D-Dimensional Generalized Unitarity and Top Quark Physics

Markus Schulze

in collaboration with Kirill Melnikov

JOHNS HOPKINS UNIVERSITY

Introduction

- I will talk about next-to-leading order calculations in QCD
  Radiative corrections split into virtual corrections and real emission corrections

- At NLO final states are modelled more realistically

- Dependencies to unphysical scales are less severe than at leading order

- Typically, NLO predictions are in excellent agreement with data

- LO predictions can be tuned by choosing appropriate scales
  (but: What is "appropriate"?, How do cuts affect those predictions?)

Introduction

Sabine Lammers (D0 collaboration) UIndiana, DPF2009 Detroit

comparison of different MC event generators with D0 data (Run II, 1fb^{-1})

- multi-d... cross sections

Precision comparisons will continue with larger dataset, W/Z+3 jet NLO calculations

Performance by	Z+jet normalization	Z+jet angles	Z+jet p_T
MCFM NLO	✔	✔	✔
Alpgen/MLM + Pythia			✔
Alpgen/MLM + Herwig			✔
Sherpa/CKKW		✔	
HERWIG			
PYTHIA			

Z+jets Measurements at D0 - July 30, 2009 21

Outline

- **Introduction**
- **Virtual corrections**
  - ► Basic ideas of unitarity based methods
  - ► OPP algorithm
  - ► *D*-dimensional generalized unitarity
  - ► Massive fermions, top quark amplitudes
- **Top quark phenomenology**
  - ► Top quark pair production and leptonic decay at NLO
  - ► Various distributions for LHC & Tevatron,
    Impact of NLO corrections to the decay

Virtual Corrections

Challenges

- Accuracy: numerical stable reduction of tensor integrals
  (avoid division by small numbers)

- Efficiency: fast evaluation on a computer
  (partonic cross section needs to be integrated
  over phase space & folded with pdfs)

Approaches

The *traditional* way to do a 1-loop calculation:

- generate all Feynman diagrams
- reduction of tensor integrals to scalar integrals
- reduction to minimal set of spin and color structures

This talk: ***D*-Dimensional Generalized Unitarity**

- completely orthogonal approach
- basic ingredients are tree level amplitudes
- better scaling with increasing number of external legs
- method is ready for phenomenology

application: **Top Quark Pair Production**

Basic ideas

***D*-Dimensional Generalized**

Unitarity

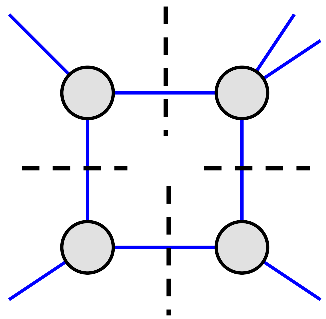
Basic ideas

cuts in $D{\neq}4$ dimensions

[van Neerven]
[Bern, Morgan]
[Giele,Kunszt,Melnikov]

cut four propagators

[Britto,Cachazo,Feng]

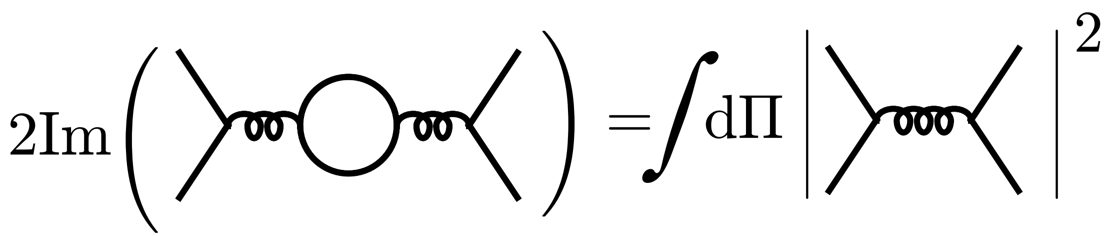


***D*-Dimensional** **Generalized**

Unitarity

[Bern,Dixon,Dunbar,Kosower]

optical theorem



Organization of the calculation

Color ordering

[Bern,Dixon,Kosower]: $A^{\text{1-loop}} = \sum_i C_i \, \mathcal{A}_i^{\text{prim.}}$

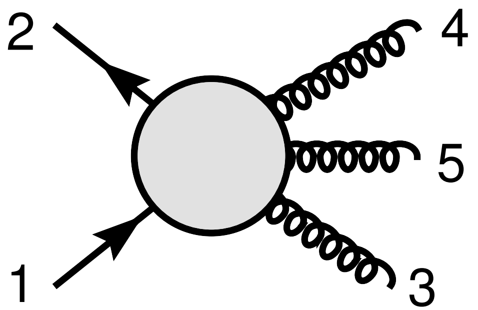
color factor

color ordered primitive amplitude

primitive amplitude:

- fixed ordering of external legs
- color factors are stripped off
- gauge invariant
- minimal set of building blocks

$\mathcal{A}^{\text{prim.}}(1_{\bar{q}}, 2_q, 4_g, 5_g, 3_g) =$



Tensor integral reduction

traditional approach: (improved) Passarino-Veltman reduction

[Ellis,Giele,Kunszt]

OPP algorithm:
[Ossola,Papadopoulos,Pittau] 2006

- meshes very well with the cut-based unitarity method
- a tensor integral reduction at the integrand level

$$\mathcal{A}^{\text{prim}} = \int \mathrm{d}^D \ell \, \boxed{\frac{\text{Num}(\ell, \{p_i\})}{D_1 ... D_N}}$$

$$D_k = (\ell + p_k)^2 - m_k^2$$

Tensor integral reduction

traditional approach: (improved) Passarino-Veltman reduction

OPP algorithm:
[Ossola,Papadopoulos,Pittau] 2006

[Ellis,Giele,Kunszt]

- meshes very well with the cut-based unitarity method
- a tensor integral reduction at the integrand level

$$\mathcal{A}^{\mathrm{prim}} = \int \mathrm{d}^D \ell \, \boxed{\frac{\mathrm{Num}(\ell, \{p_i\})}{D_1 ... D_N}} \qquad D_k = (\ell + p_k)^2 - m_k^2$$

for the moment:

OPP algorithm

1. partial fractioning:

master formula

$$\frac{\mathrm{Num}(\ell,\{p_i\})}{D_1 \ldots D_N} = \sum_{[i|j|k|l]} \frac{\bar{d}_{ijkl}(\ell)}{D_i D_j D_k D_l} + \sum_{[i|j|k]} \frac{\bar{c}_{ijk}(\ell)}{D_i D_j D_k} + \sum_{[i|j]} \frac{\bar{b}_{ij}(\ell)}{D_i D_j} + \sum_{[i]} \frac{\bar{a}_i(\ell)}{D_i}$$

2. choose vector basis that spans *physical space* $\{p_i\}$ + *transverse space* $\{n_i\}$ and express all momenta in this basis,

⇨ coefficients can be decomposed into *integral coefficients* and *spurious terms*

$$\bar{c}(\ell) = c + \tilde{c}_{1..r} \times \underbrace{\left(n_1^{\mu_1} \ldots n_r^{\mu_r} \ell_{\mu_1} \ldots \ell_{\mu_r}\right)}_{\text{vanish after loop integration}} \qquad \boxed{n_i \cdot p_j = 0}$$

$$c \;\rightarrow\; c \times \int \frac{\mathrm{d}^D \ell}{D_1 D_2 D_3}$$

OPP algorithm

1. partial fractioning:

master formula

$$\frac{\mathrm{Num}(\ell, \{p_i\})}{D_1 ... D_N} = \sum_{[i|j|k|l]} \frac{\bar{d}_{ijkl}(\ell)}{D_i D_j D_k D_l} + \sum_{[i|j|k]} \frac{\bar{c}_{ijk}(\ell)}{D_i D_j D_k} + \sum_{[i|j]} \frac{\bar{b}_{ij}(\ell)}{D_i D_j} + \sum_{[i]} \frac{\bar{a}_i(\ell)}{D_i}$$

3. assuming that we know the LHS, i.e. $\mathrm{Num}(\ell, \{p_i\})$,
we can evaluate this equation for different values of ℓ, so that the determination of the coefficients is a purely algebraic problem

OPP algorithm

1. partial fractioning:

master formula

$$\frac{\mathrm{Num}(\ell,\{p_i\})}{D_1 \dots D_N} = \sum_{[i|j|k|l]} \frac{\bar{d}_{ijkl}(\ell)}{D_i D_j D_k D_l} + \sum_{[i|j|k]} \frac{\bar{c}_{ijk}(\ell)}{D_i D_j D_k} + \sum_{[i|j]} \frac{\bar{b}_{ij}(\ell)}{D_i D_j} + \sum_{[i]} \frac{\bar{a}_i(\ell)}{D_i}$$

3. assuming that we know the LHS, i.e. $\mathrm{Num}(\ell,\{p_i\})$,
we can evaluate this equation for different values of ℓ , so that
the determination of the coefficients is a purely algebraic problem

4. OPP tell us that we should choose ℓ such that some denominators vanish
⇨ this leads to an efficient recursive determination of the coefficients

