B.C.F.W. recursion relations and the link to Feynman graphs

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Fermilab Theory Seminar
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introduction

- Recursion relations very useful in the past
- Schwinger-Dyson rec. relations / n-point functions
- ALPHA algorithm / C.M. algorithm
- Berends-Giele recursion relations
- The new hype:

  E. Witten [hep-th/0312171]: duality between the perturbative regime of N=4 SYM and the perturbative regime of topological string theory model on twistor space.

  F. Cachazo, P. Svrcek and E. Witten [hep-th/0406177]: “twistor inspired” construction of Yang Mills amplitudes with off-shell MHV diagrams as building blocks

  R. Britto, F. Cachazo and B. Feng [hep-th/0412308]: “twistor inspired” recursion relations for Yang Mills amplitudes with on-shell amplitudes as building blocks

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why seeking a link with Feynman graphs?

- BCFW recursion relations can be used to calculate **analytically** tree level color-ordered amplitudes
- Application at NLO has given previously unknown results and there is more coming
- Previously known results were obtained in **simpler** form with much less effort.

- the origin of cancellations?
- gauge dependence?
- the role of **Yang-Mills** vertices?
color stripped amplitudes

• External helicities are fixed
• Familiar trace decomposition into color ordered graphs

\[ \mathcal{A}(1, \ldots, n) = \sum_{\sigma \in S_{n-1}} \text{Tr} \left( T^{a_1} T^{a_{\sigma_2}} \ldots T^{a_{\sigma_n}} \right) A(1, \sigma_2, \ldots, \sigma_n) \]

notation:
BCFW recursion relation

\[ A(p_1^{h_1}, \ldots, p_n^{h_n}) = \sum_{j=2}^{n-1} \sum_h \left[ A(\hat{p}_1^{h_1}, \ldots, p_j^{h_j}, -\hat{p}_1^{h_j}) \right]_{z=z_j} \frac{1}{P_{1\ldots j}^2} \left[ A(\hat{P}_{1\ldots j}^{-h}, p_{j+1}^{h+1}, \ldots, p_n^{h_n}) \right]_{z=z_j} \]

\[ \hat{p}^\mu = p^\mu + z\epsilon^\mu \quad p \neq p_n \]

\[ \hat{p}^\mu = p^\mu - z\epsilon^\mu \quad p = p_n \]

\[ \epsilon \cdot \sigma_{a\dot{a}} \equiv \epsilon_{a\dot{a}} = n_a \tilde{1}_{\dot{a}} \]

\( z \) such that the propagator is on-shell

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spinors semiology

\[ p^\mu q_\mu = p_\mu q_\nu g^{\mu\nu} = \frac{1}{2} p_\mu q_\nu \sigma^{\mu}_{a\dot{a}} \sigma^{\nu}_{b\dot{b}} \epsilon^{ab} \epsilon^{\dot{a}\dot{b}} \]

\[ = \frac{1}{2} p_a \tilde{p}_{\dot{a}} q_b \tilde{q}_{\dot{a}} \epsilon^{ab} \epsilon^{\dot{a}\dot{b}} \]

\[ = \frac{1}{2} \langle pq \rangle [pq] \]

\[ p^2 = 0 \rightarrow p_\mu \cdot \sigma^{\mu}_{a\dot{a}} = p_a \tilde{p}_{\dot{a}} \]

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spinors semiology

\[ \epsilon_\mu(k) \cdot k = 0 \quad \epsilon^+ \cdot \epsilon^{++} = -1 \quad \epsilon^+ \cdot \epsilon^+ = 0 \]

\[ \epsilon_\mu^+(k) \epsilon^+_{\nu}*(k) + \epsilon_\mu^-(k) \epsilon^-_{\nu}*(k) = D_{\mu\nu} \]

\[ \epsilon_{a\bar{a}}^+(p) \equiv \epsilon_\mu^+(p) \sigma_{a\bar{a}}^\mu = \sqrt{2} \frac{q_a \tilde{p}_{\bar{a}}}{\langle pp \rangle} \]

