Modeling Quench Propagation in Inductively Coupled System of Coils:
Dump Resistors Connected in Parallel to Each Coil in the System

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I. Introduction

In an attempt to improve existing tool for modeling quench protection (QP) of the SS2 focusing lens of the HINS linac front end [1] and make the QP technique more flexible and reliable, a QP case was studied where each coil in a system of inductively coupled coils has a dump resistor permanently connected in parallel to the coil. In the previously developed approach [2], only two dump resistors were used, which were connected to the main coils **or** to the bucking coils. Although quite adequate for the SS2 lens system, this approach could not be applied to an arbitrary system of coils without making significant reservations. Having the same system of coils as a polygon for studying the improved (new) approach, we had an opportunity to verify results obtained by the new modeling tool by comparing them with what was obtained earlier in [2]. The most challenging part in the development of the new tool was to properly describe magnetic field on turns of all quenching coils, partly because in this case none of the lens coils' currents coincide with the total circuit current.

Although the approach described in this note is exemplified by analysis of the SS2 focusing lens, and the modeling program is mostly an expansion to previously completed work [2], it is valid for other systems, e.g. Mu2e or ITER CS coil system.

To cover a QP problem in depth, the next effects must be taken into account and embedded in the modeling tool:

- each coil in the system is equipped with its own dump resistor;
- quenches can happen in any of the coils in the system, initially and/or with some time delay;
- a diode or a combination of a diode and a resistor is used instead of a dump resistor;
- thermal insulation between the coils can transfer heat from coil to coil;
- active heaters are included in the system and an option of using the active heaters as dumps is provided;
- inductive heating of metal structure of the coil system and transfer of generated heat to the coils.

At the moment, only the first two features of the list are implemented in the code. Implementation of the next three features does not seem too challenging, and can be made if a need to include the features becomes obvious. It is more difficult to introduce the last feature in the list, although this is also quite feasible.

II. The modeling code

A modified circuit for modeling QP of the SS2 focusing lens of the HINS linac front end is shown in Fig. 1 below.

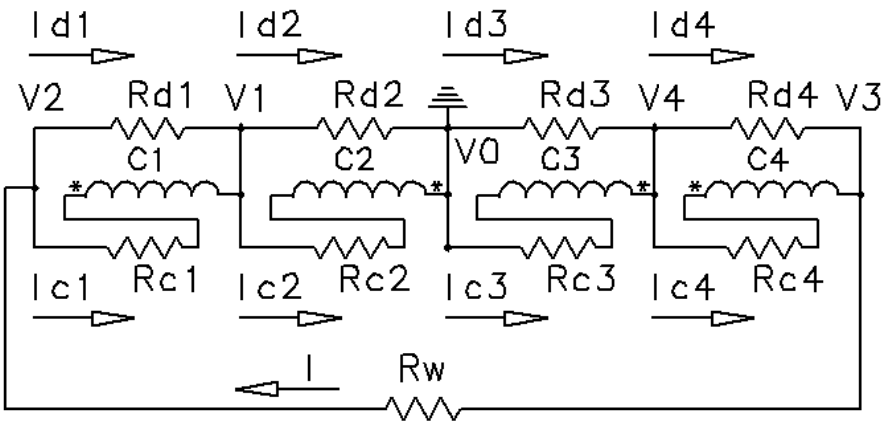


Fig. 1: Current discharge circuit in the case when quench happens in every coil and each coil is equipped with a dump resistor.

For the purpose of easy expansion to a different system of coupled coils, all the coils in the scheme are numbered, and functional difference of the coils is not reflected in the scheme. This simplifies generation of equations to solve the system. For the purpose of analysis of the SS2 system, and to simplify comparison with the case studies in [2], this assumption will not be strictly followed though.

The program is written in MATLAB and consists of several M-Files: an Input File, a Main File, an Output File, and several files which describe functions used in the program.

In the Input File, most of the geometrical parameters and program constants are defined. Since the bucking coils and main coils have different properties, two variations of some parameters are created: B_X for the bucking coil and M_X for the main coil. For example, the inner diameter for the bucking coil is defined as B_Di, while the inner diameter for the main coil is M_Di. Some parameters are different for each coil, so they will be named B1_X, B2_X, M1_X, or M2_X. For example, the temperature matrix for the first bucking coil is labeled as B1_Tturn. Other relevant parameters include the values of the dump resistance, which are labeled Rd#, where # is determined by the coil to which the dump resistor is connected (numbering begins at the first bucking coil and ends at the last bucking coil, as shown in Fig. 1). All parameters can be modified by users.