OPP algorithm

Example: 4 particle process

master formula for N=4

$$\frac{\mathrm{Num}(\ell,\{p_i\})}{D_1 ... D_4} = \frac{\bar{d}_{1234}(\ell)}{D_1 D_2 D_3 D_4} + \sum_{1<i<j<k<4} \frac{\bar{c}_{ijk}(\ell)}{D_i D_j D_k} + \sum_{1<i<j<4} \frac{\bar{b}_{ij}(\ell)}{D_i D_j} + \sum_{1<i<4} \frac{\bar{a}_i(\ell)}{D_i}$$

Step 1: multiply by $D_1 D_2 D_3 D_4$ and choose ℓ such that $D_1 = D_2 = D_3 = D_4 = 0$

⇨ there are two complex solutions $\ell_\pm$

$$\boxed{\mathrm{Num}(\ell_\pm,\{p_i\}) = \bar{d}_{1234}(\ell_\pm)}$$

OPP algorithm

Example: 4 particle process

master formula for N=4

$$\frac{\mathrm{Num}(\ell,\{p_i\})}{D_1 ... D_4} = \frac{\bar{d}_{1234}(\ell)}{D_1 D_2 D_3 D_4} + \sum_{1<i<j<k<4} \frac{\bar{c}_{ijk}(\ell)}{D_i D_j D_k} + \sum_{1<i<j<4} \frac{\bar{b}_{ij}(\ell)}{D_i D_j} + \sum_{1<i<4} \frac{\bar{a}_i(\ell)}{D_i}$$

Step 1: multiply by $D_1 D_2 D_3 D_4$ and choose ℓ such that $D_1 = D_2 = D_3 = D_4 = 0$

⇨ there are two complex solutions $\ell_\pm$

$$\boxed{\mathrm{Num}(\ell_\pm,\{p_i\}) = \bar{d}_{1234}(\ell_\pm) = d_{1234} + \tilde{d}_{1234}(\ell)\ \ell.n_1}$$

⇨ evaluate equation for two solutions $\ell_\pm$ and solve for $d_{1234},\ \tilde{d}_{1234}$

OPP algorithm

Example: 4 particle process

Step 2:

$$\frac{\mathrm{Num}(\ell,\{p_i\})}{D_1 ... D_4} - \frac{\bar{d}_{1234}(\ell)}{D_1 D_2 D_3 D_4} =$$
$$\sum_{1<i<j<k<4} \frac{\bar{c}_{ijk}(\ell)}{D_i D_j D_k} + \sum_{1<i<j<4} \frac{\bar{b}_{ij}(\ell)}{D_i D_j} + \sum_{1<i<4} \frac{\bar{a}_i(\ell)}{D_i}$$

multiply by $D_1 D_2 D_3$ and choose ℓ such that $D_1 = D_2 = D_3 = 0$

⇨ there are infinite complex solutions for ℓ

$$\frac{\mathrm{Num}(\ell,\{p_i\})}{D_4} - \frac{\bar{d}_{1234}(\ell)}{D_4} = \bar{c}_{123}(\ell)$$

OPP algorithm

Example: 4 particle process

Step 2:

$$\frac{\mathrm{Num}(\ell,\{p_i\})}{D_1 ... D_4} - \frac{\bar{d}_{1234}(\ell)}{D_1 D_2 D_3 D_4} =$$

$$\sum_{1<i<j<k<4} \frac{\bar{c}_{ijk}(\ell)}{D_i D_j D_k} + \sum_{1<i<j<4} \frac{\bar{b}_{ij}(\ell)}{D_i D_j} + \sum_{1<i<4} \frac{\bar{a}_i(\ell)}{D_i}$$

multiply by $D_1 D_2 D_3$ and choose ℓ such that $D_1 = D_2 = D_3 = 0$

⇨ there are infinite complex solutions for ℓ

$$\frac{\mathrm{Num}(\ell,\{p_i\})}{D_4} - \frac{\bar{d}_{1234}(\ell)}{D_4} = \bar{c}_{123}(\ell) = c_{123}$$
$$+\tilde{c}^1_{123}\,(n_1.\ell) + \tilde{c}^2_{123}\,(n_2.\ell) + \tilde{c}^3_{123}\,(n_1.\ell)(n_2.\ell) + \tilde{c}^4_{123}\,(n_1.\ell)^2(n_2.\ell)$$
$$+\tilde{c}^5_{123}\,(n_1.\ell)(n_2.\ell)^2 + \tilde{c}^6_{123}\,((n_1.\ell)^2 - (n_2.\ell)^2)$$

OPP algorithm

Step 3...: solve for all coefficients recursively

Finally: take the loop integral over $\frac{\mathrm{Num}(\ell,\{p_i\})}{D_1 D_2 D_3 D_4}$

⇨ all spurious terms vanish

$$\mathcal{A}^{\mathrm{prim}} = \int \mathrm{d}^D\ell \, \frac{\mathrm{Num}(\ell,\{p_i\})}{D_1 ... D_4} = d_{1234} I^4_{1234} + \sum_{1<i<j<k<4} c_{ijk} I^3_{ijk} + \sum_{1<i<j<4} b_{ij} I^2_{ij} + \sum_{1<i<4} a_i I^1_i$$

- coefficients $d_{ijkl}, c_{ijk}, b_{ij}, a_i$ have been calculated
- scalar integrals $I^n_{i_1..i_n}$ can be taken from an integral library

QCD 1-loop package
[Ellis, Zanderighi]

OPP algorithm

Step 3...: solve for all coefficien... ursi...

Finally:

QUESTION:

How do we evaluate

$$\mathrm{Num}(\ell, \{p_i\})$$

in the reduction steps ?

$$\mathcal{A}^{\rm prim} = \ldots \sum_{\ldots<i<j<k<4} c_{ijk} I^3_{ijk} \ldots \sum_{1<i<j<4} b_{ij} I^2_{ij} + \sum_{1<i<4} a_i I^1_i$$

- coe...nts $d_{ijkl}, c_{ijk}, b_{ij}, a_i$ have been calculated
- ...alar integrals $I^n_{i_1..i_n}$ can be taken from an integral library

QCD 1-loop package
[Ellis, Zanderighi]

Unitarity

remember: we extract the coefficients by considering only those loop momenta for which certain sets of inverse propagators vanish

⇨ virtual particles go on-shell

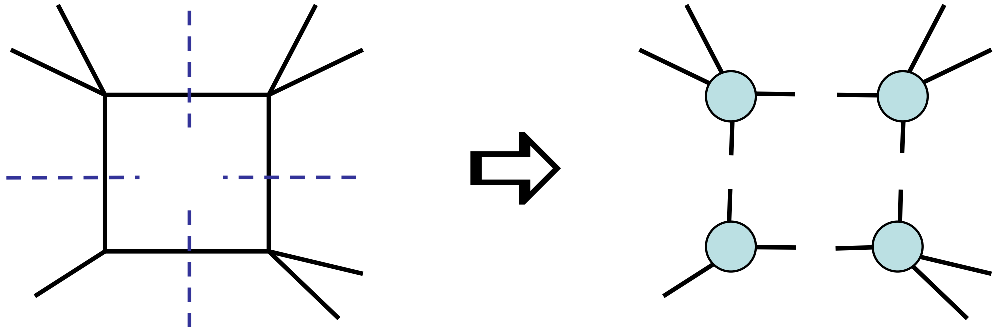
Unitarity

remember: we extract the coefficients by considering only those loop momenta for which certain sets of propagators functions vanish

⇨ virtual particles go on-shell

⇨ equivalent to unitarity cuts

this is where
OPP + Unitarity
mesh



$$\mathrm{Num}(\ell, \{p_i\}) \rightarrow \Pi_k A_k^{\mathrm{tree}}(\ell, \{p_i\})$$

Unitarity

remember: we extract the coefficients by considering only those

this is w

OPP +

remember Step 1:

$$\mathrm{Num}(\ell_\pm, \{p_i\}) = \bar{d}_{1234}(\ell_\pm) = d_{1234} + \tilde{d}_{1234}(\ell)$$

$$A_1^{\mathrm{tree}}(\ell, \{p_i\}) \times A_2^{\mathrm{tree}}(\ell, \{p_i\}) \times A_2^{\mathrm{tree}}(\ell, \{p_i\}) \times A_4^{\mathrm{tree}}(\ell, \{p_i\})$$

$$\mathrm{Num}(\ell, \{p_i\}) \to \Pi_k A_k^{\mathrm{tree}}(\ell, \{p_i\})$$

Unitarity

➪ **our basic ingredients are tree level amplitudes**
with complex on-shell momenta

D-Dimensional Unitarity

Rational terms

rational terms

the full truth: $$A^{1\text{-loop}} = \sum_j c_j I_j + \mathcal{R}$$

QCD needs regularization: $D = 4 - 2\varepsilon$ (Dim. Reg.)

rational terms: originate from ε -dependent terms in integral coefficients

Dim. Reg. requires ε to be a complex parameter

⇨ hard for numerical implementation

solution*:*

Using our knowledge about the particular structure of *D*-dependence of a one loop amplitude, we can construct copies of QCD in **integer-** dimensionsional (*D*>4) spaces to interpolate the result in $D = 4 - 2\varepsilon$.

