Charged Current Deep Inelastic scattering at three loops

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Content

- Introduction to the Deep Inelastic Scattering

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- Experiments

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- Calculation of the third order QCD corrections

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- Experiments
- Calculation of the third order QCD corrections
- Outlook: application to $N = 4$ Super Yang-Mills theory

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- Deep-inelastic lepton-hadron scattering ($e^{\pm}p$, $e^{\pm}n$, νp, $\bar{\nu}p$, ... - collisions)

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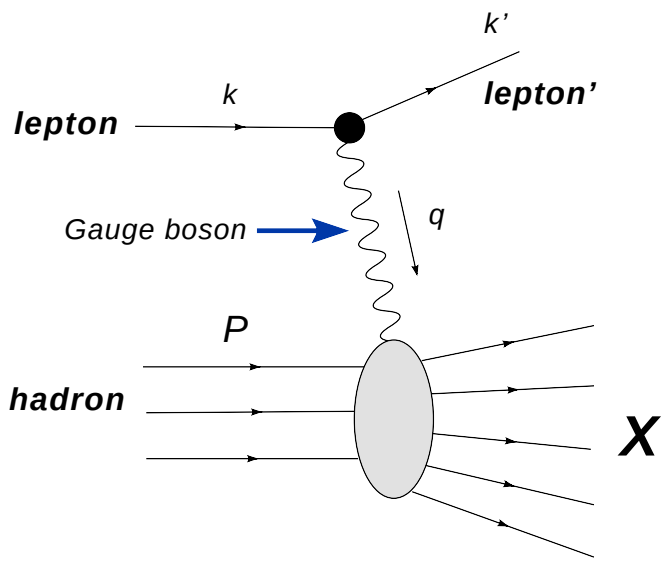

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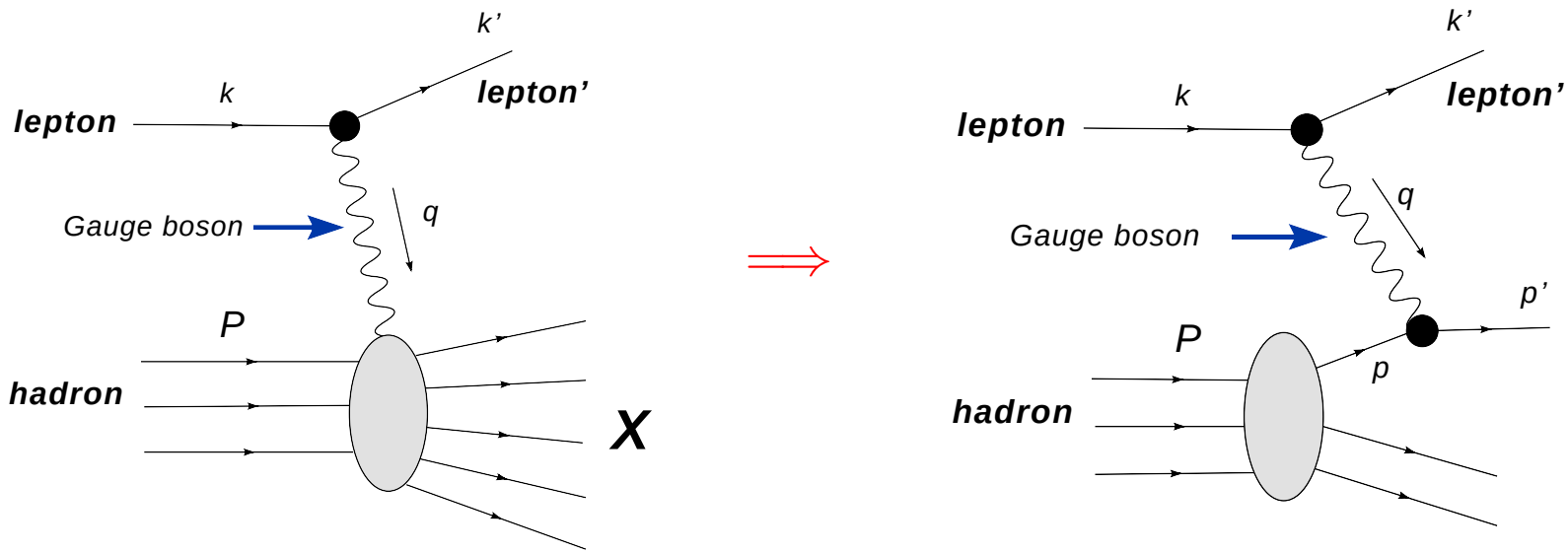

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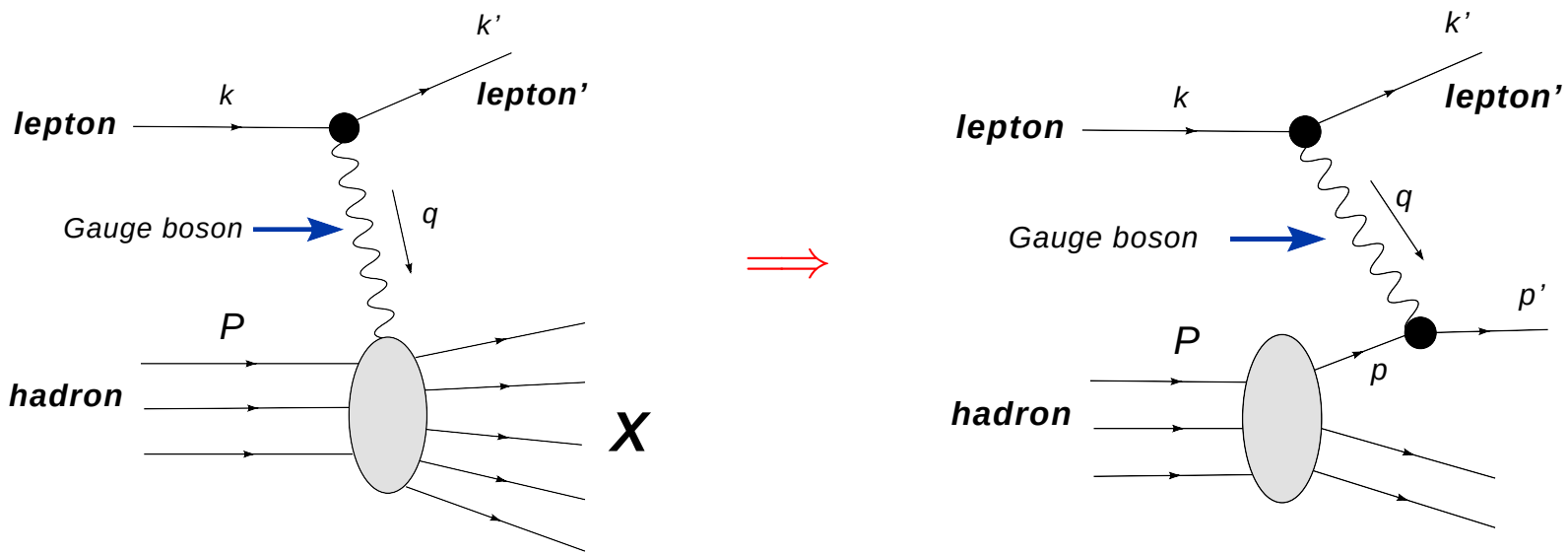


- *Gauge boson*:

