagger dU)^3\big]\right) = 0$$

$$d(U^\dagger U) = 0$$

In SU(3)/SU(2), no such current:

$$(\Phi^\dagger d\Phi)^3 = 0$$

$$d\big[\Phi^\dagger d\Phi d\Phi^\dagger d\Phi\big] \neq 0$$

reflects the fact that $\pi_3(SU(N)) = Z$ but $\pi_3(S^5) = 0$

SUMMARY

- fully gauged anomalous action for SU(3)/SU(2), complete with U(1) factors, gauge-invariant operators
- interesting equivalences, eating and decoupling, factor of 2 in quantization, absence of skyrmion
- applications to extensions of EW standard model, like little higgs
- topological derivation of anomalous action for SM-like Higgs field