Magnetic Field of Magnetized Ellipsoids

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As the quality factor of accelerating cavities in LCLS-II cryomodules is expected to exceed $3 \cdot 10^{10}$, magnetic field hygiene becomes a never ending theme for multiple discussions. The level of magnetic field in the cryomodules is significantly reduced by using steel as a material for the vacuum vessel and compensating coils to effectively cancel the longitudinal component of the magnetic field; nevertheless, different sub-systems in the cryomodule can use magnetized parts (e.g. step motors), or hardened magnetic materials (e.g. bearings), or soft magnetics (e.g. steel bolts). Some non-magnetic materials or materials with low permeability can undergo phase transformation after unintentional heat treatment (e.g. by welding) or after cooling down below the phase transition threshold temperature. To get a simple gauge of potential threat to the expected performance of the cryomodules, a study was made with the goal to find a set of expressions that could be used for evaluation of the magnetic field introduced by the magnetized parts or the parts made of magnetic materials in a background magnetic field. For simplicity, ellipsoid was used to reproduce the shape of the parts as well-known theoretical expressions can be used; main conclusions of the study do not depend of the shape though. Practical system of units will be used, where the field in the magnetic material can be written as

$$\mathbf{B} = \boldsymbol{\mu_0 \mu_r} \mathbf{H} + \mathbf{M} \qquad /1/$$

It is known (e.g. see [1]) that the magnetic potential of a magnetic dipole $\boldsymbol{m}$ in vacuum can be expressed as:

$$\mathrm{Um} = 1/(4\pi\mu_0)\cdot[\boldsymbol{m} \times \boldsymbol{r}]/r^3 = 1/(4\pi\mu_0)\cdot[3(\boldsymbol{m}\boldsymbol{r^0})\cdot\boldsymbol{r^0}/r^3 - \mathbf{m}/r^3] \qquad /2/$$

Here the bold font is used to show vectors; $\boldsymbol{r^0}$ is the unit vector in the direction of the radius r. This expression converges to the following expressions for the z-component of the magnetic field in the **z** direction (vector $\mathbf{z^0}$ is parallel to the long axis 2a of an ellipsoid, which is magnetized along this direction) in the perpendicular direction $\boldsymbol{r^0}$ along the short axis 2b:

Along Z: $H_z = 2m/(4\pi\mu_0 r^3)$; /3a/

Along R: $H_z = -m/(4\pi\mu_0 r^3)$. /3b/

For the flat contour, $\boldsymbol{m} = \mu_0\mu_r \boldsymbol{I}\cdot\boldsymbol{A}$, where $\boldsymbol{A}$ is the surface area (vector) enclosed by the contour.

For a magnetized ellipsoid with unit permeability, we can use the magnetization **M** to evaluate the efficient current density: $j_\tau = M/\mu_0$. In this case it is close to the value of the internal magnetic field H_{in}. Changing contour surface area is taken care by integration; as a result, we have

$$m \approx 4/3\pi\mu_r\mu_0\cdot ab^2 H_{in} \ \text{ or } \ m \approx \mu_r\mu_0\cdot V\cdot H_{in} \qquad /4/$$

As an example, let's use the next values for the half-axis of the ellipsoid: a = 5 mm, b = 1 mm.

Let's first consider a fully magnetized steel piece with the remnant field of 1 T: $\mu_r = 1$ and $\mu_0 \cdot H_{in}$=1 T. Fig. 1 shows a graph comparing results of modeling (COMSOL) and using analytical expression for the Z field component along the Z axis. The field drops two orders of magnitude just 50 mm from the center of the ellipsoid.

One can see that the field is proportional to the magnetization and to the volume of the sample, so it is straightforward to make scaling when needed.