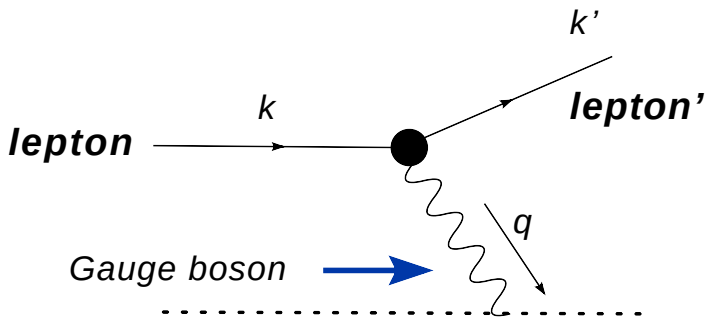
  γ, Z^0 - NC

  $W^{\pm}$ - CC

- **Kinematic variables**
  - momentum transfer $Q^2 = -q^2 > 0$
  - Bjorken variable $x = Q^2/(2P \cdot q)$
  - Inelasticity $y = (P \cdot q)/(P \cdot k)$

Leptonic tensor

$$d\sigma \sim L^{\mu\nu} W_{\mu\nu}$$



$\Rightarrow$ $L^{\mu\nu}$-leptonic tensor

$$L^{\mu\nu} = A \times \left(k^\mu k'^\nu + k^\nu k'^\mu - k \cdot k' g^{\mu\nu}\right) + B \times \left(\mathrm{i}\epsilon^{\mu\nu\alpha\beta} k^\alpha k'^\beta\right)$$

- Coefficients A and B are real and depend on the process.

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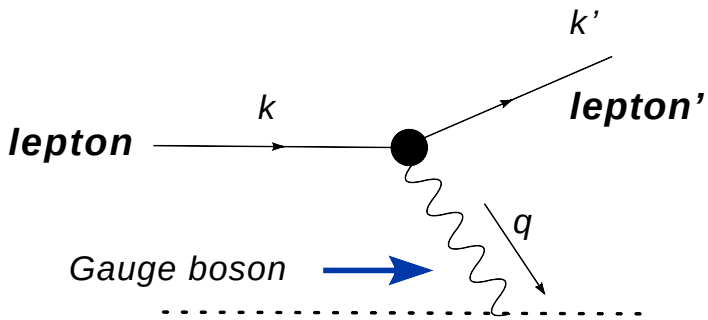

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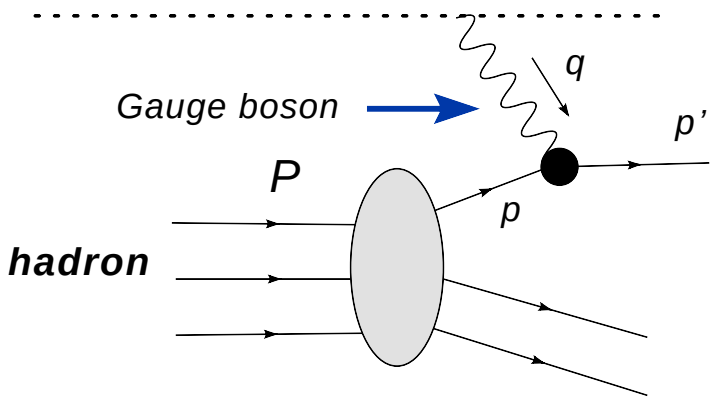
- Coefficients A and B are real and depend on the process.
- $B \neq 0$ for NC with Z^0, CC or polarized processes (e.g., $e^\pm$ at HERA)

Hadronic tensor



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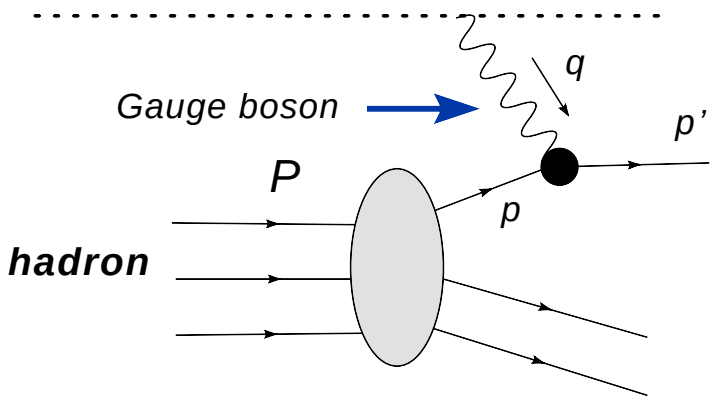

$\Rightarrow$ $W_{\mu\nu}$-hadronic tensor

$$W_{\mu\nu} = e_{\mu\nu}\frac{1}{2x}F_L(x,Q^2) + d_{\mu\nu}\frac{1}{2x}F_2(x,Q^2) + \mathrm{i}\epsilon_{\mu\nu\alpha\beta}\frac{P^\alpha q^\beta}{P\cdot q}F_3(x,Q^2)$$

- $e_{\mu\nu}$, $d_{\mu\nu}$ - tensors, depend on P, q. symmetric under $\mu \leftrightarrow \nu$

Structure functions

$$F_{2,L,3}(x,Q^2) = \sum_{p\,(\text{partons})} e_p^2 \int_x^1 dz\, \hat{f}_p\left(\frac{x}{z}\right)\, \hat{F}^p_{2,L,3}(z,Q^2)$$

$$\equiv \sum_p e_p^2 \hat{f}_p\,(y) \otimes \hat{F}^p_{2,L,3}(z,Q^2),\;\; z = Q^2/(2p\cdot q)$$

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- Naive Parton Model: partons (quarks) are non-interacting point particles
  - e.g. pure electromagnetism

$$\hat{F}^p_2(z,Q^2) = \delta(1-z) \Rightarrow F_2(x,Q^2) = \textstyle\sum_p e_p^2\; \hat{f}_p\,(x)$$

- Neutral Current $e^{\pm}p$ cross section

$$\frac{d^2\sigma^{NC}(e^{\pm}p)}{dxdQ^2} = \frac{2\pi\alpha^2}{xQ^4}[(1+(1-y)^2)F_2 - y^2 F_L \mp (1-(1-y)^2)xF_3]$$

$$F_2 = \sum_q A_q\, x\,(q+\bar{q}) \qquad \text{with} \qquad A_q = e_{q^2} + \mathcal{O}\left(\frac{Q^2}{Q^2+M_Z^2}\right)$$

$$F_3 = \sum_q B_q\, x\,(q-\bar{q}) \qquad \text{with} \qquad B_q = \mathcal{O}\left(\frac{Q^2}{Q^2+M_Z^2}\right)$$

- Charged Current $e^{\pm}p$ cross section $\longrightarrow$ flavour separation:
  - $$\frac{d^2\sigma^{CC}(e^{+}p)}{dxdQ^2} = \frac{G_F^2}{2\pi}\left(\frac{M_W^2}{M_W^2+Q^2}\right)^2 [\bar{u}+\bar{c}+(1-y)^2(d+s)]$$
  - $$\frac{d^2\sigma^{CC}(e^{-}p)}{dxdQ^2} = \frac{G_F^2}{2\pi}\left(\frac{M_W^2}{M_W^2+Q^2}\right)^2 [u+c+(1-y)^2(\bar{d}+\bar{s})]$$