Quench event is initialized by assigning a temperature at a point in one of the coils higher than the critical temperature; the point (or a set of the points) is chosen prior to the program run. By default, this point is at the location of greatest magnetic field, but any other location can be used instead. In general, an arbitrary array of quenching points can be chosen, including points in several (e.g. all) coils.

In the Main File, all the calculations necessary for modeling are carried out. First, the Input File is called and all the commands inside it are completed. Then, arrays are initialized that will be used to store the values of certain variables at evenly-spaced times of quench propagation modeling. These arrays provide information about the evolution of variables throughout the duration of the quench, and later will be used to generate visual output. Before calculations begin, a list of computational properties and user settings are displayed for convenience. Next, a "for" loop begins which will run for the duration of quench propagation modeling specified by the user. The modeling is based on time steps; the length of the step can be specified in the Input File. A smaller time step will result in greater computational time, but will provide more accurate results. Parameters found in the Input File determine initial conditions used by the loop. Each cycle of the loop uses conditions calculated by the previous cycle.

In the loop:

First the magnetic field in each coil is evaluated. Earlier analysis (Appendix, #1) determined that calculating the magnetic field using an analytical approach (Appendix, #2) would not be accurate, since it would not take into account the flux return made of soft steel. Instead, Comsol was used to model the magnetic system with the flux return. Data was extracted at various ratios of currents in the coils to the total current in the circuit. This data can be found in the M-File labeled "Magnetic_Field." To calculate the magnetic field in the bucking coil, only the current ratio of the bucking coil and the current ratio of the nearest half of the main coil are important. For a half of the main coil, the ratios of this half coil, of the nearest bucking coil, and of the other half of the main coil are needed. The data computed at various current ratios is compiled into multi-dimensional arrays. Interpolation using the current ratios calculated during the time step is carried out on these data arrays. The magnetic field data was collected at the 1 A circuit current because the dependency of the magnetic field on the circuit current is very close to linear. Interpolated data effectively forms a transfer function that must be multiplied by the total current in the circuit to find the actual magnetic field in the coil.

The resistivity of the turns in the coils is then evaluated. A function has been created for this purpose (the use of functions provides more flexibility for changes to the magnet system, such as the addition and removal of coils or use of a different superconducting material). This function takes into account factors such as coil temperature and magnetic field, and is stored in a separate M-File ("TurnResFunc"). If the working point in the Temperature - Magnetic Field - Current Density space at the location of each turn is inside the region bounded by the critical surfaces, the resistivity is set to 0. Otherwise, the resistivity calculated for a normally conducting material applies. The resistivity of each turn is used to find the resistance of the quenching coil layers. Then, the total resistance developed in the quenching coil is evaluated as the sum of layer resistances.

Next, the current derivatives for the present time step are calculated. A system of equations can be set up based on the circuit diagram in Fig. 1. As all parameters of the circuit, namely the mutual inductances between the coils, the currents in the coils, the currents in the dump resistors, the dump resistances, and the resistances developed in the quenching coils are known, the system

of equations can be solved to find the values of all the current derivatives in the present time step. A full explanation of this method can be found in (Appendix, #3).

The voltages in the system can then be found. In each coil, the voltage of each layer relative to the innermost layer is calculated as the sum of the voltages of the inner layers. The voltage across a dump resistor is equal to the product of the resistance and the current in the resistor. Using the circuit diagram in Fig. 1, the voltages to ground in the quenching coil and in any point of the circuit can be found. To find the voltage to ground of each layer in a bucking coil, the previously calculated layer voltage relative to the innermost layer can be added to the voltage to ground in the proper location of the circuit.

The instantaneous power in a dump resistor is calculated as the square of the current in the resistor multiplied by the resistance. The total energy dissipated in the dump resistor can then be found as the sum of the previous total of energy dissipation and the energy dissipated in the current time step, which is evaluated as the instantaneous power multiplied by the length of the time step.

Since now we know the values of all variables in the current time step, their values can be stored in the previously prepared storage arrays. However, they are not stored during each iteration of the loop. Instead, the variables are entered into the arrays in an evenly-spaced manner so that the total number of points stored is equal to the parameter "Nslice" found in the Input File. Modifying the value of "Nslice" will alter the size of the storage arrays, which will ultimately determine the resolution of the output graphs (having about 100 data points usually provides good resolution).