D-dimensional unitarity

there are two sources of *D*-dependence

dimensionality of loop momentum	number of spin and polarization states

⇨ *D*-dep. from loop integration

extra dimensions in ℓ enter isotropically:

$$\mathcal{A}(\ell_D) = \mathcal{A}(\ell_4, \tilde{\ell}^2), \tilde{\ell}^2 = \sum_{i=5}^{D} \ell_i^2$$

⇨ loop momentum can be restricted to 5-dimensional subspace

⇨ • OPP needs to be extended to pentagon contributions
• some additional master integrals

⇨ *D*-dep. from contraction of metric tensors or gamma matrices

dependence on *D* is linear:

$$\mathcal{A}(D_s) = \mathcal{A}_0 + (D_s - 4)\mathcal{A}_1$$

⇨ evaluate at $D_s = 6$ and $D_s = 8$ to obtain the full D_s -dependence

D-dimensional unitarity

recap: **Shopping list for**

D-dimensional generalized unitarity

all we need: tree level amplitudes for complex on-shell momenta with

- spinors, gamma matrices and polarization vectors in $D_s = 6, 8$ dimensions
- loop momenta restricted to 5-dimensional subspace
- external momenta restricted to 4-dimensional subspace

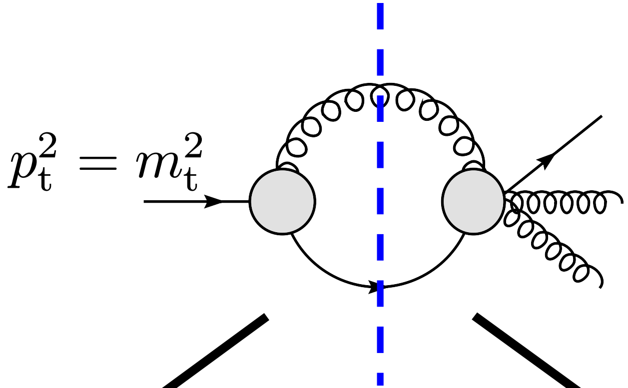
⇨ **this allows us to fully reconstruct a one-loop amplitude in dimensional regularized QCD**

Massive Fermions

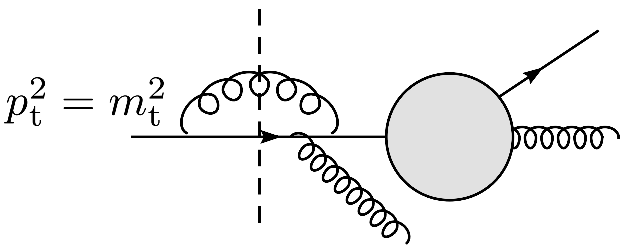
Unitarity and self-energy corrections on massive quarks lines:

(general issue for unitarity based methods!)

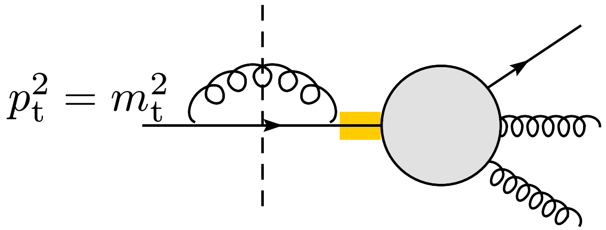
double cut:



regular contribution



external self-energy contribution!



leads to on-shell propagator in tree amplitude

solution: • discard terms that lead to those cuts
(truncate BG recurrence relations)

• add wave function renormalization constants δZ_t later

Massive Fermions

Implementation:

- *Rocket*-like `Fortran90` program:

$$0 \to \mathrm{t\bar{t}} + N \text{ gluons}$$

$$0 \to \mathrm{t\bar{t}} + q\bar{q} + N \text{ gluons}$$

at 1-loop QCD

(including N_f-terms with massive top quark loop)

- all one needs for: $\mathrm{pp} \to \mathrm{t\bar{t}}$, $\mathrm{pp} \to \mathrm{t\bar{t}} + \mathrm{jet}$
- fully numerical implementation:

  helicity amplitudes via Berends-Giele recurrence relations
- caching of Berends-Giele-currents
- control over numerical stability: switch to quadruple precision if necessary
- helicity formalism allows us to implement top decay matrix elements

Top Quark

Phenomenology

Top Quark Pheno

Top quark phenomenology is rich:

- large cross section
- top quark mass, spin, charge, branching fractions
- spin correlations, forward-backward asymmetry
- sensitive to new physics

Furthermore:

- top pairs are standard candles at LHC (constrain gluon pdf at large x)
- background to Higgs searches

Top Quark Pheno

$t\bar{t}$ production beyond leading order QCD:

first analytic calculation: [Nason, Dawson, Ellis, 1990] total $t\bar{t}$ production cross section
+ various threshold corrections, electroweak corrections,...
[Bernreuther, Brandenburg, Si, Uwer, 2001] production+decay

programs: MCFM: NLO $t\bar{t}$ production, no top decay
(LO $t\bar{t}$ production, LO top decay)

PowHeg, MC@NLO: NLO $t\bar{t}$ production, LO top decay

our program: NLO $t\bar{t}$ production and NLO leptonic decay

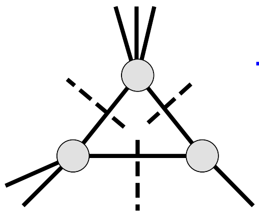
JHEP 0908:049, 2009
arXiv:0907.3090 [hep-ph]

flexible MC program:

- accounts for all spin correlations
- allows for arbitrary cuts

Top Quark Pheno

NLO $t\bar{t}$ production and NLO leptonic decay



top production: • virtual corrections using *D*-dimensional generalized unitarity

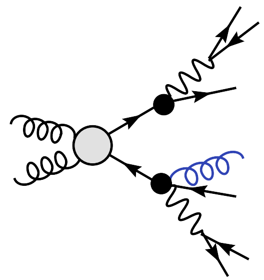
• real corrections using dipole subtraction method

leptonic top decay: • on-shell approximation, error $\mathcal{O}\left(\frac{\Gamma_t}{m_t}\right)$

• retain all spin correlations

• include virtual and real corrections to decay

• neglect non-factorizable contributions



Results:

Predictions for:
- Tevatron
- LHC @ 10TeV

Realistic final states: di-lepton final state

require two b-jets (k_{T}-clustering with $R = 0.4$)

Cuts:

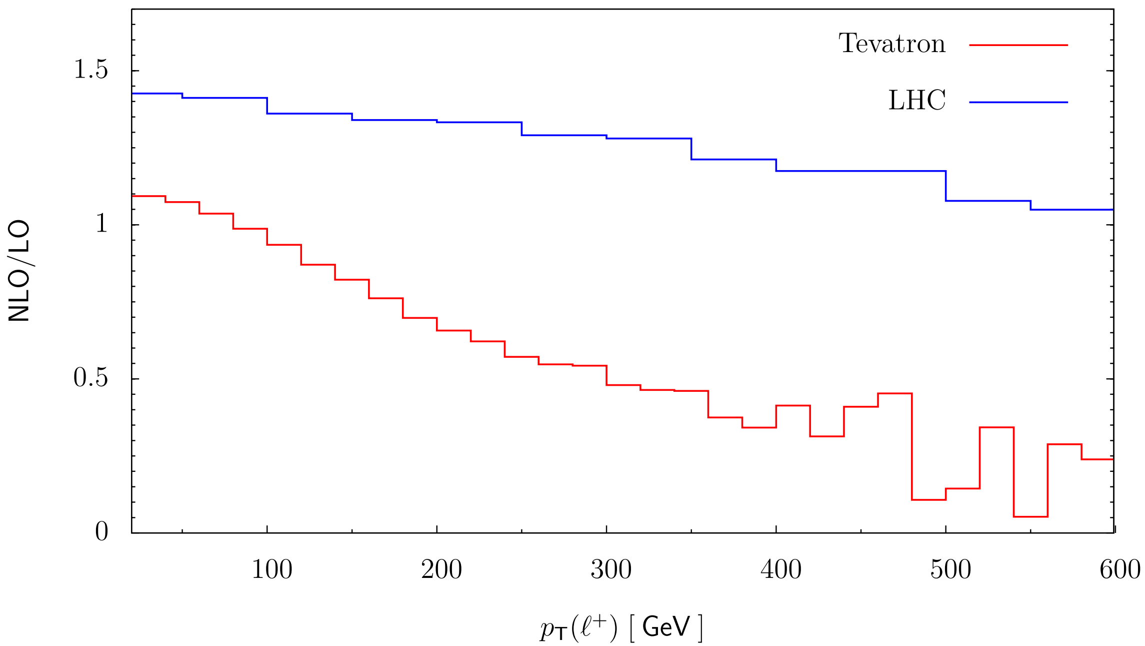
$$p_{\mathrm{T}}^{\mathrm{b-jet}} > 20 \ \mathrm{GeV}$$

$$p_{\mathrm{T}}^{\ell ep} > 20 \ \mathrm{GeV}$$

$$p_{\mathrm{T}}^{\mathrm{miss}} > 40 \ \mathrm{GeV}$$

Results:

K - factor



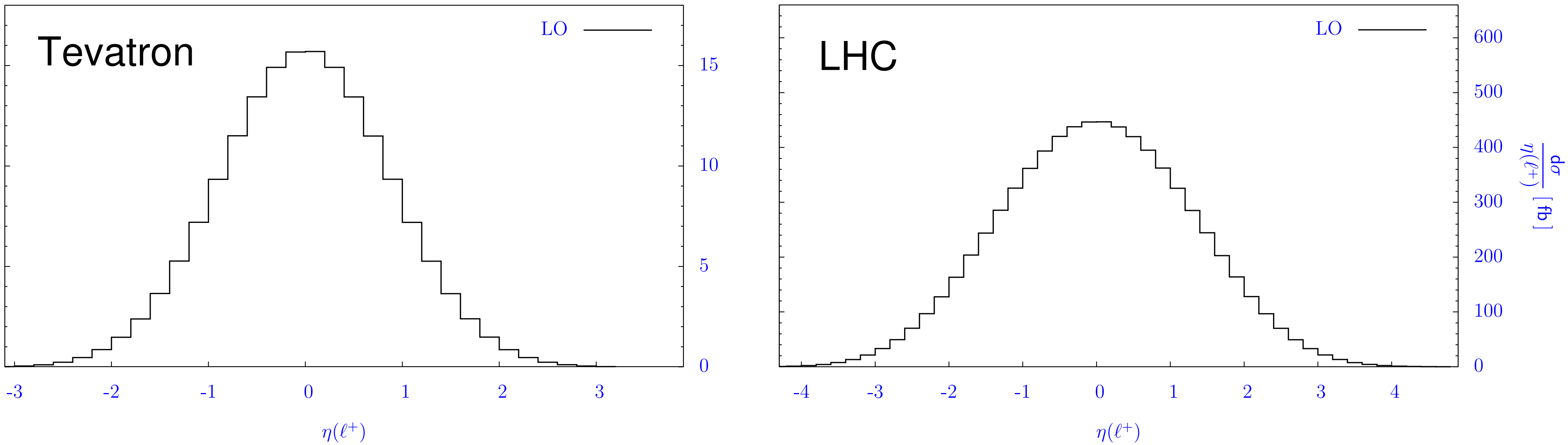
K-factors are not constant!

Tevatron: NLO change pT-distribution significantly

LHC: smaller change but non-negligible

Results:

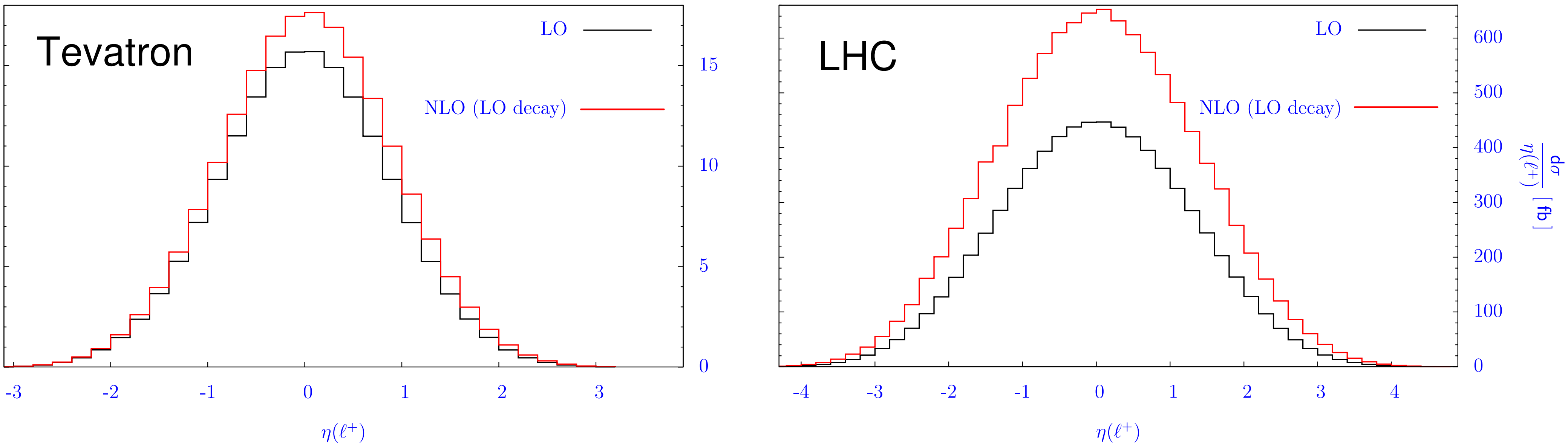
lepton rapidity distribution



- NLO corrections to rapidity distribution are important
- NLO correction to decay shifts rapidity distributions significantly

Results:

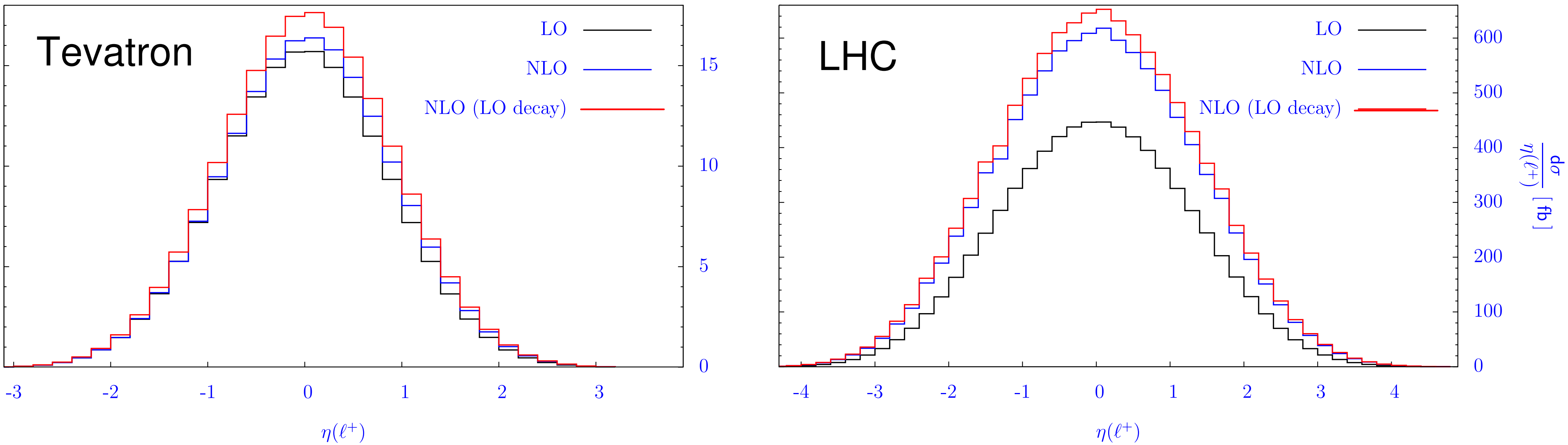
lepton rapidity distribution



- NLO corrections to rapidity distribution are important
- NLO correction to decay shifts rapidity distributions significantly

Results:

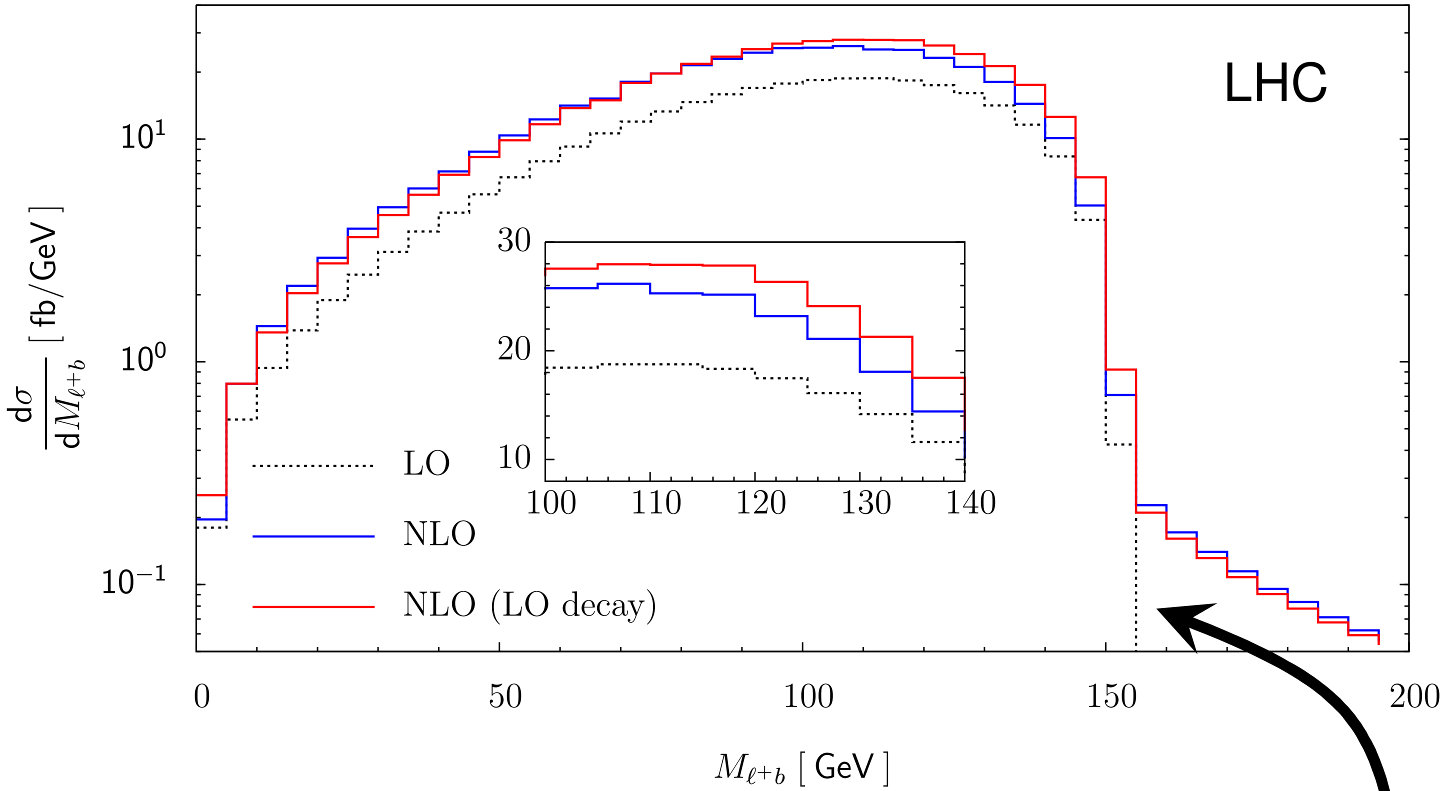
lepton rapidity distribution



- NLO corrections to rapidity distribution are important
- NLO correction to decay shifts rapidity distributions significantly

Results:

invariant mass of lepton and b-jet $M^2_{\ell^+b} = \big(p(\ell^+) + p(\text{b-jet})\big)^2$

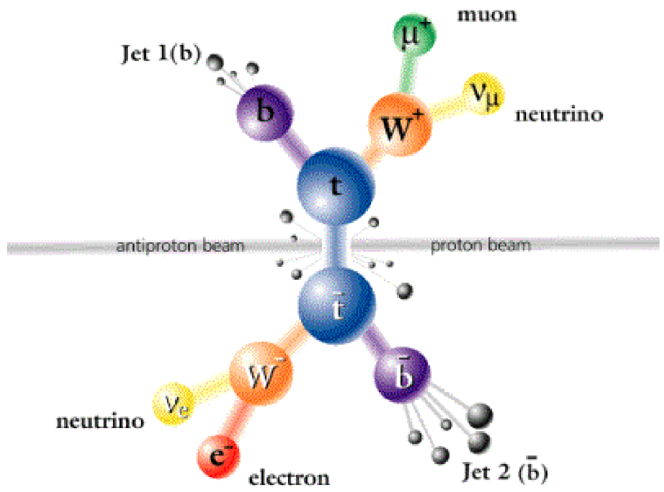


- boundary is top mass dependent
- spin studies for BSM particles
- NLO induces a tail

$$\max(M^2_{\ell^+b}) = m^2_{\text{top}} - m^2_W$$

Results:

spin correlations



- large mass and short life time prevent hadronization effects to wash out spin information

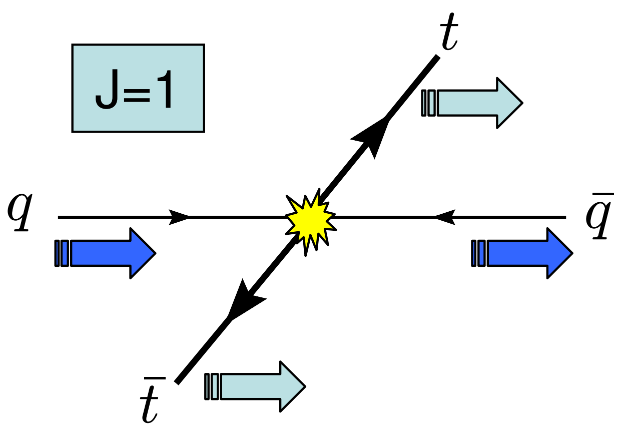
  ⇨ top spin correlations induced by production process are conserved

- spin correlations of top quarks are passed to decay products

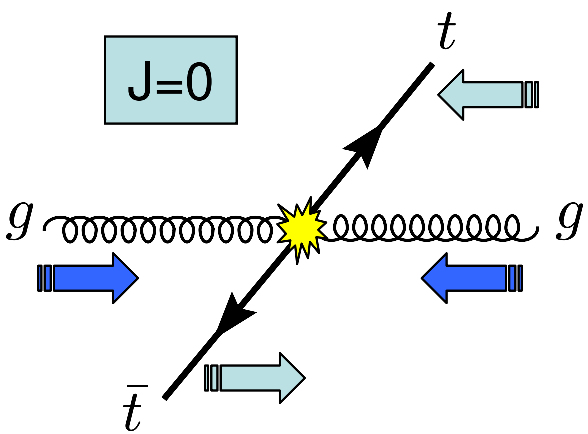
  ⇨ leptons prefer to fly parallel or anti-parallel wrt. each other

Results:

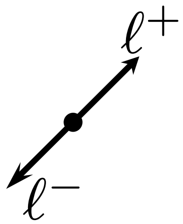
spin correlations



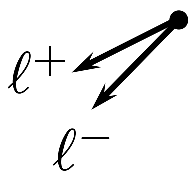
close to threshold:
S-wave production
(L=0)



⇨ leptons preferably anti-parallel



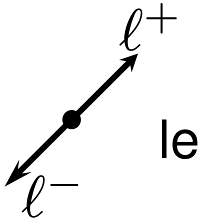
⇨ leptons preferably parallel



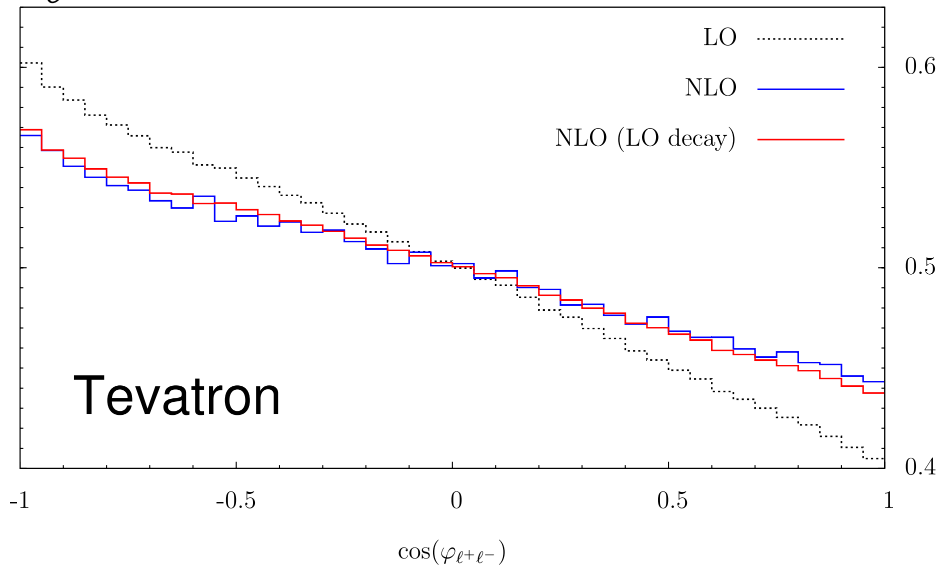
typical observable:

$$\frac{1}{\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}\cos(\varphi_{\ell^+\ell^-})}$$

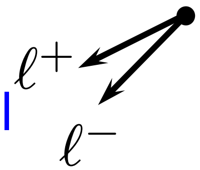
$\varphi_{\ell^+\ell^-}$: angle between the directions of flight of leptons in the corresponding ***top rest frame***