QCD improved parton model

$$F_2(x,Q^2) = \sum_{p\,(\text{partons}\,-q,\bar{q},g)} \left[PDF_p\left(\alpha_s(Q^2),Q^2\right) \otimes C_{2,p}\left(\alpha_s(Q^2)\right)\right](x)$$

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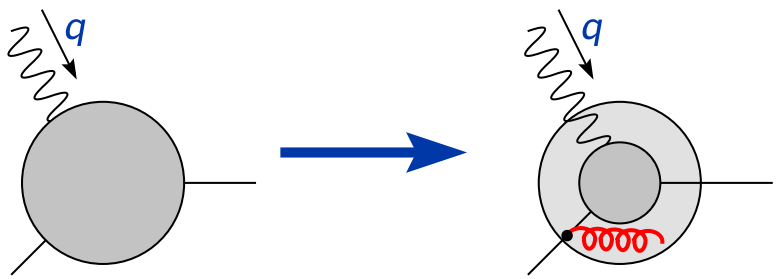
- $C_{2,p}$ - Wilson coefficients, calculable in perturbative QCD
- PDF_p - Parton Distribution Functions
  - NOT calculable in p.QCD.
  - Extracted from data [HERA, Tevatron, fixed target exp.]
  - Evolution of PDF's is calculable in p.QCD from evolution equation

$$\frac{d}{d\ln Q^2} PDF_{p_1} = \left[P_{p_1 p_2}\left(\alpha_s(Q^2)\right) \otimes PDF_{p_2}\left(\alpha_s(Q^2),Q^2\right)\right](x)$$

- $P_{p_1 p_2}$ - Splitting functions, calculable in p.QCD

$$\Longrightarrow$$

Evolution (the physical picture)



- Proton:
  - probed with increasing resolution $1/Q$
  - lower momentum partons are resolved

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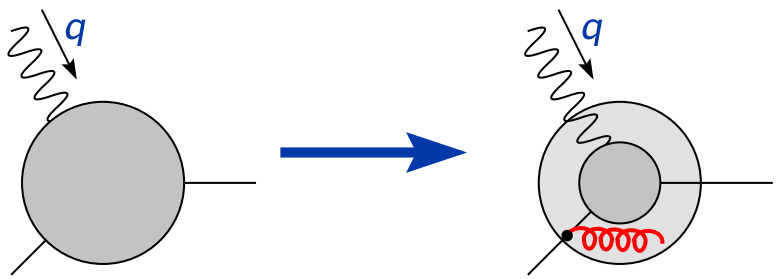

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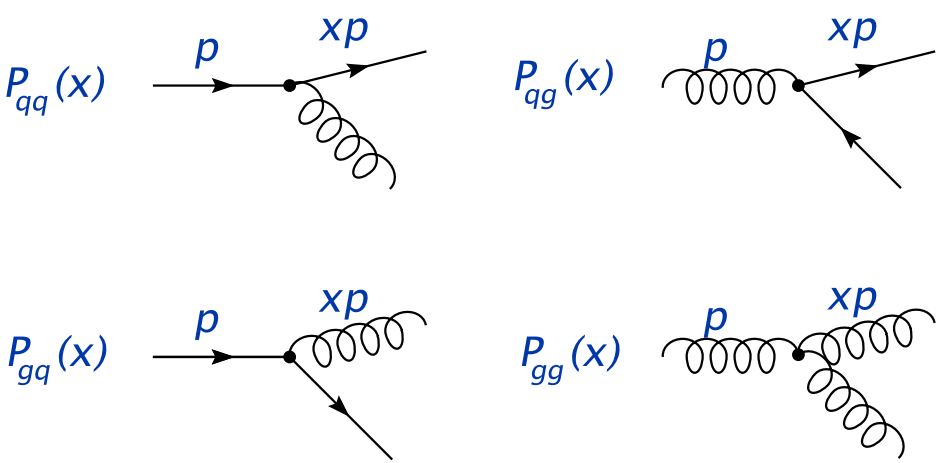


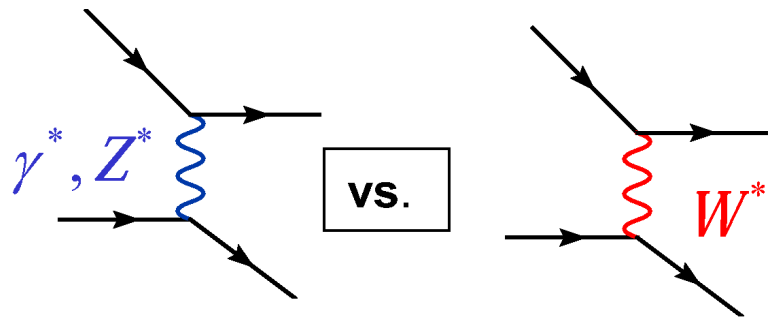
- Feynman diagrams at leading order

- DGLAP:
  - can be solved, if PDFs are given at input scale Q_0 as function of x

$$\frac{Q^2}{\partial Q^2}\begin{bmatrix} q \\ g \end{bmatrix} = \begin{bmatrix} P_{\mathrm{qq}} & P_{\mathrm{qg}} \\ P_{\mathrm{gq}} & P_{\mathrm{gg}} \end{bmatrix} \otimes \begin{bmatrix} q \\ g \end{bmatrix}$$

Experiments

- **EW unification: neutral vs . charged current at HERA**



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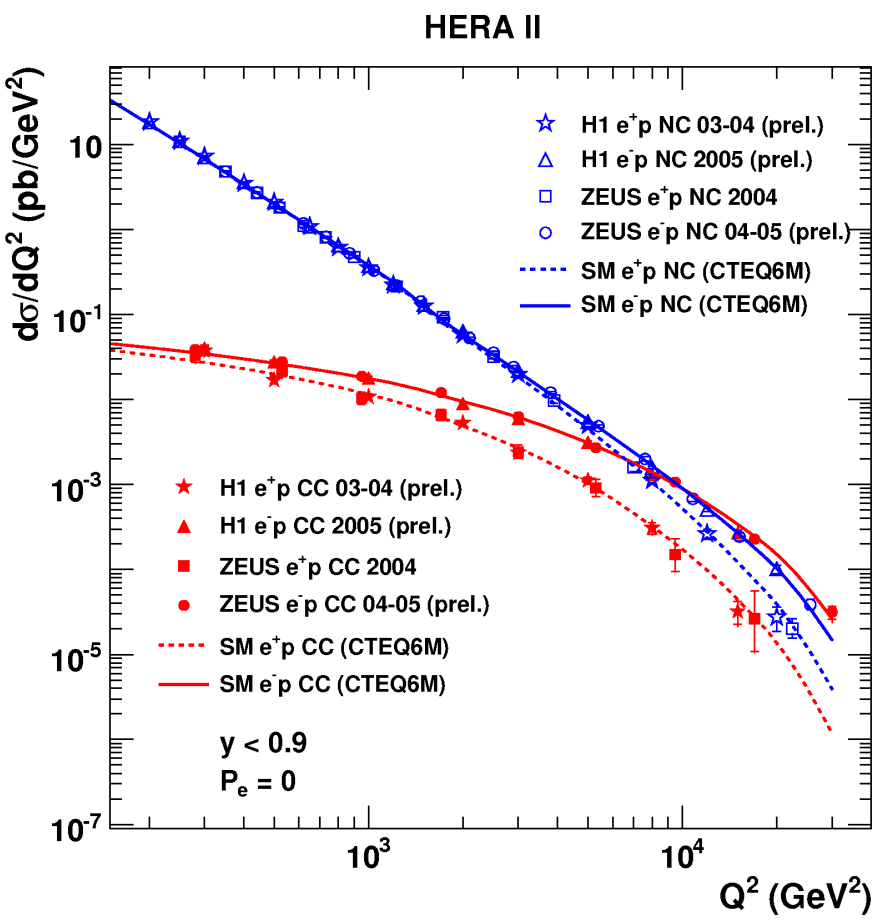
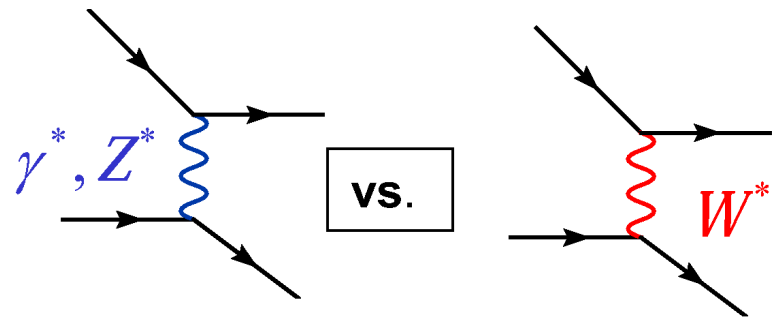