\[ \epsilon_{a\bar{a}}^-(p) \equiv \epsilon_\mu^-(p) \sigma_{a\bar{a}}^\mu = -\sqrt{2} \frac{p_a \tilde{q}_{\bar{a}}}{[pq]} \]

for every polarization vector, a different (null) gauge vector can be chosen, i.e. a different spinor \( q \)
null vector axial gauge

\[
D_{\mu\nu} \sigma^\mu_{a\dot{a}} \sigma^\nu_{b\dot{b}} = -2 \frac{p \cdot \sigma_{a\dot{a}} q \cdot \sigma_{b\dot{b}} + q \cdot \sigma_{a\dot{a}} p \cdot \sigma_{b\dot{b}}}{\langle qp \rangle \langle pq \rangle} - \epsilon_{\alpha \beta} \epsilon_{\dot{\alpha} \dot{\beta}} =
\]

\[
= \{-g_{\mu\nu} + \frac{p_\mu q_\nu + p_\nu q_\mu}{p \cdot q}\} \sigma^\mu_{a\dot{a}} \sigma^\nu_{b\dot{b}}
\]

changing the gauge vector amounts to a gauge trs.

\[
\epsilon^\mu_+ \rightarrow \epsilon^\mu_+ + cp^\mu
\]

\[
\epsilon^+_a(p) \rightarrow \epsilon^+_a(p) + cp_a p_a = \sqrt{2} \frac{p_a \tilde{q}'_a}{\langle p' p \rangle}
\]

\[
q'_a = q_a + \frac{c}{\sqrt(2)} \langle qp \rangle p_a
\]

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the Parke-Taylor formula

MHVs can be used as building blocks of BCFW decompositions.

$$A_{MHV^+}(1^+, \ldots, i^-, \ldots, j^-, \ldots, n^+) = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n - 1 n \rangle \langle n1 \rangle}$$

$$A_{MHV^-}(1^-, \ldots, i^+, \ldots, j^+, \ldots, n^-) = \frac{[ij]^4}{[12][23] \cdots [n - 1 n][n1]}$$

But one only needs the three-vertex

$$A(a^+, b^+, c^-) = \frac{[ab]^3}{[bc][ca]} \quad \quad A(a^+, b^-, c^-) = \frac{\langle bc \rangle^3}{\langle ca \rangle \langle ab \rangle}$$
BCFW graph

\[ A(p_1^{h_1}, \ldots, p_n^{h_n}) = \sum_{j=2}^{n-1} \sum_h \left[ A(p_1^{h_1}, \ldots, p_j^{h_j}, -\hat{P}_{1\ldots j}^h) \right] z = z_j \frac{1}{P_{1\ldots j}^2} \left[ A(\hat{P}_{1\ldots j}^{-h}, p_j^{h+1}, \ldots, p_n^{h_n}) \right] z = z_j \]

\[ \hat{p}^\mu = p^\mu + z \epsilon^\mu \quad p \neq p_n \]
\[ \hat{p}^\mu = p^\mu - z \epsilon^\mu \quad p = p_n \]

\[ \epsilon \cdot \sigma_{a\dot{a}} \equiv \epsilon_{a\dot{a}} = n_a \tilde{1}_{\dot{a}} \]

z such that the propagator is on-shell

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trivial BCFW example

\[ \hat{P}^2 = (p_1 + p_2 + z\epsilon)^2 = 2p_1 \cdot p_2 + 2z(p_1 + p_2) \cdot \epsilon = 0 \Rightarrow z = -\frac{\langle 12 \rangle \langle 12 \rangle}{\langle 24 \rangle \langle 21 \rangle} = -\frac{\langle 12 \rangle}{\langle 42 \rangle} \]

\[ \hat{P} = q\tilde{q} + 2\tilde{s} - \frac{\langle 12 \rangle}{\langle 42 \rangle} 4\tilde{t} = 2(\tilde{t} + \frac{\langle 41 \rangle}{\langle 42 \rangle} \tilde{t}) \]