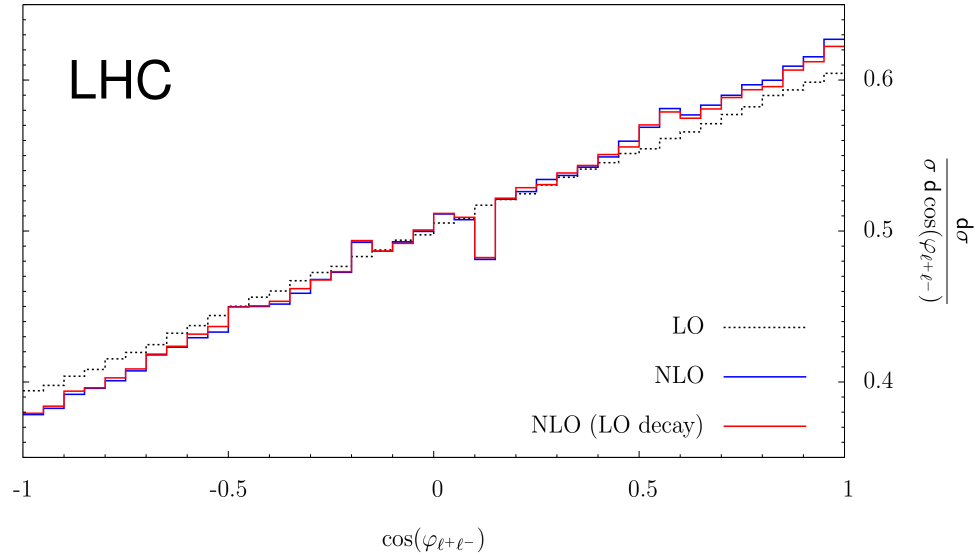


leptons preferably anti-parallel



leptons preferably parallel



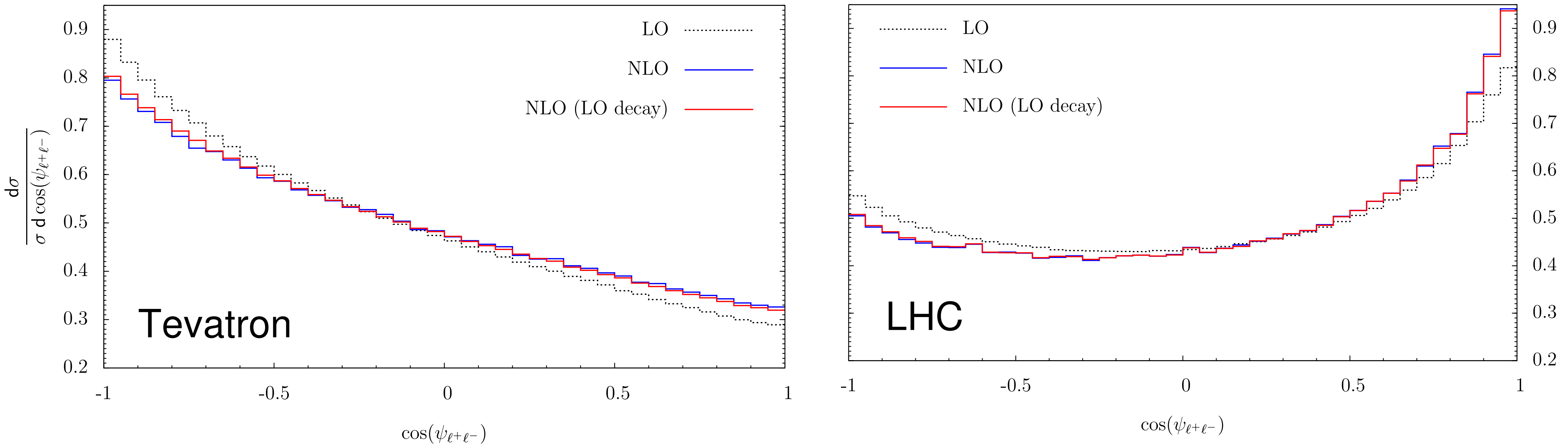


- substantial angular correlations, even at NLO
- NLO effects at Tevatron are significant

simpler observable:

$$\frac{1}{\sigma}\frac{d\sigma}{d\cos(\psi_{\ell^+\ell^-})}$$

$\psi_{\ell^+\ell^-}$: opening angle of the leptons in the ***laboratory frame***



- top quark rest frames need not to be reconstructed
- angular correlations remain, stronger NLO effects at LHC

Summary

D-dimensional generalized unitarity...

... is a robust and transparent method to calculate 1-loop corrections

... basic ingredients are on-shell tree amplitudes

... is ready for phenomenology

Top quark pair production

flexible MC program for NLO $t\bar{t}$ production and NLO leptonic decay

... accounts for all spin correlations

... interesting distributions sensitive to spin correlations

Extras

Giele, Kunszt, Melnikov
JHEP 0804:049,2008.
arXiv:0801.2237 [hep-ph]

dimensional space time, the number of spin eigenstates changes. For example, massless spin-one particles in D_s dimensions have $D_s - 2$ spin eigenstates while spinors in D_s dimensions have $2^{(D_s-2)/2}$ spin eigenstates. In the latter case, D_s should be even.

The spin density matrix for a massless spin-one particle with momentum l and polarization vectors $e_\mu^{(i)}$ is given by

$$\sum_{i=1}^{D_s-2} e_\mu^{(i)}(l) e_\nu^{(i)}(l) = -g_{\mu\nu}^{(D_s)} + \frac{l_\mu b_\nu + b_\mu l_\nu}{l \cdot b}, \tag{5}$$

where b_μ is an arbitrary light-cone gauge vector associated with a particular choice of polarization vectors. Similarly, the spin density matrix for a fermion with momentum l and mass m is given by

$$\sum_{i=1}^{2^{(D_s-2)/2}} u^{(i)}(l)\overline{u}^{(i)}(l) = \not{l} + m = \sum_{\mu=1}^{D} l_\mu \gamma^\mu + m \; . \tag{6}$$

While, as we see from these examples, the number of spin eigenstates depends explicitly on the space-time dimensionality, the loop-momentum l itself has implicit D-dependence. We can define the loop momentum as a D-dimensional vector, with the requirement $D \le D_s$ [35]. We now extend the notion of dimensional dependence of the one-loop scattering amplitude in Eq. (1) by taking the sources of all unobserved particles in D_s-dimensional space-time

$$\mathcal{A}_{(D,D_s)}(\{p_i\},\{J_i\}) = \int \frac{d^D\, l}{i(\pi)^{D/2}} \frac{\mathcal{N}^{(D_s)}(\{p_i\},\{J_i\};l)}{d_1 d_2 \cdots d_N}. \tag{7}$$

The numerator function $\mathcal{N}^{(D_s)}(\{p_i\},\{J_i\};l)$ depends explicitly on D_s through the number of spin eigenstates of virtual particles. However, the dependence of the numerator function on the loop momentum dimensionality D emerges in a peculiar way. Since external particles are kept in four dimensions, the dependence of the numerator function on $D-4$ components of the loop momentum l appears only through its dependence on l^2. Specifically

$$l^2 = \overline{l}^2 - \widetilde{l}^2 = l_1^2 - l_2^2 - l_3^2 - l_4^2 - \sum_{i=5}^{D} l_i^2 \; , \tag{8}$$

where $\overline{l}$ and $\widetilde{l}$ denote four- and $(D-4)$-dimensional components of the vector l. It is apparent from Eq. (8) that there is no preferred direction in the $(D-4)$-dimensional subspace of the D-dimensional loop momentum space.

A simple, but important observation is that in one-loop calculations, the dependence of scattering amplitudes on D_s is *linear*. This happens because, for such dependence to appear,

Giele, Kunszt, Melnikov
JHEP 0804:049,2008.
arXiv:0801.2237 [hep-ph]

we need to have a closed loop of contracted metric tensors and/or Dirac matrices coming from vertices and propagators. Since only a single loop can appear in one-loop calculations, we find

$$\mathcal{N}^{(D_s)}(l) = \mathcal{N}_0(l) + (D_s - 4)\mathcal{N}_1(l). \tag{9}$$

We emphasize that there is no explicit dependence on either D_s or D in functions $\mathcal{N}_{0,1}$.

For numerical calculations we need to separate the two functions $\mathcal{N}_{0,1}$. To do so, we compute the left hand side of Eq. (9) for $D_s = D_1$ and $D_s = D_2$ and, after taking appropriate linear combinations, obtain

$$\begin{aligned}\mathcal{N}_0(l) &= \frac{(D_2-4)\mathcal{N}^{(D_1)}(l) - (D_1-4)\mathcal{N}^{(D_2)}(l)}{D_2 - D_1},\\ \mathcal{N}_1(l) &= \frac{\mathcal{N}^{(D_1)}(l) - \mathcal{N}^{(D_2)}(l)}{D_2 - D_1}.\end{aligned} \tag{10}$$

Because both D_1 and D_2 are integers, amplitudes are numerically well-defined. We will comment more on possible choices of $D_{1,2}$ in the forthcoming sections; here suffice it to say that if fermions are present in the loop, we have to choose *even* D_1 and D_2.

Having established the D_s-dependence of the amplitude, we discuss analytic continuation for sources of unobserved particles. We can interpolate D_s either to $D_s \to 4-2\epsilon$ (the t'Hooft-Veltman (HV) scheme) [34] or to $D_s \to 4$ (the four-dimensional helicity (FDH) scheme) [35]. The latter scheme is of particular interest in supersymmetric (SUSY) calculations since all SUSY Ward identities are preserved. We see from Eq. (9) that the difference between the two schemes is simply $-2\epsilon\mathcal{N}_1$.

We now substitute Eq.(10) into Eq. (7). Upon doing so, we obtain explicit expressions for one-loop amplitudes in HV and FDH schemes. We derive

$$\begin{aligned}\mathcal{A}^{\rm FDH} &= \left(\frac{D_2-4}{D_2-D_1}\right)\mathcal{A}_{(D,D_s=D_1)} - \left(\frac{D_1-4}{D_2-D_1}\right)\mathcal{A}_{(D,D_s=D_2)},\\ \mathcal{A}^{\rm HV} &= \mathcal{A}^{\rm FDH} - \left(\frac{2\epsilon}{D_2-D_1}\right)\left(\mathcal{A}_{(D,D_s=D_1)} - \mathcal{A}_{(D,D_s=D_2)}\right).\end{aligned} \tag{11}$$

We emphasize that $D_s = D_{1,2}$ amplitudes on the r.h.s. of Eq. (11) are conventional one-loop scattering amplitudes whose numerator functions are computed in higher-dimensional space-time, i.e. all internal metric tensors and Dirac gamma matrices are in integer $D_s = D_{1,2}$ dimensions. The loop integration is in $D \leq D_s$ dimensions. It is important that explicit dependence on the regularization parameter $\epsilon = (4-D)/2$ is not present in these amplitudes.