Charged and neutral deep inelastic scattering cross sections become comparable when Q^2 reaches the electroweak scale

$\Longleftarrow$



- Polarized charged current DIS at HERA

CC cross section modified by polarization:

$$\sigma_{CC}^{e^{\pm}p}(P_e) = (1 \pm P_e) \cdot \sigma_{CC}^{e^{\pm}p}(P_e = 0) \qquad P_e = \frac{N_R - N_L}{N_R + N_L}$$

- Cross section is linearly proportional to polarization
- Standard model predicts zero cross section for $P_e = +1(-1)$ in $e^{-(+)}$ scattering

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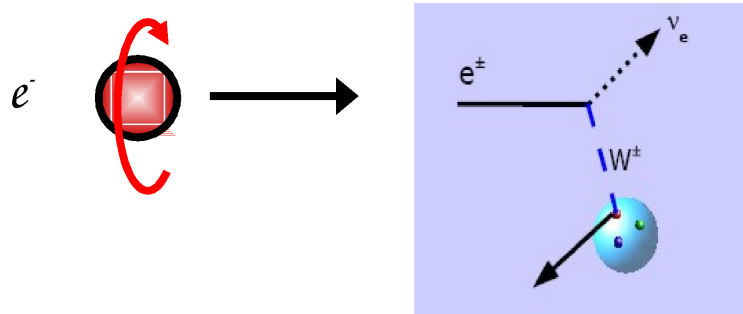
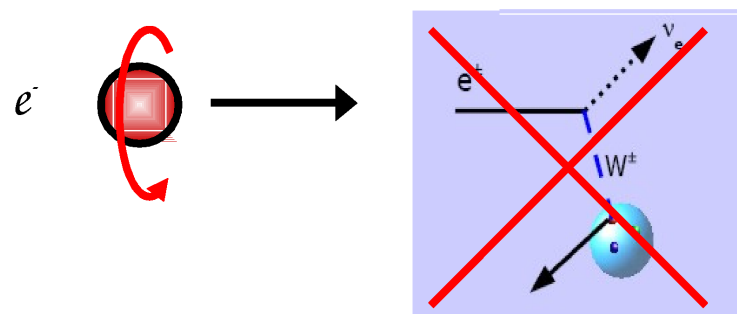


- *righthanded electrons do not!*



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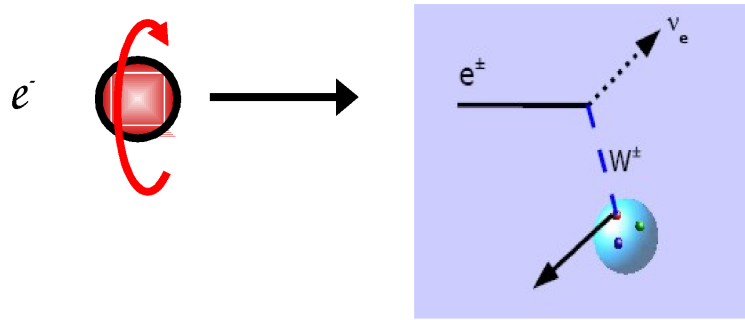

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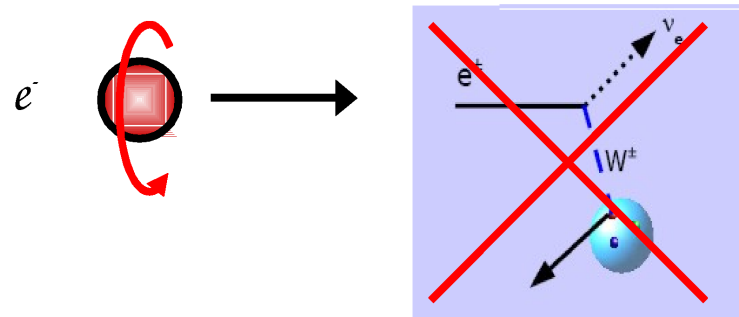