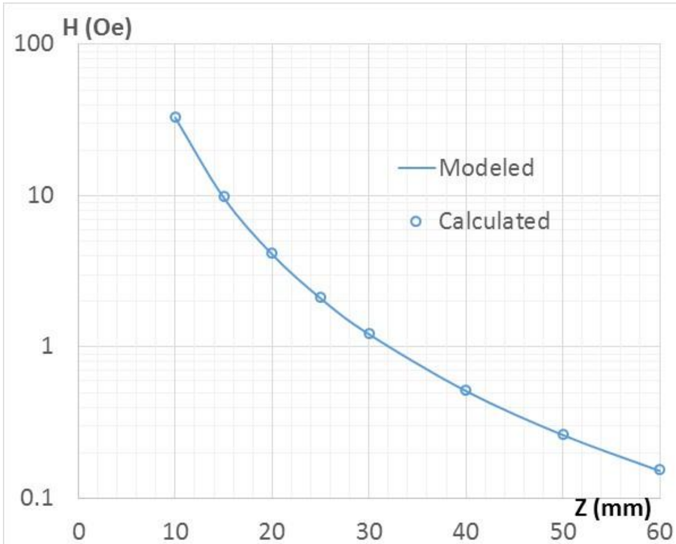

Fig. 1. Magnetic field of the test ellipsoid with magnetization 1 T along the long axis

In the case of the magnetization induced in the ellipsoid by the external field H_0, we need to take into account the following expressions for the magnetization:

$$M = \mu_0(\mu_r - 1)H_i \qquad /5/$$

$$H_i = H_0/[1+N\cdot(\mu_r - 1)] \qquad /6/$$

Here N is a demagnetization factor that can be found for ellipsoids using the next formula that uses parameter p = a/b when it is higher than 1:

$$N = 1/(p^2-1)\cdot\{p/\sqrt{(p^2-1)}\cdot\ln[p+\sqrt{(p^2-1)}]-1\} \qquad /7/$$

Graph on Fig. 2 shows values of the parameter p in the range from 0 to 10.

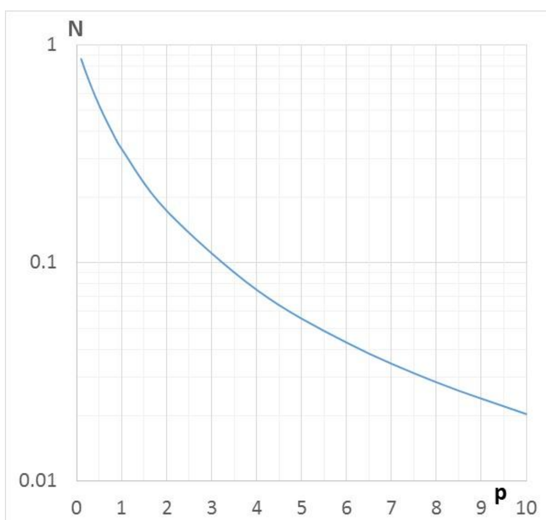

Fig. 2. Demagnetization factor for a range of parameter p = a/b from 0 to 10

For the previously used ellipsoid, p = 5, and N = 0.056.
Then magnetization

$$M = \mu_0(\mu_r - 1)H_0/[1+N\cdot(\mu_r - 1)] \qquad /8/$$

The magnetic moment is then

$$m = \mu_r/(\mu_r - 1)V\cdot M = V\cdot \mu_0 H_0(\mu_r - 1)/[1+N\cdot(\mu_r - 1)] \qquad /9/$$

Let's assume the external field of 1 G (1E-4 T or 80 A/m) and $\mu_r = 200$
Then m = 3.5E^{-11} T·m^3. Graphs that compare results obtained by modeling (COMSOL) and by using the theoretical expression for the field of a magnetic dipole can be found in Fig. 3.
As the internal field flux density in this case is 17 G, it is ~600 times smaller than in the previous case, we expect the same ratio of the two magnetic moments and the fields, which we sure have.

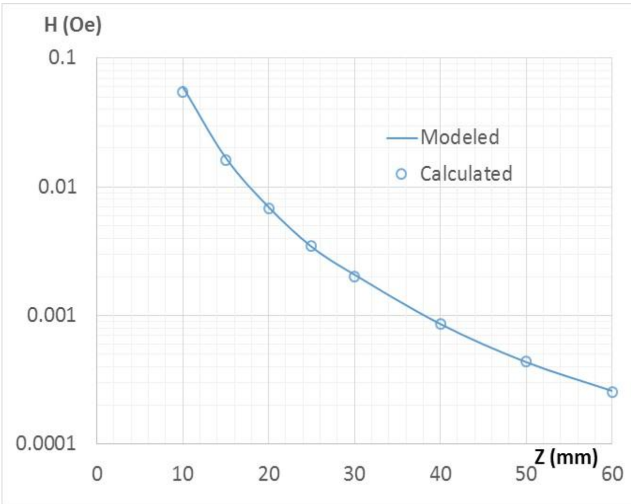

Fig. 3. Magnetic field of the test ellipsoid with μ_r = 200 placed in the external 1 G field.