Giele, Kunszt, Melnikov
JHEP 0804:049,2008.
arXiv:0801.2237 [hep-ph]

function. Hence the integrand of the N-particle amplitude in Eq. (1) can be parameterized as

$$\frac{\mathcal{N}^{(D_s)}(l)}{d_1 d_2 \cdots d_N} = \sum_{[i_1|i_5]} \frac{\overline{e}^{(D_s)}_{i_1 i_2 i_3 i_4 i_5}(l)}{d_{i_1} d_{i_2} d_{i_3} d_{i_4} d_{i_5}} + \sum_{[i_1|i_4]} \frac{\overline{d}^{(D_s)}_{i_1 i_2 i_3 i_4}(l)}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} + \sum_{[i_1|i_3]} \frac{\overline{c}^{(D_s)}_{i_1 i_2 i_3}(l)}{d_{i_1} d_{i_2} d_{i_3}} + \sum_{[i_1|i_2]} \frac{\overline{b}^{(D_s)}_{i_1 i_2}(l)}{d_{i_1} d_{i_2}} + \sum_{[i_1|i_1]} \frac{\overline{a}^{(D_s)}_{i_1}(l)}{d_{i_1}} . \tag{13}$$

where the dependence on the external momenta and sources are suppressed. From four-dimensional unitarity we know that computation of each cut of the scattering amplitude is simplified if convenient parameterization of the residue is chosen. We now discuss how these parameterizations change when D_s-dimensional unitarity cuts are considered.

A. Pentuple residue

To calculate the pentuple residue, we choose momentum l such that five inverse propagators in Eq. (13) vanish. We define

$$\overline{e}^{(D_s)}_{ijkmn}(l_{ijkmn}) = \mathrm{Res}_{ijkmn}\left(\frac{\mathcal{N}^{(D_s)}(l)}{d_1 \cdots d_N}\right) . \tag{14}$$

The momentum l_{ijkmn} satisfies the following set of equations $d_i(l_{ijkmn}) = \cdots = d_n(l_{ijkmn}) = 0$. The solution is given by

$$l^{\mu}_{ijkmn} = V_5^{\mu} + \sqrt{\frac{-V_5^2 + m_n^2}{\alpha_5^2 + \cdots + \alpha_D^2}} \left(\sum_{h=5}^{D} \alpha_h n_h^{\mu}\right) , \tag{15}$$

where m_n is the mass in the propagator d_n which is chosen to be as $d_n = l^2 - m_n^2$ by adjusting the reference vector q_0. The parameters α_h can be chosen freely. The four-dimensional vector V_5^{μ} depends only on external momenta and propagator masses. It is explicitly constructed using the Vermaseren-van Neerven basis as outlined in Ref. [24]. The $D-4$ components of the vector l_{ijkmn} are necessarily non-vanishing; for simplicity we may choose l_{ijkmn} to be five-dimensional, independent of D_s. We will see below that this is sufficient to determine pentuple residue.

To restrict the functional form of the pentuple residue $\overline{e}_{ijkmn}(l)$ we apply the same reasoning as in four-dimensional unitarity case, supplemented with the requirement that $\overline{e}^{(D_s)}_{ijkmn}(l)$

Giele, Kunszt, Melnikov
JHEP 0804:049,2008.
arXiv:0801.2237 [hep-ph]

depends only on even powers of s_e; this requirement is a necessary consequence of the discussion around Eq. (8). These considerations lead to the conclusion that the pentuple residue is independent of the loop momentum

$$\overline{e}_{ijkmn}^{(D_s)}(l) = e_{ijkmn}^{(D_s,(0))}. \tag{16}$$

To calculate $e^{(0)}$ in the FDH scheme, we employ Eq. (10) and obtain

$$e_{ijkmn}^{(0),\mathrm{FDH}} = \left(\frac{D_2-4}{D_2-D_1}\right)\mathrm{Res}_{ijkmn}\left(\frac{\mathcal{N}^{(D_1)}(l)}{d_1\cdots d_N}\right) - \left(\frac{D_1-4}{D_2-D_1}\right)\mathrm{Res}_{ijkmn}\left(\frac{\mathcal{N}^{(D_2)}(l)}{d_1\cdots d_N}\right). \tag{17}$$

The calculation of the residues of the amplitude on the r.h.s. of Eq. (17), is simplified by their factorization into products of tree amplitudes

$$\mathrm{Res}_{ijkmn}\left(\frac{\mathcal{N}^{(D_s)}(l)}{d_1\cdots d_N}\right) = \sum \mathcal{M}(l_i;p_{i+1},\ldots,p_j;-l_j)\times\mathcal{M}(l_j;p_{j+1},\ldots,p_k;-l_k) \tag{18}$$
$$\times\mathcal{M}(l_k;p_{k+1},\ldots,p_m;-l_m)\times\mathcal{M}(l_m;p_{m+1},\ldots,p_n;-l_n)\times\mathcal{M}(l_n;p_{n+1}\ldots,p_i;-l_i).$$

Here, the summation is over all different quantum numbers of the cut lines. In particular, we have to sum over polarization vectors of the cut lines. This generates explicit D_s dependence of the residue, as described in the previous section. Note that the complex momenta $l_h^\mu = l^\mu + q_h^\mu$ are on-shell due to the unitarity constraint $d_h = 0$.

B. Quadrupole residue

The construction of the quadrupole residue follows the discussion of the previous subsection and generalizes the four-dimensional case studied in [23, 24]. We define

$$\overline{d}_{ijkn}^{(D_s)}(l) = \mathrm{Res}_{ijkn}\left(\frac{\mathcal{N}^{(D_s)}(l)}{d_1\cdots d_N} - \sum_{[i_1|i_5]}\frac{e_{i_1i_2i_3i_4i_5}^{(D_s,(0))}}{d_{i_1}d_{i_2}d_{i_3}d_{i_4}d_{i_5}}\right), \tag{19}$$

where the last term in the r.h.s. is the necessary subtraction of the pentuple cut contribution. We now specialize to the FDH scheme. In this case, the most general parameterization of the quadrupole cut is given by

$$\overline{d}_{ijkn}^{\mathrm{FDH}}(l) = d_{ijkn}^{(0)} + d_{ijkn}^{(1)}s_1 + (d_{ijkn}^{(2)} + d_{ijkn}^{(3)}s_1)s_e^2 + d_{ijkn}^{(4)}s_e^4, \tag{20}$$

where $s_1 = l\cdot n_1$. We used the fact that, in renormalizable quantum field theories, the highest rank of a tensor integral that may contribute to a quadrupole residue is four and

Giele, Kunszt, Melnikov
JHEP 0804:049,2008.
arXiv:0801.2237 [hep-ph]

find

$$\int \frac{\mathrm{d}^D l}{(i\pi)^{D/2}} \frac{s_e^2}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} = -\frac{D-4}{2} I^{D+2}_{i_1 i_2 i_3 i_4},$$
$$\int \frac{\mathrm{d}^D l}{(i\pi)^{D/2}} \frac{s_e^4}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} = \frac{(D-2)(D-4)}{4} I^{D+4}_{i_1 i_2 i_3 i_4},$$
$$\int \frac{\mathrm{d}^D l}{(i\pi)^{D/2}} \frac{s_e^2}{d_{i_1} d_{i_2} d_{i_3}} = -\frac{(D-4)}{2} I^{D+2}_{i_1 i_2 i_3},$$
$$\int \frac{\mathrm{d}^D l}{(i\pi)^{D/2}} \frac{s_e^2}{d_{i_1} d_{i_2}} = -\frac{(D-4)}{2} I^{D+2}_{i_1 i_2}. \tag{26}$$

$$s_e^2 = (\tilde{l} \cdot n_5)^2$$

Using Eq. (26), we arrive at the following representation of the scattering amplitude

$$\begin{aligned}\mathcal{A}_{(D)} &= \sum_{[i_1|i_5]} e^{(0)}_{i_1 i_2 i_3 i_4 i_5}\; I^{(D)}_{i_1 i_2 i_3 i_4 i_5} \\ &+ \sum_{[i_1|i_4]} \left(d^{(0)}_{i_1 i_2 i_3 i_4}\; I^{(D)}_{i_1 i_2 i_3 i_4} - \frac{D-4}{2} d^{(2)}_{i_1 i_2 i_3 i_4}\; I^{(D+2)}_{i_1 i_2 i_3 i_4} + \frac{(D-4)(D-2)}{4} d^{(4)}_{i_1 i_2 i_3 i_4}\; I^{(D+4)}_{i_1 i_2 i_3 i_4} \right) \\ &+ \sum_{[i_1|i_3]} \left(c^{(0)}_{i_1 i_2 i_3}\; I^{(D)}_{i_1 i_2 i_3} - \frac{D-4}{2} c^{(9)}_{i_1 i_2 i_3}\; I^{(D+2)}_{i_1 i_2 i_3} \right) \\ &+ \sum_{[i_1|i_2]} \left(b^{(0)}_{i_1 i_2}\; I^{(D)}_{i_1 i_2} - \frac{D-4}{2} b^{(9)}_{i_1 i_2}\; I^{(D+2)}_{i_1 i_2} \right) + \sum_{i_1=1}^{N} a^{(0)}_{i_1}\; I^{(D)}_{i_1}\;. \end{aligned} \tag{27}$$

We emphasize that the explicit D-dependence on the r.h.s. of Eq. (27) is the consequence of our choice of the basis for master integrals in Eq. (26).