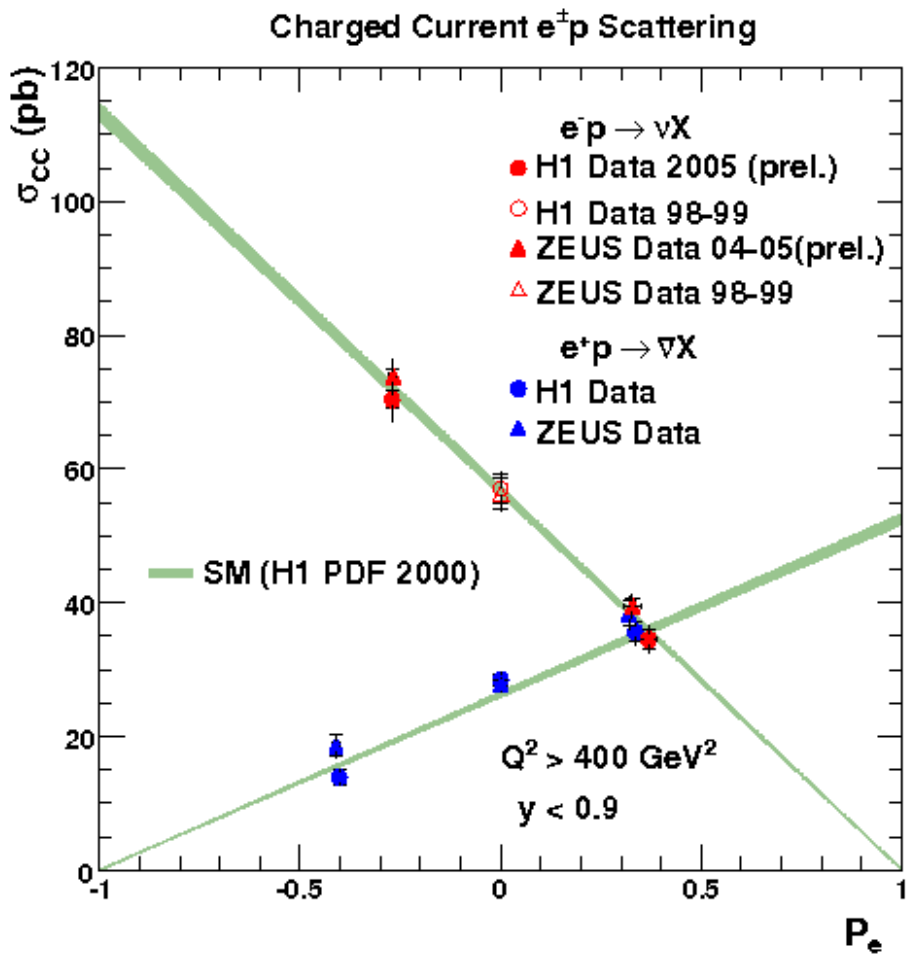

NuTeV experiment

- *The Paschos-Wolfenstein relation*

  Exact relation for massless quarks and isospin zero target

  Paschos,Wolfenstein'73, Llewelin Smith'83

$$R^- = \frac{\sigma^{\nu}_{NC} - \sigma^{\bar{\nu}}_{NC}}{\sigma^{\nu}_{CC} - \sigma^{\bar{\nu}}_{CC}} = \frac{1}{2} - \sin^2\theta_W$$

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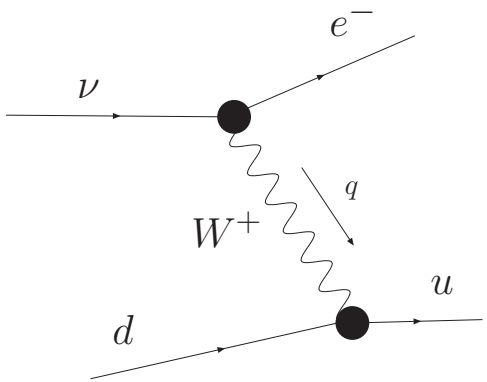
- QCD corrections to *the Paschos-Wolfenstein relation*
  Use second moments of PDFs $q^- = \int dx\, x(q - \bar{q})$ and expansion in isoscalar combination $u^- + d^-$
  Davidson, Forte, Gambino, Rius, Strumia hep-ph/0112302
  Dobrescu, Ellis hep-ph/0310154

$$R^- = \frac{1}{2} - \sin^2\theta_W + \left[1 - \frac{7}{3}\sin^2\theta_W + \frac{8\alpha_s}{9\pi}\left(\frac{1}{2} - \sin^2\theta_W\right)\right] \times \left(\frac{u^- - d^-}{u^- + d^-} - \frac{s^-}{u^- + d^-} + \frac{c^-}{u^- + d^-}\right)$$

- Main uncertainties in s^-: e.g. fit of MRST
  Martin, Roberts,Stirling, Thorne hep-ph/0411040

Calculations of the third order QCD corrections

- Parton level ⇒



- Exchange via $W^\pm$ gauge boson
  - Vector and Axial-Vector interaction: $a\gamma^\mu + b\gamma^\mu\gamma^5$

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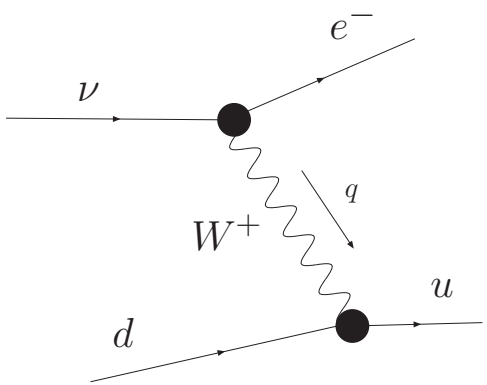

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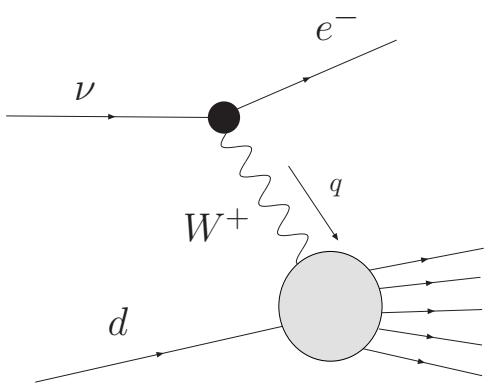
- Beyond LO ⇒



- One has to take into account all possible final states up to fixed order in α_s (including virtual corrections)
  - Our calculations: up to α_s^3

Current state in the determination of the structure functions

- F_i^{ep} via one photon exchange:
  LO - Gross, Wilczek'73;Altarelli ,Parisi'77 (*)
  NLO - Bardeen, Buras, Duke, Muta'78 (**)
  NNLO - Zijlstra, van Neerven'92 (***)
  N^{3}LO - Moch, Vogt, Vermaseren'05 (****)
  $i = 2, L$-needs even Mellin moments (MM), $i = 3$ needs odd MM

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- $F_{2,L}^{\nu p+\bar{\nu} p}$ and $F_3^{\nu p+\bar{\nu} p}$:
  LO, NLO - (**), NNLO - (***), N^{3}LO - (****)
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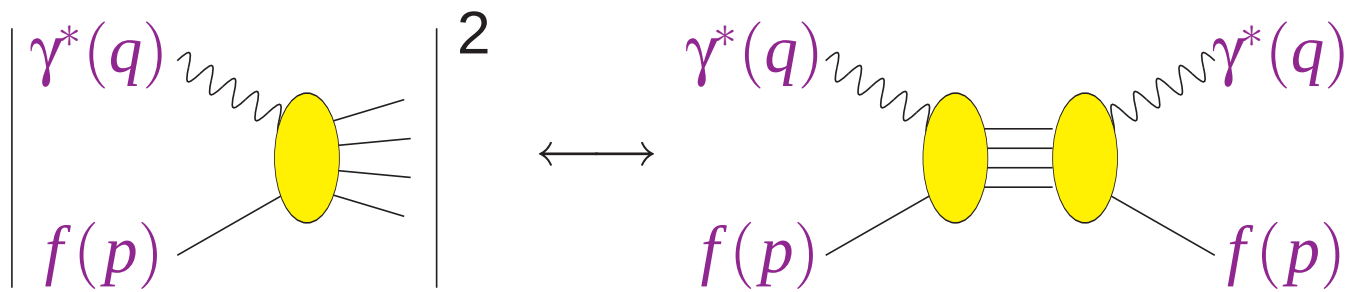
- $F_{2,L}^{\nu p-\bar{\nu} p}$ and $F_3^{\nu p-\bar{\nu} p}$:
  NEW odd MM for the first case and even for the second case (#). It is the main difference in the determination of these structure functions.
  LO, NLO - (**), NNLO - (***)
  Our calculation - up to N^{3}LO ($\sim {\alpha_s}^3$) for the coefficient functions .