Field is decaying very fast near the sharp end of the ellipsoid; just 2 mm off the end, the field changes from ~17 G to ~1.2 G, which is illustrated by the following figure.

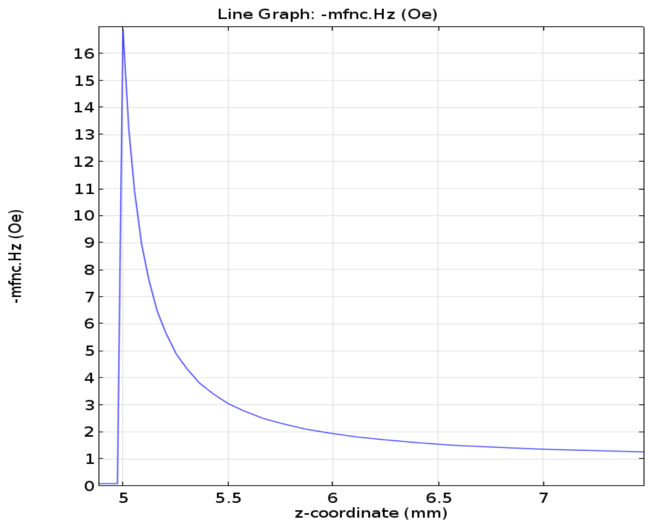

Fig. 4. Fast field decay near the end of the ellipsoid.

In the transverse direction the field near the ellipsoid is lower than the background; it reaches back to the background at $r \approx 10$ mm, which is the scale of the system.

If we use a magnetized sphere with $\mu_r \approx 1$, the results are fully compatible with what was found earlier for the ellipsoid.

If a sphere with $\mu_r > 1$ is placed in a background magnetic field, the induced magnetization can be found using the following expression:

$$M = 3\mu_0 H_0 \cdot (\mu_r - 1)/(\mu_r + 2), \qquad /10/$$

which is what expression /5/ can be converted to with N = 1/3 and p = 1 in /7/. If $\mu_r >> 1$, the magnetization is three times stronger than the background field. The value of the induced magnetic dipole is found by integration of the magnetization though the volume of the sphere, which gives

$$m = 3V \cdot \mu_0 H_0 \cdot (\mu_r - 1)/(\mu_r + 2) = 4\pi \cdot a^3 \cdot \mu_0 H_0 \cdot (\mu_r - 1)/(\mu_r + 2)$$

This expression can also be obtained by using /9/ with N = 1/3.

Graphs that compare results obtained by modeling (COMSOL) and by using the theoretical expression for the field in this case can be found in Fig. 5. Flux density inside the sphere in this case is ~3 G.

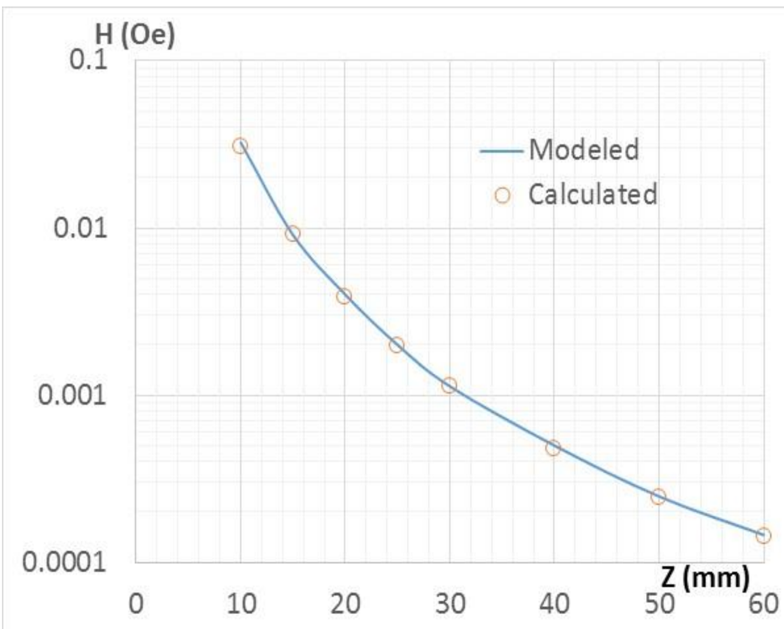

Fig. 5. Magnetic field of the test sphere (r0 = 2.5 mm, $\mu_r = 200$) placed in the external 1 G field.