We note that the above decomposition is valid for any value of D. We can now interpolate the loop integration dimension D to $D \to 4 - 2\epsilon$. The extended basis of master integrals that we employ provides a clear separation between cut-constructible and rational parts of the amplitude. The cut-constructible part is given by the integrals in D-dimensions in Eq. (27), while the rational part is given by the integrals in $D+2$ and $D+4$ dimensions. However, it is possible to use smaller basis of master integrals by rewriting integrals $\{I^{(D+4)}_{i_1 i_2 i_3 i_4}, I^{(D+2)}_{i_1 i_2 i_3 i_4}, I^{(D+2)}_{i_1 i_2 i_3}, I^{(D+2)}_{i_1 i_2}\}$ in terms of $\{I^{(D)}_{i_1 i_2 i_3 i_4}, I^{(D)}_{i_1 i_2 i_3}, I^{(D)}_{i_1 i_2}\}$ using the integration-by-parts techniques.

Since we are interested in NLO computations, we only need to consider the limit $\epsilon \to 0$ in Eq. (27) and neglect contributions of order ϵ. This leads to certain simplifications. First, in this limit, we can re-write the scalar 5-point master integral as a linear combination of four-point master integrals up to $\mathcal{O}(\epsilon)$ terms. If we employ this fact in Eq. (27), we obtain

$$\lim_{D \to 4-2\epsilon} \left(\sum_{[i_1|i_5]} e^{(0)}_{i_1 \cdots i_5}\; I^{(D)}_{i_1 \cdots i_5} + \sum_{[i_1|i_4]} d^{(0)}_{i_1 \cdots i_4}\; I^{(D)}_{i_1 \cdots i_4} \right) = \sum_{[i_1|i_4]} \tilde{d}^{(0)}_{i_1 \cdots i_4}\; I^{(4-2\epsilon)}_{i_1 \cdots i_4} + \mathcal{O}(\epsilon)\;. \tag{28}$$

	$\mathrm{pp} \to \mathrm{t\bar{t}} + X$		$\mathrm{pp} \to \mathrm{t\bar{t}} + \mathrm{jet} + X$	
partonic channels:	LO + 1-loop: $gg, q\bar{q}$		LO + 1-loop: $gg, q\bar{q}, gq, g\bar{q}$	
1-loop topologies:	$0 \to \mathrm{t\bar{t}}gg$	$0 \to \mathrm{t\bar{t}}q\bar{q}$	$0 \to \mathrm{t\bar{t}}ggg$	$0 \to \mathrm{t\bar{t}}q\bar{q}g$
diagrams:	*31*	*10*	*354*	*94*
primitive ampl.:	*8*	*5*	*36*	*18*
evaluation time /prim.ampl./helicity `[Intel Xeon 2.8GHz]`	*3-5* msec		*10-40* msec	

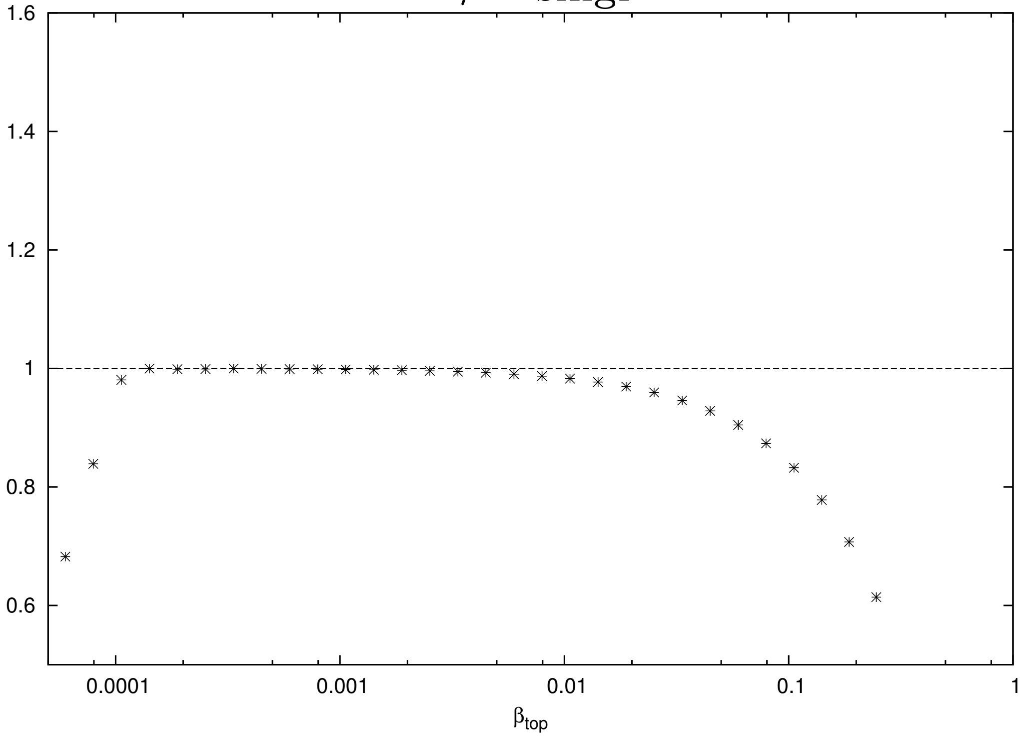
$0 \to \mathrm{t\bar{t}}gggg$: ≈*300* msec

Numerical stability:

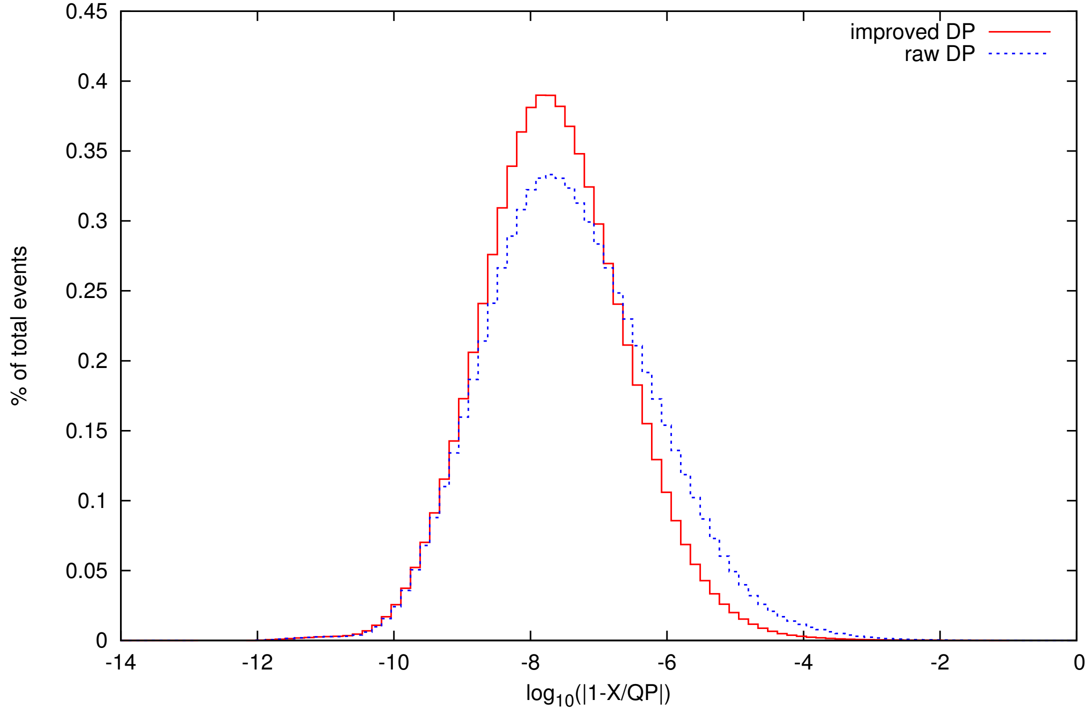
$\mathrm{t\bar{t}}:$
- no stability issues
- checked threshold effects

$\mathrm{t\bar{t} + jet}:$
- switch to quadruple precision if necessary

$$A^{\text{1-loop}} / A^{\text{soft}}_{\text{singl}}$$



$$A^{\text{finite}}_{\text{L}}(1_{\bar{t}}, 4_g, 3_g, 5_g, 2_t)$$



$$A^{\text{soft}}_{\text{singl}} = \frac{\alpha_s \pi}{2} A^{\text{tree}} \frac{C_F}{\beta}, \qquad \beta = \sqrt{1 - 4m^2_{\text{top}}/\hat{s}}$$