Now, the values of variables in the next time step must be computed, starting with the coil temperature. If the coil is not quenching, there is no heat generation due to energy dissipation, which would cause a temperature increase. A coil's temperature could still rise if some heat comes from adjacent, quenching coils through insulation or from active heaters. If the coil is quenching, the temperature change is calculated in a function ("TempFunc") which takes into account strand heating due to resistance and heat propagation through conduction. The change in temperature for the time step is added to the current coil temperature to find the coil temperature for the next time step.

To find the currents in the coils in the next time step, the change in current (described by dIdt*time) is added to the currents in the present time step. The currents in the dump resistors in the next time step also must be determined. A system of equations is set up using the circuit diagram in Fig. 1. Knowing the values of the coil currents in the next time step and the constant dump resistances, the system of equations can be solved to find the currents in the dump resistors in the next time step. This method is similar to the technique used earlier in calculating the current derivatives, and is described in (Appendix, #3).

All the needed values for the next time step are now known, so the "for" loop can be carried out for the next cycle. The program commands in the loop will be completed for each time step until the duration of quench propagation modeling specified by the user has been reached. Then,

several ending commands may be called, such as a call to the Output File so that data is displayed.

The Output File contains commands which will visualize an assortment of data. For example, graphs of currents vs. time, voltages vs. time, instantaneous temperature in coil, etc. may be called. These commands may be adjusted so that related data (e.g. in other coils) is displayed. The Output File is provided for convenience, and any additional commands may be added to the file or called directly in Matlab after the program has finished running.

A block diagram of the program is shown below.