Optical theorem

- The hadronic tensor is related to the imaginary part of the forward Compton scattering amplitude
  - α_s^3 calculation in DIS with help of loop technology



$$W_{\mu\nu}(p,q) = \frac{1}{2\pi} Im T_{\mu\nu}(p,q)$$

- Calculation of Mellin moments

$$F^N(Q^2) = \int_0^1 dx x^{N-1} F(x, Q^2)$$

Operator product expansion

- Optical theorem relates hadronic tensor $W_{\mu\nu}$ to imaginary part of Compton amplitude $T_{\mu\nu}$

$$W_{\mu\nu} = e_{\mu\nu}\frac{1}{2x}F_L(x,Q^2) + d_{\mu\nu}\frac{1}{2x}F_2(x,Q^2) + \mathrm{i}\epsilon_{\mu\nu\alpha\beta}\frac{P^\alpha q^\beta}{P\cdot q}F_3(x,Q^2)$$

- Operator product expansion of $T_{\mu\nu}$ for short distances $z^2 \simeq 0$ in Bjorken limit $Q^2 \to \infty$, x fixed
  Christ, Hasslacher, Mueller '72

$$\begin{aligned} T_{\mu\nu} &= \mathrm{i}\int d^4z\, \mathrm{e}^{\mathrm{i}q\cdot z}\langle \mathrm{P}|\, T\left(J_\mu^\dagger(z)J_\nu(0)\right)|\mathrm{P}\rangle \\ &= \sum_{N,j}\left(\frac{1}{2x}\right)^N \left[e_{\mu\nu}\, C_{L,j}^N\left(\frac{Q^2}{\mu^2},\alpha_s\right) + d_{\mu\nu}\, C_{2,j}^N\left(\frac{Q^2}{\mu^2},\alpha_s\right)\right. \\ &\quad \left. +\mathrm{i}\epsilon_{\mu\nu\alpha\beta}\frac{p^\alpha q^\beta}{p\cdot q}C_{3,j}^N\left(\frac{Q^2}{\mu^2},\alpha_s\right)\right] A_{\mathrm{P},N}^j\left(\mu^2\right) + \text{ higher twists} \end{aligned}$$

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- hard scattering coefficient functions in Mellin space $C_{i,j}^N$ **(our interest)**

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- Optical theorem relates hadronic tensor $W_{\mu\nu}$ to imaginary part of Compton amplitude $T_{\mu\nu}$

$$W_{\mu\nu} = e_{\mu\nu}\frac{1}{2x}F_L(x,Q^2) + d_{\mu\nu}\frac{1}{2x}F_2(x,Q^2) + \mathrm{i}\epsilon_{\mu\nu\alpha\beta}\frac{P^\alpha q^\beta}{P\cdot q}F_3(x,Q^2)$$

- Operator product expansion of $T_{\mu\nu}$ for short distances $z^2 \simeq 0$ in Bjorken limit $Q^2 \to \infty$, x fixed
  Christ, Hasslacher, Mueller '72

$$\begin{aligned} T_{\mu\nu} &= \mathrm{i}\int d^4z\, \mathrm{e}^{\mathrm{i}q\cdot z}\langle \mathrm{P}|\, T\left(J_\mu^\dagger(z)J_\nu(0)\right)|\mathrm{P}\rangle \\ &= \sum_{N,j}\left(\frac{1}{2x}\right)^N \left[e_{\mu\nu}\, C_{L,j}^N\left(\frac{Q^2}{\mu^2},\alpha_s\right) + d_{\mu\nu}\, C_{2,j}^N\left(\frac{Q^2}{\mu^2},\alpha_s\right)\right. \\ &\quad \left. +\mathrm{i}\epsilon_{\mu\nu\alpha\beta}\frac{p^\alpha q^\beta}{p\cdot q}C_{3,j}^N\left(\frac{Q^2}{\mu^2},\alpha_s\right)\right] A_{\mathrm{P},N}^j\left(\mu^2\right) + \text{ higher twists} \end{aligned}$$

- hard scattering coefficient functions in Mellin space $C_{i,j}^N$ **(our interest)**
- leading twist matrix elements $A_{\mathrm{P},N}^i$ of parton operators O^i, $i = \mathrm{ns}, \mathrm{q}, \mathrm{g}$

Operator basis

$$O^{\alpha,\{\mu_1,\cdots,\mu_N\}} = \overline{\psi}\lambda^\alpha\gamma^{\{\mu_1}D^{\mu_2}\cdots D^{\mu_N\}}\psi, \quad \alpha = 1, 2, ..., (n_f^2 - 1)$$

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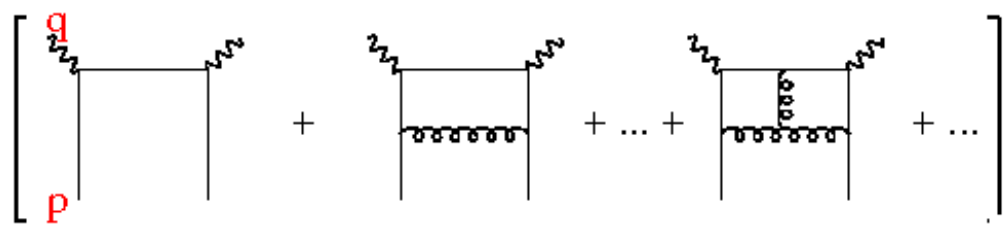
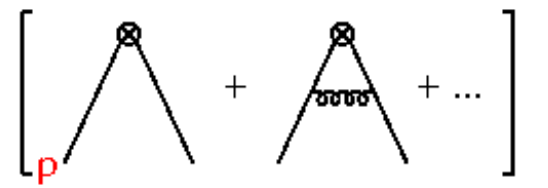
Method of projection

Gorishnii, Larin, Tkachev '83; Gorishnii, Larin '87

- Operator product expansion $\longrightarrow$ Green functions with ternal partons



$$= \sum_N \left(\frac{1}{2x}\right)^N \left\{ C^N_{i,q} Z^{qq} \right.$$



$$\left. \right\}$$

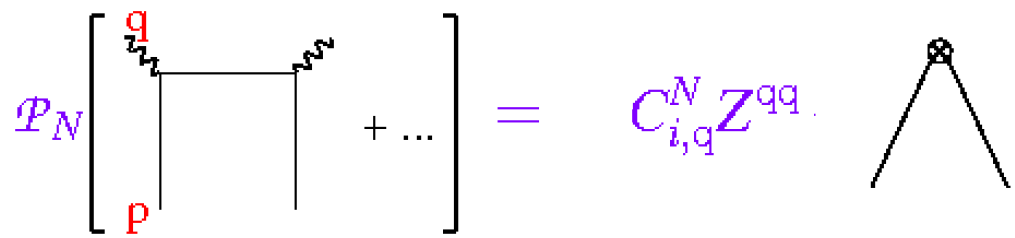
- Operator projects out N-th moment

$$\mathcal{P}_N \equiv \left[\frac{q^{\{\mu_1} \cdots q^{\mu_N\}}}{N!} \frac{\partial^N}{\partial p^{\mu_1} \cdots \partial p^{\mu_N}} \right] \Bigg|_{p=0}$$

- Projection $\mathcal{P}_N \Rightarrow$

  - only tree level operator matrix elements survive

  - method needs dimensional regularization $D = 4 - 2\epsilon$



The calculation

- Big number of diagrams $\Rightarrow$ need of automatization
  e.g. DIS structure functions $F_{2,L}^{\nu p \pm \bar{\nu} p}$, $F_3^{\nu p \pm \bar{\nu} p}$ - 1314 diagrams up to 3 loops

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Calculation of diagrams $\mapsto$

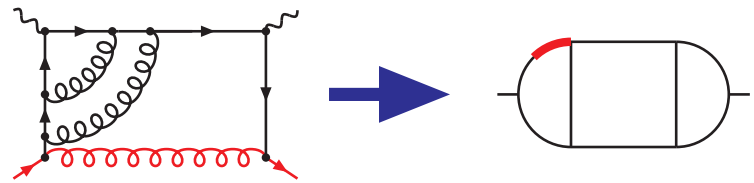
- MINCER in FORM [Larin, Tkachev, Vermaseren'91]

What does MINCER do?