\[ \hat{p}_1 = 1\tilde{t} - \frac{\langle 12 \rangle}{\langle 42 \rangle} 4\tilde{t} = 1\tilde{t} + \frac{\langle 24 \rangle}{\langle 42 \rangle} 1\tilde{t} + \frac{\langle 41 \rangle}{\langle 42 \rangle} 2\tilde{t} = \frac{\langle 41 \rangle}{\langle 42 \rangle} 2\tilde{t} \]

\[ \hat{p}_4 = 4\tilde{t} + \frac{\langle 12 \rangle}{\langle 42 \rangle} 4\tilde{t} = 4(\tilde{t} + \frac{\langle 12 \rangle}{\langle 42 \rangle} \tilde{t}) \]

\[
\begin{align*}
\frac{[\hat{P}]^3}{[P_2][2\hat{t}]} & \frac{1}{(p_1 + p_2)^2} \frac{\langle \hat{P}_4 \rangle^3}{\langle 43 \rangle \langle 3\hat{P} \rangle} = \frac{[12]^3}{\langle 41 \rangle \langle 12 \rangle [21]} \frac{1}{\langle 12 \rangle [12]} \frac{\langle 24 \rangle^3}{\langle 43 \rangle \langle 32 \rangle} = \frac{\langle 24 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}
\end{align*}
\]

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BCFW producing mhv's

A mostly plus MHV can be either

or
\[ A(1^+, 2^+, 3^+, 4^-, 5^+, 6^+, 7^-) = \frac{\langle 72 \rangle}{\langle 71 \rangle \langle 12 \rangle} A(2^+, 3^+, 4^-, 5^+, 6^+, 7^-) = \]

\[ = \frac{\langle 72 \rangle}{\langle 71 \rangle \langle 12 \rangle} \frac{\langle 73 \rangle}{\langle 72 \rangle \langle 23 \rangle} A(3^+, 4^-, 5^+, 6^+, 7^-) = \]

\[ = \frac{\langle 72 \rangle}{\langle 71 \rangle \langle 12 \rangle} \frac{\langle 73 \rangle}{\langle 72 \rangle \langle 23 \rangle} \frac{\langle 74 \rangle}{\langle 73 \rangle \langle 34 \rangle} A(4^-, 5^+, 6^+, 7^-) = \]

\[ = \frac{\langle 74 \rangle}{\langle 71 \rangle \langle 12 \rangle \langle 23 \rangle \langle 34 \rangle} A(5^+, 6^+, 7^-, 4^-) = \]

\[ = \frac{\langle 74 \rangle}{\langle 71 \rangle \langle 12 \rangle \langle 23 \rangle \langle 34 \rangle} \frac{\langle 46 \rangle}{\langle 45 \rangle \langle 56 \rangle} A(6^+, 7^-, 4^-) = \]

\[ = \frac{\langle 74 \rangle}{\langle 71 \rangle \langle 12 \rangle \langle 23 \rangle \langle 34 \rangle} \frac{\langle 46 \rangle}{\langle 45 \rangle \langle 56 \rangle} \frac{\langle 74 \rangle^3}{\langle 46 \rangle \langle 67 \rangle} \]
An example beyond the MHVs: the six point amplitude

![Graphs](image)

Things get uglier, but still the result is more compact than the previously calculated one.
the BCFW proof

• relies heavily on the analyticity properties of the color-ordered amplitudes
• turned the attention back to the pole structure of the amplitude
• but shed little light to the connection with Feynman diagrams

what about t-channel graphs?
kinematical identities

A complex function with simple poles equals the sum over residues

\[
\frac{1}{\hat{p}_1^2 \hat{p}_2^2 \cdots \hat{p}_k^2} = \sum_{j=1..k} \left[ \frac{1}{\hat{p}_1^2 \hat{p}_2^2 \cdots \hat{p}_{j-1}^2 \hat{p}_{j+1} \cdots \hat{p}_k^2} \right] z = z_j \frac{1}{\hat{p}_j^2}
\]