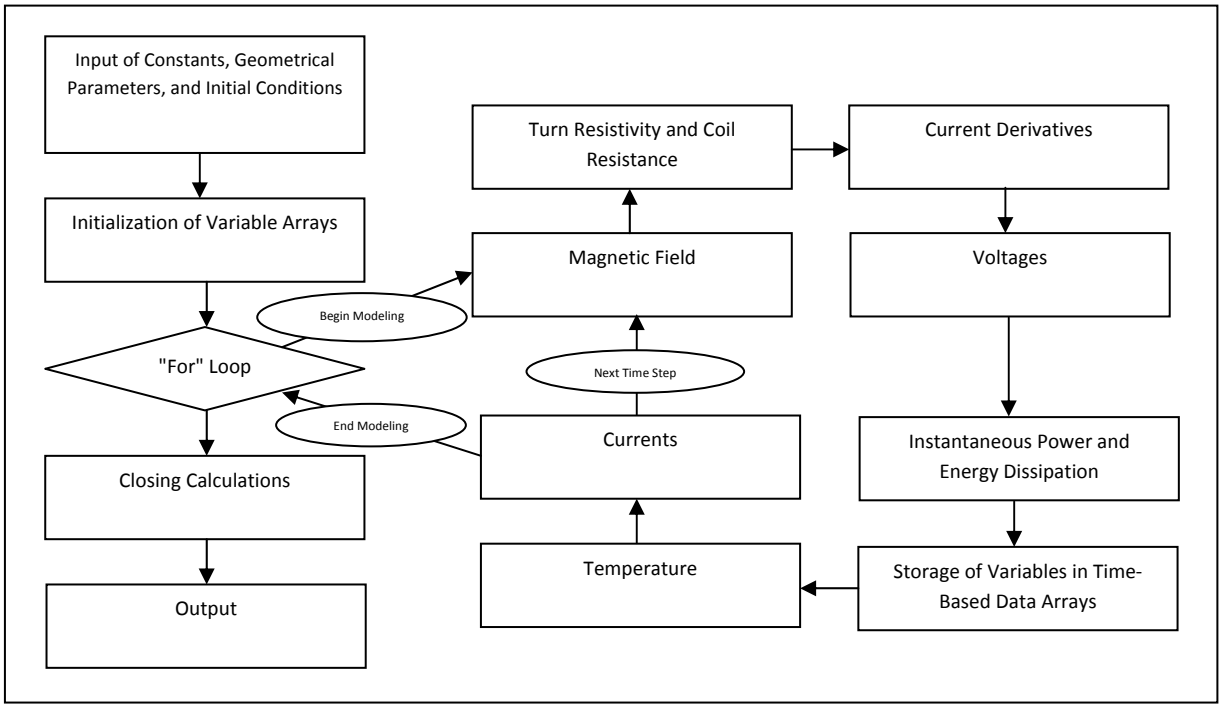


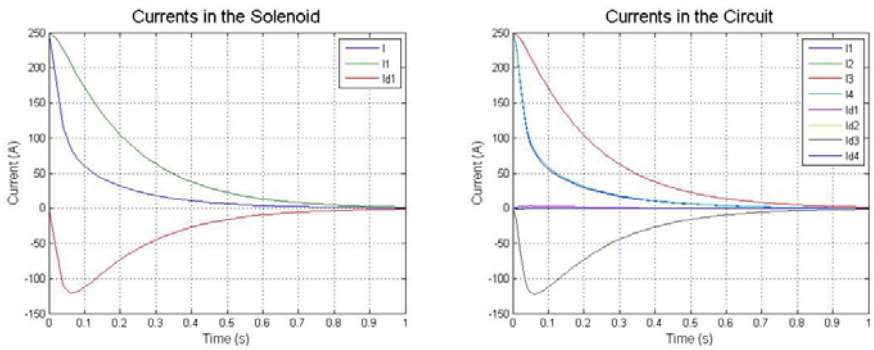
Fig. 2. Block diagram of the quench propagation code

III. Verification of the code

To verify the accuracy of the program, the results can be compared to the previously completed work [2], which did not use a general approach to quench protection. A lack of dump resistors connected to halves of the main or to the bucking coils can be simulated by assigning a high value to the corresponding dump resistors (e.g. 100 Ohms). Because of the high resistance, a minute amount of current will flow through the dump resistor so it will be almost ineffective in dissipating energy. The following program parameters were used: the starting current *I0* = 249 A, the values of dump resistors: *Rd1* = 100 Ohm, *Rd2* = 1.3 Ohm, *Rd3* = 1.3 Ohm, *Rd4* = 100 Ohm, the step size ***stepT*** = 10^{-4} s, the length of the output array, *NSlice* = 100. Figures 3 to 9 compare results obtained in [2] with the results calculated using the general approach for the BC-MC case (quench in a bucking coil and dump resistors connected in parallel around the main coils). Similarly, Figures 10 to16 compare the results for the MC-MC case (quench in a main coil and dump resistors connected in parallel around the main coils). Table 1 provides a summary of the

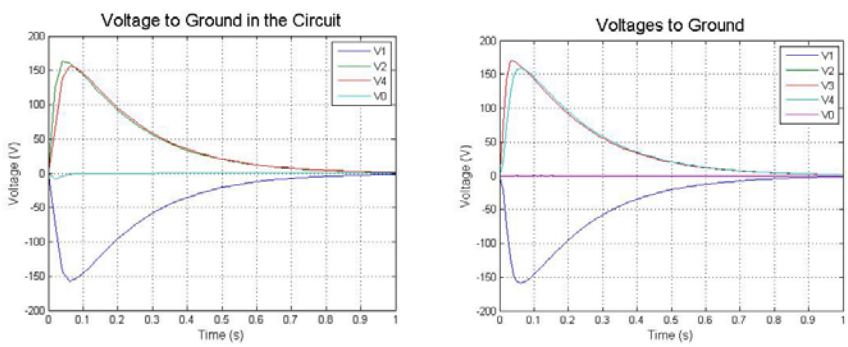
results obtained by both codes for the BC-MC case, while Table 2 contains a summary of the MC-MC case.

BC-MC: Quench In Bucking Coil. Dump Resistors around Main Coils



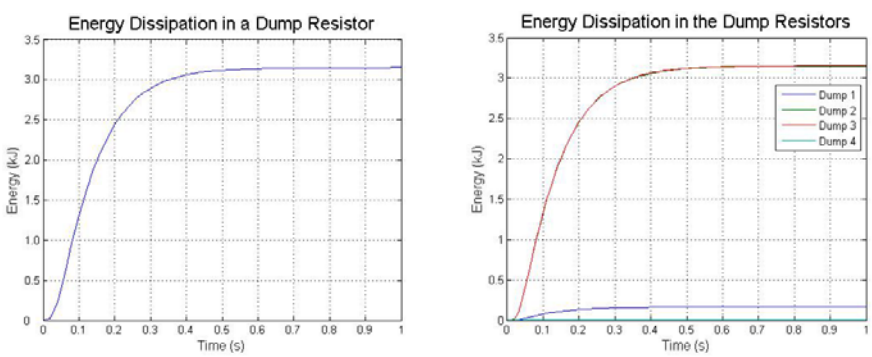
a) b)