Let's take a look at two examples that illustrate how to use the suggested approach to analyze possible threats of using magnetized objects in a cryomodule.

Example 1

Let's assume that we have a steel bolt with the aspect ratio 5 used somewhere in a cryomodule with the external magnetic field B_0. As the material is in the relatively low field, the permeability of steel is also relatively low – we assume $\mu_r = 200$. To evaluate the impact of this element on the field in the cryomodule, we will use the formulas for the ellipsoid with the aspect ratio $p = a/b = 5$. The external field is directed along the long axis.

Let's find the maximum allowed volume (or diameter, which is equivalent) of the bolt assuming that at the distance $r_0 = 0.1$ m the disturbance of the background field must not exceed 10%, that is the magnetic field associated with the induced magnetization is ten times lower than the background field: $H_m < 0.1H_0$. Using expression /9/ for the induced magnetic moment:

$$m = (\mu_r\text{-}1)\mu_0 V H_0 \,/\, [1+N(\mu_r\text{-}1)],$$

and expression /3a/ for the magnetic field along the long axis of the magnetized ellipse, we come to the next expression for the total volume of the material of this particular shape to induce the dangerous field disturbance:

$$V = 2\pi\cdot(r_0)^3\cdot H_i/H_0\cdot[1+N(\mu_r - 1)]/(\mu_r - 1) \qquad /11/$$

With $r_0 = 0.1$ m, $H_i/H_0 = 0.1$, $\mu_r = 200$, and $N = 0.056$, we get $V \approx 35$ cm^3, which corresponds to the equivalent **cylinder** with the length ~11 cm and the diameter ~2 cm.

In the direction perpendicular to the long axis, the background field will be reduced by ~5%.

If the background field is 0.1 G, the magnetic field inside of this object will reach ~1.6 G. If to measure the field in the vicinity of the object in the environment of the earth field (0.5 G), one can see ~8 G field. This a near zone where the formulas /3/ cannot be applied, but they will give close results though if the distance from the end exceeds several diameters of the object. For this example, where the length of the **ellipsoid** is ~125 mm and diameter is ~25 mm, the increase of the field at 100 mm from the center should be higher than 10%. Magnetic modeling gives ~15% increase: magnetic field measured at 100 mm from the center of the ellipsoid placed in 8 A/m background field is 9.3 A/m.

Example 2

Let's evaluate allowed magnetization of steel ball (Ø6 mm) of a ball bearing so that the field generated at the distance 50 mm from its center would not exceed one tenth of the background field $B_0 = 100$ mG.

In accordance with /4/, $m = B_{in}V$, where B_{in} is the flux density inside the ball (and hence the maximum field measured near its surface). Using /3a/ we can get the following equation

$$B_0/10 > B_{in}V/2\pi r^3 \qquad /12/$$

So the field measured at the surface of the ball must be

$$B_{in} < \pi r^3 B_0/5V = 3/20\cdot(r/r_0)^3\cdot B_0 \approx 69 \text{ G} \qquad /13/$$

If the balls in a bearing are magnetized stochastically, mush higher magnetization is probably acceptable. If the several balls in a bearing are magnetized in the same direction, some attention is needed as resulting shape becomes more complicated and the characteristic dimension becomes larger and expression /3/ becomes valid at larger distances.

Conclusion

As a result of this study the following conclusions can be made:

1. Magnetic field generated by a magnetized object can be evaluated using the “magnetic moment” approach.
2. Using magnetic modeling, it is possible to find equivalent magnetic moment of any object.
3. The magnetic moment scales linearly with the volume of the object and with the magnetization.
4. Measuring magnetic field in the vicinity of a part that is suspected to be a source of undesired magnetic field does not provide reliable information about the strength of its induced magnetic moment, but it still can be used to evaluate residual flux density in a magnetic object.

The presence of a magnetic shielding around cavities in a cryomodule requires using entirely different approach. The shield itself usually significantly reduce the background field in the area around the cavity; nevertheless technological holes in the shields must be taken into account during direct and specific modeling when parts or assemblies are suspected in being magnetic pollutants. This kind of modeling is being made; results will be posted in future notes.

References

1. K. Simonyi, Theoretische Electrotechnik, Berlin, 1989