Taking the limit \( z \) to zero at both sides

\[
\frac{1}{p_1^2 p_2^2 \cdots p_k^2} = \sum_{j=1..k} \left[ \frac{1}{p_1^2 p_2^2 \cdots p_{j-1}^2 \hat{p}_{j+1}^2 \cdots p_k^2} \right] z = z_j \frac{1}{p_j^2}
\]

Similarly for a power of \( z \) in the numerator...

\[
\frac{z^\rho}{\hat{p}_1^2 \hat{p}_2^2 \cdots \hat{p}_k^2} = \sum_{j=1..k} \left[ \frac{z^\rho}{\hat{p}_1^2 \hat{p}_2^2 \cdots \hat{p}_{j-1}^2 \hat{p}_{j+1} \cdots \hat{p}_k^2} \right] z = z_j \frac{1}{\hat{p}_j^2}
\]
...which shows that for rho smaller than k

\[ \sum_{j=1..k} \left[ \frac{z^\rho}{\hat{p}_1^2 \hat{p}_2^2 \cdots \hat{p}_{j-1}^2 \hat{p}_j \cdots \hat{p}_k^2} \right] \Bigg|_{z=z_j} \frac{1}{p_j^2} = 0 \]

For rho equals k, though, we need to add the limit at infinity to the residue sum

\[ \frac{z^k}{\hat{p}_1^2 \hat{p}_2^2 \cdots \hat{p}_k^2} - \lim_{z \to \infty} \frac{z^k}{\hat{p}_1^2 \hat{p}_2^2 \cdots \hat{p}_k^2} = \sum_{j=1..k} \left[ \frac{z^k}{\hat{p}_1^2 \hat{p}_2^2 \cdots \hat{p}_{j-1}^2 \hat{p}_j \cdots \hat{p}_k^2} \right] \Bigg|_{z=z_j} \frac{1}{p_j^2} \]

... and then taking z to zero gives a constant term on LHS.

\[ -\lim_{z \to \infty} \frac{z^k}{\hat{p}_1^2 \hat{p}_2^2 \cdots \hat{p}_k^2} = \sum_{j=1..k} \left[ \frac{z^k}{\hat{p}_1^2 \hat{p}_2^2 \cdots \hat{p}_{j-1}^2 \hat{p}_j \cdots \hat{p}_k^2} \right] \Bigg|_{z=z_j} \frac{1}{p_j^2} \]

\[ \frac{-1}{\prod_{j=1}^{k} 2\epsilon \cdot p_j} = \sum_{j=1..k} \left[ \frac{z^k}{\hat{p}_1^2 \hat{p}_2^2 \cdots \hat{p}_{j-1}^2 \hat{p}_j \cdots \hat{p}_k^2} \right] \Bigg|_{z=z_j} \frac{1}{p_j^2} \]
choosing gauge vectors

\[ \overline{p}_1 = p_n \text{ and } \overline{p}_n = p_1 \]

\[ \epsilon_1^+ = \sqrt{2} \frac{n\tilde{1}}{\langle n1 \rangle} \]

\[ \epsilon_n^- = \sqrt{2} \frac{n\tilde{1}}{[n1]} \]

\[ \epsilon_1 \cdot \epsilon_n = 0 = \epsilon \cdot \epsilon_1 = \epsilon \cdot \epsilon_n \]

\[ \epsilon_{+1} \cdot p_n = 0 = \epsilon_{-n} \cdot p_1. \]

\[ \epsilon = n_{\alpha} \tilde{1}_{\dot{\alpha}} \]

- The shift vector is proportional to the polarization vectors of the selected legs
- The polarization vectors of the first and last leg are left invariant under the shifting

\[ \langle n1 \rangle \rightarrow \langle n1 \rangle + z\langle nn \rangle = \langle n1 \rangle \]

\[ [n1] \rightarrow [n1] - z[11] = [n1] \]