Fig. 3. Currents in the circuit: a) old method; b) new method



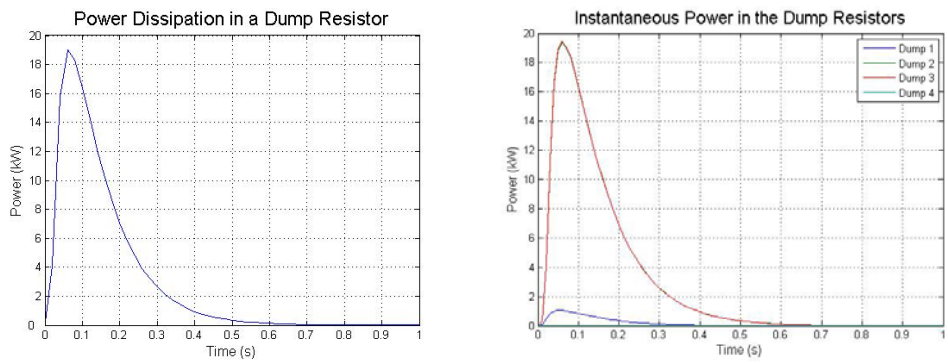
a) b)

Fig. 4. Voltages to ground in the circuit: a) old method; b) new method



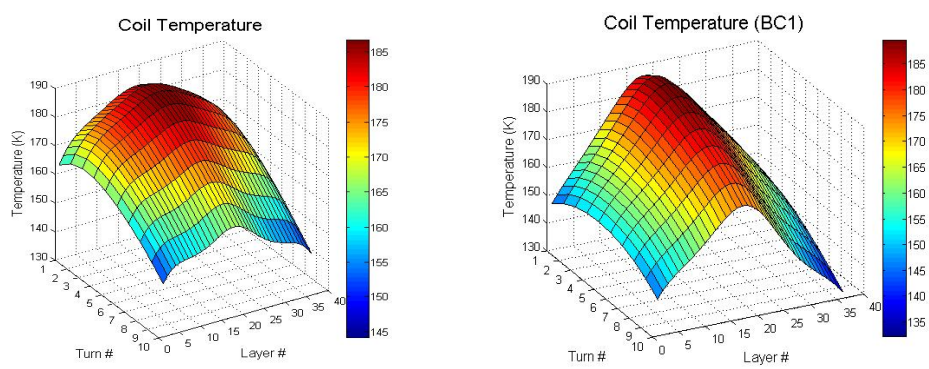
a) b)

Fig. 5. Energy dissipation in the dump resistors: a) old method; b) new method



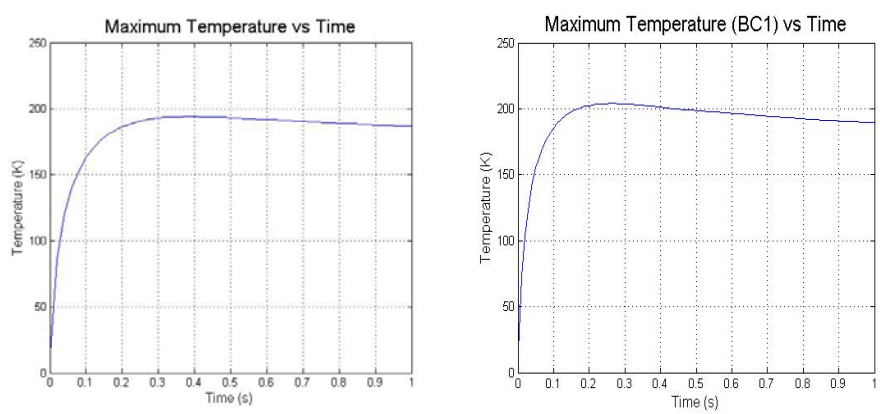
a) b)