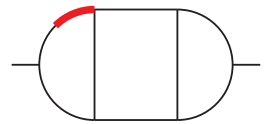
MINCER minces integrals

Method of projection in pictures

- Identify scalar topologies

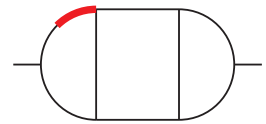


- Scalar diagram with external momenta P and Q



$$= \int \prod_{n}^{3} d^D l_n \frac{1}{(P-l_1)^2} \frac{1}{l_1^2 \ldots l_8^2}$$

- N-th moment:
  $\longrightarrow$ coefficient of $(2P \cdot Q)^N$



$$= \frac{(2\,P \cdot Q)^N}{(Q^2)^{N+\alpha}} C_N$$

- Taylor expansion

$$\frac{1}{(P-l_1)^2} = \sum_i \frac{(2P \cdot l_1)^i}{(l_1^2)^{i+1}} \longrightarrow \frac{(2P \cdot l_1)^N}{(l_1^2)^N}$$

- Feed scalar two-point functions in MINCER

Mincer

- $\int dP \frac{\partial}{\partial P^\mu} \left[(P - l_j)^\mu \times I(l_1, .., P, ...) \right] = 0$ - integration by part identities
  t'Hooft, Veltman'72; Chetyrkin , Tkachov '81
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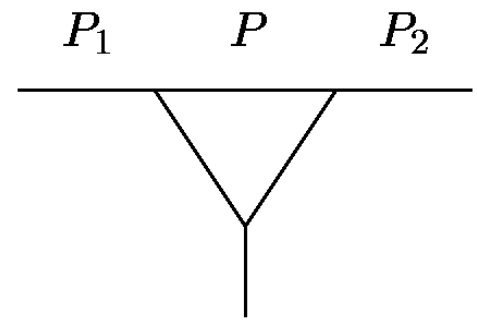

Triangle rule

Define

$$I(\alpha_0, \beta_1, \beta_2, \alpha_1, \alpha_2) = \int d^D P \frac{1}{(P^2)^{\alpha_0}((P+P_1)^2)^{\beta_1}(P_1^2)^{\alpha_1}((P+P_2)^2)^{\beta_2}(P_2^2)^{\alpha_2}}$$

and act the integrand with $\frac{\partial}{\partial P_\mu} P_\mu = D + P_\mu \frac{\partial}{\partial P_\mu}$. Result $\Rightarrow$

Recursion relation:

$$I(\alpha_0, \beta_1, \beta_2, \alpha_1, \alpha_2) \times (D - 2\alpha_0 - \beta_1 - \beta_2) =$$
$$\beta_1 (I(\alpha_0 - 1, \beta_1 + 1, \beta_2, \alpha_1, \alpha_2) - I(\alpha_0, \beta_1 + 1, \beta_2, \alpha_1 - 1, \alpha_2))$$
$$\beta_2 (I(\alpha_0 - 1, \beta_1, \beta_2 + 1, \alpha_1, \alpha_2) - I(\alpha_0, \beta_1, \beta_2 + 1, \alpha_1, \alpha_2 - 1))$$

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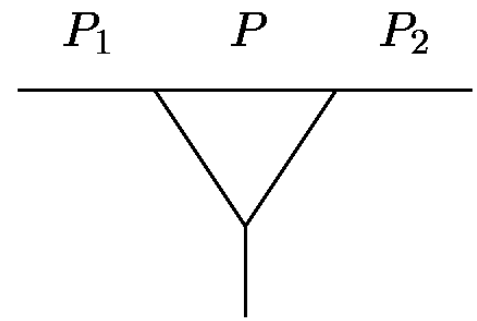

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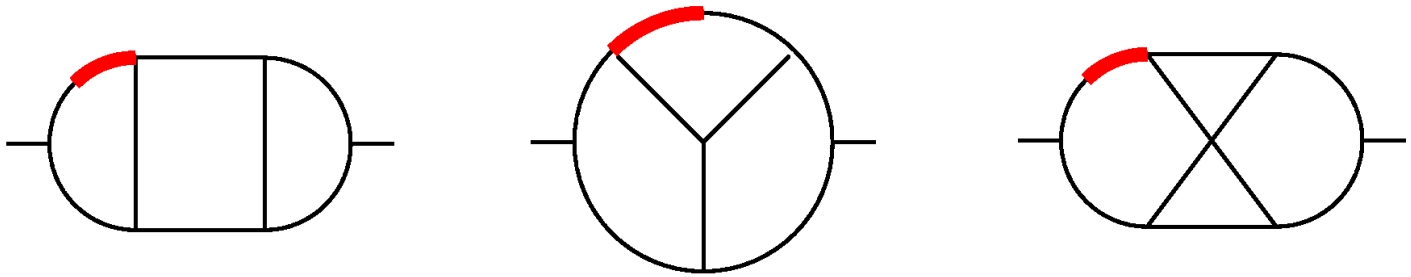
In pictures

Classification of loop integrals

- Classify according to topology of underlying two-point function
  - top-level topology types ladder, benz, non-planar



- Using **IBP** identities more complicated topologies are reduced to simpler topologies

Renormalization - a subtle point for F_3

Larin, Vermaseren'91

- γ^5 with the Larin prescription

$$\gamma_\mu\gamma_5 = \mathrm{i}\frac{1}{3!}\epsilon_{\mu\nu\sigma\tau}\gamma^\rho\gamma^\sigma\gamma^\tau$$

- Violates the axial Ward identity. To be restored by an additional renormalization Z_A:(in $\overline{MS}$)

$$Z_A = 1 + \left(\frac{\alpha_s(\mu^2)}{4\pi}\right)^2 \frac{1}{\varepsilon}\left[\frac{22}{3}C_A C_F - \frac{4}{3}C_F n_f\right] + \mathcal{O}({\alpha_s}^3)$$

- The treatment of γ^5 in $D = 4 - 2\varepsilon$ introduces an extra finite renormalization with Z_5 . It is derived in $\overline{MS}$ from

$$(R_{\overline{MS}}V_\mu)\gamma_5 = Z_5(R_{\overline{MS}}A_\mu),$$

$R_{\overline{MS}}$ - denotes renorm. operator in $\overline{MS}$ to remove UV divergencies

$$Z_5 = 1 - \frac{\alpha_s(\mu^2)}{\pi}C_F + \left(\frac{\alpha_s(\mu^2)}{4\pi}\right)^2\left[22C_F^2 - \frac{107}{9}C_A C_F + \frac{2}{9}C_F n_f\right] + \mathcal{O}({\alpha_s}^3)$$

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$$\mathrm{i}\frac{-g^{\mu\nu} + (1-\xi)q^\mu q^\nu}{q^2 - \mathrm{i}\epsilon}$$