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choosing gauge vectors

$$\bar{p}_1 = p_n \text{ and } \bar{p}_n = p_1$$

$$\epsilon^+_1 = \sqrt{2} \frac{n\tilde{1}}{\langle n1 \rangle}$$

$$\epsilon^-_n = \sqrt{2} \frac{n\tilde{1}}{[n1]}$$

$$\epsilon_1 \cdot \epsilon_n = 0 = \epsilon \cdot \epsilon_1 = \epsilon \cdot \epsilon_n$$

$$\epsilon_{+1} \cdot p_n = 0 = \epsilon_{-n} \cdot p_1.$$
making the shift

\[ \epsilon_1^+ \quad p_1 \rightarrow p_1 + z\epsilon \quad p_n \rightarrow p_n - z\epsilon \quad \epsilon_n^- \]

with momentum conservation all propagators along the line get their momentum shifted
Expanding the shifted vertex

\[
\hat{V}_{\mu\nu\rho} = g_{\mu\nu}(\hat{p}_1 - p_2)_{\rho} + g_{\nu\rho}(p_2 - \hat{p}_3)_\mu + g_{\rho\mu}(\hat{p}_3 - \hat{p}_1)_\nu
\]

\[
= g_{\mu\nu}(p_1 - p_2)_{\rho} + g_{\nu\rho}(p_2 - p_3)_\mu + g_{\rho\mu}(p_3 - p_1)_\nu
\]

\[-z\varepsilon^\sigma (2g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho} - g_{\mu\nu}g_{\sigma\rho})
\]

\[
= V_{\mu\nu\rho} - z\varepsilon^\sigma V_{\mu\nu\rho\sigma}
\]
cutting along a propagator

- multiply with $\hat{P}^2$
- set $z = z_0$ such that $\hat{P}^2 = 0$
- divide with $P^2$

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a simple case: make the shift

- $z$
- $z$
- $+z^2$

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a simple case: cut along the (only) propagator

\[ z_0 = -\frac{p_{1A}^2}{2(p_{1A} \epsilon)} \]
a simple case: after the cut
a simple case: after the cut

\[ \text{The same holds if instead of external lines we have full subamplitudes in the upper lines.} \]

We can talk about classes of diagrams with the same partitioning of the remaining n-2 ext. legs.
the next to simplest case
the next to simplest case
the next to simplest case
the next to simplest case

\[ \sum \text{cuts} \]
the next to simplest case
the next to simplest case

\[ \sum \left( \begin{array}{c}
\text{cuts}
\end{array} \right) = \begin{array}{c}
\text{graphs}
\end{array} \]
the general case

\[ \sum \text{cuts} \]
the general case

- Two or more four vertices = sum over cuts of their shifted equivalent.
- One four vertex = sum over cuts of their shifted version plus terms
- Zero four vertex = sum over cuts of their shifted version minus terms
summing up

The sum of all Feynman diagrams

= 

The sum of (the sum over cuts along the main line) of the shifted version of all Feynman diagrams
regrouping in BCFW decompositions

• grouping together all shifted diagrams with the same propagator cut gives a BCFW decomposition!!!!

P. Draggiotis, R. Kleiss, C. Papadopoulos, A. L.

→ light cone coordinates at the Lagrangian level, space-cone gauge and the largest time equation

D. Vaman, Y. P. Yao [hep-th/0512031]
efficiency in tree level

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<th>7</th>
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Table 1: CPU time in seconds for the computation of the $n$ gluon amplitude on a standard PC (2 GHz Pentium IV), summed over all helicities.

Dinsdale, Ternick and Weinzierl [hep-ph:0602204]

- explicit sum over helicities and color ordering
- algebraically recursive, numerically ??? (shifted momenta)
- Recent treatment of color to avoid factorial growth

Duhr, Hoche, and Maltoni [hep-ph/0607057]
prospects?

- Further advances in the front of NLO, (NNLO??) calculations
- Possibility for a more efficient tree level implementation (??)
- Using the shifting / analytic continuation idea as a tool in specific analytical calculations