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- Sum rules calculations in DIS
  ▲ Gross - Llewellyn Smith sum rule and Bjorken sum rule
  [Larin, Vermaseren,91]

$$\int_0^1 dx(F_1^{\bar{\nu}p}(x,Q^2)-F_1^{\nu p}(x,Q^2))=1,\ \int_0^1 dx(F_3^{\bar{\nu}p}(x,Q^2)+F_3^{\nu p}(x,Q^2))=6$$

  ▲ Adler sum rule, Gottfried sum rule [Broadhurst, Kataev, Maxwell’04]
  Conjucture of colour coefficients of coefficient functions “even" - “odd"
  Difference $\sim (C_F - C_A/2)$ subleading color

Preliminary

$$c_{3,q}^{(0),ns}(4) = 1 \quad c_{3,q}^{(1),ns}(4) = C_F * (73/20)$$

$$c_{3,q}^{(2),ns}(4) = C_F * n_f * (-1073981/108000)$$

$$+C_F * C_A * (+3575579/54000 - 227/5 * \zeta_3)$$

$$+{C_F}^2 * (-59219099/6480000 + 28 * \zeta_3)$$

$$c_{3,q}^{(3),ns}(4) = C_F * {n_f}^2 * (+12195323/1749600 + 628/405 * \zeta_3)$$

$$+C_F * C_A * n_f * (-529878917/3499200 - 314/15 * \zeta_4 + 29266/405 * \zeta_3)$$

$$+C_F * {C_A}^2 * (+8293616147/17496000 + 1430/3 * \zeta_5$$

$$+1439/150 * \zeta_4 - 1625431/2025 * \zeta_3)$$

$$+{C_F}^2 * n_f * (-18625311191/109350000 + 314/15 * \zeta_4 + 38021/675 * \zeta_3)$$

$$+{C_F}^2 * C_A * (+1003904196083/1749600000 - 208 * \zeta_5$$

$$-1439/50 * \zeta_4 - 185929/2250 * \zeta_3)$$

$$+{C_F}^3 * (-48030418393/5832000000 - 704/3 * \zeta_5$$

$$+1439/75 * \zeta_4 + 9183239/40500 * \zeta_3)$$

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Outlook

- We can use it for other applications

$N = 4$ Super Yang-Mills theory

- Maximally supersymmetric Yang-Mills theory in four dimensions (MSYM)
  - renewed interest from AdS/cfT and from twistor space methods
  - simple planar limit for large n_c
  - vanishing β-function (conformal invariance)

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- set MSYM identification for color coefficients $C_A = C_F = n_c$
  - n_f-terms do not contribute at highest transcendentality

Anomalous dimensions of MSYM

- KLOV obtained universal anomalous dimension $\gamma_{\rm ij}^{(n)}(N) \; = \; - \int_0^1 dx \; x^{\,N-1} \, P_{\rm ij}^{(n)}(x)$ in MSYM to three loops using QCD results of Moch, Vermaseren, Vogt'04

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  - at l-loops harmonic sums of weight $w = 2l - 1$ are the terms of "leading transcendentality"

is

$$\gamma(j) \equiv \gamma_{uni}(j) \;=\; \hat{a}\gamma_{uni}^{(0)}(j) + \hat{a}^2\gamma_{uni}^{(1)}(j) + \hat{a}^3\gamma_{uni}^{(2)}(j) + ..., \qquad \hat{a} = \frac{\alpha N_c}{4\pi}\,, \tag{9}$$

where

$$\frac{1}{4}\,\gamma_{uni}^{(0)}(j+2) \;=\; -S_1, \tag{10}$$

$$\frac{1}{8}\,\gamma_{uni}^{(1)}(j+2) \;=\; \Big(S_3 + \overline{S}_{-3}\Big) - 2\,\overline{S}_{-2,1} + 2\,S_1\Big(S_2 + \overline{S}_{-2}\Big), \tag{11}$$

$$\begin{aligned}\frac{1}{32}\,\gamma_{uni}^{(2)}(j+2) \;=\;& 2\,\overline{S}_{-3}\,S_2 - S_5 - 2\,\overline{S}_{-2}\,S_3 - 3\,\overline{S}_{-5} + 24\,\overline{S}_{-2,1,1,1} \\ &+6\Big(\overline{S}_{-4,1} + \overline{S}_{-3,2} + \overline{S}_{-2,3}\Big) - 12\Big(\overline{S}_{-3,1,1} + \overline{S}_{-2,1,2} + \overline{S}_{-2,2,1}\Big) \\ &-\Big(S_2 + 2\,S_1^2\Big)\Big(3\,\overline{S}_{-3} + S_3 - 2\,\overline{S}_{-2,1}\Big) - S_1\Big(8\,\overline{S}_{-4} + \overline{S}_{-2}^{\,2} \\ &+4\,S_2\,\overline{S}_{-2} + 2\,S_2^2 + 3\,S_4 - 12\,\overline{S}_{-3,1} - 10\,\overline{S}_{-2,2} + 16\,\overline{S}_{-2,1,1}\Big)\end{aligned} \tag{12}$$

and $S_a \equiv S_a(j),\; S_{a,b} \equiv S_{a,b}(j),\; S_{a,b,c} \equiv S_{a,b,c}(j)$ are harmonic sums

- The results were checked by
Bern, Dixon, Smirnov' 05
from the value of the $1/\varepsilon^2$ in the scattering amplitude.

Large spin limit $N \to \infty$ for $\gamma(N) \Longleftrightarrow x \to 1$ limit for $P_{\rm ij}^{(n)}(x)$ splitting function

$$P^{(n)}_{aa,x\to 1} = \frac{A^a_{n+1}}{(1-x)_+} + B^a_{n+1}\delta(1-x) + C^a_{n+1}\ln(1-x) + \mathcal{O}(1)$$

Moch, Vermaseren, Vogt

$$A_1^q = 4C_F \cdot \mathbf{1}$$

$$A_2^q = 8C_F\left[\left(\frac{67}{18} - \zeta_2\right)C_A - \frac{5}{9}n_f\right]$$

$$A_3^q = 16C_F C_A^2[\frac{245}{24} - \frac{67}{9}\zeta_2 + \frac{11}{6}\zeta_3 + \frac{11}{5}\zeta_2^2] + 16C_F n_f[-\frac{55}{24} + 2\zeta_3]$$

$$+16C_F C_A n_f[-\frac{209}{108} + \frac{10}{9}\zeta 2 - \frac{7}{3}] + 16C_F n_f^2[-\frac{1}{27}]$$

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$$\Updownarrow$$

Bern, Dixon, Smirnov

$$f_0^{(1)} = \mathbf{1}, \qquad f_0 = -\zeta_2, \qquad f_0^{(3)} = \frac{11}{2}\zeta_4 = \frac{11}{5}\zeta_2^2$$

- also checked by all orders proposal of
  Eden, Staudacher hep-th/0603157

$$f_0(g) = 4g^2 - \frac{2}{3}\pi^2 g^4 + \frac{11}{45}\pi^4 g^6 - \left(\frac{73}{630}\pi^6 - 4\zeta_3^2\right) g^8 + \ldots$$

- Direct prediction for highest transcendentality term in A_4^q

$$A_4^q \rightarrow C_F\, C_A^3 \left(\frac{73}{630}\pi^6 - 4\zeta_3^2\right)$$

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  - in the far future : attack A_4^q