Fig. 6. Instantaneous power in the dump resistors: a) old method; b) new method



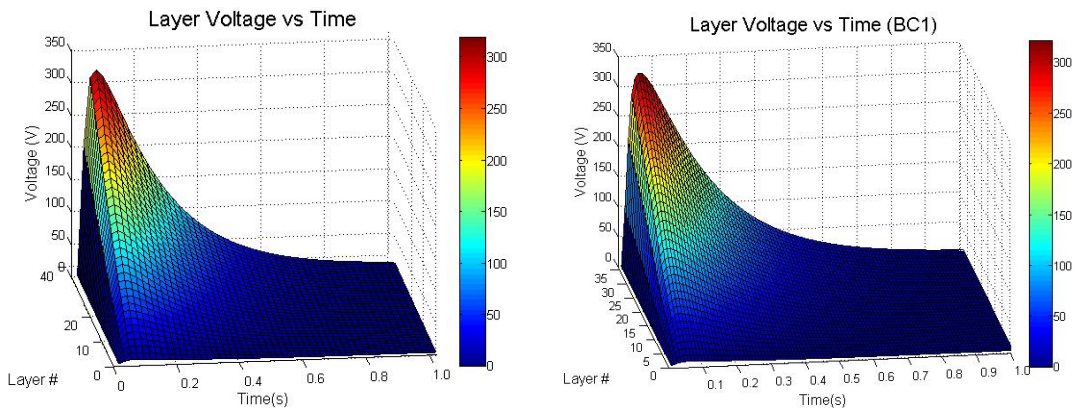
a) b)

Fig. 7. Temperature distribution in quenching coil, at 1 sec: a) old method; b) new method



a) b)

Fig. 8. Maximum temperature in the quenching coil: a) old method; b) new method



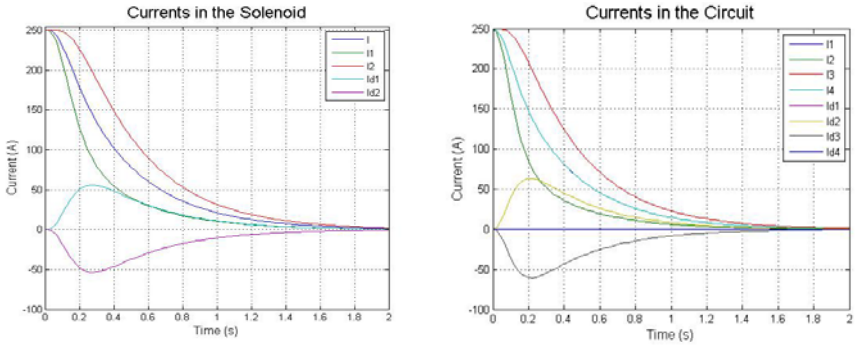
a) b)

Fig. 9. Voltage in the layers of the quenching coil relative to the inner layer: a) old method; b) new method

Table 1. Comparison of the results of the two methods for the BC-MC Case

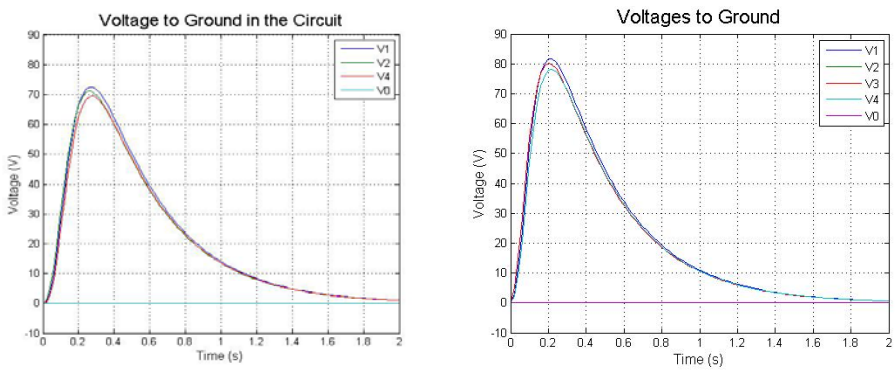
Method	Max. Coil Resistance (Ohm)	Max. Coil Temperature (K)	Max. Voltage to Ground (V)	Max. Voltage across Coil (V)	Max. Power in Dump 2 (kW)	Max. Power in Dump 3 (kW)	Energy Dissipation in Dump 2 (kJ)	Energy Dissipation in Dump 3 (kJ)	Energy Dissipation in Coil (kJ)
Old	6.32	194	163	318	19.0	19.0	3.15	3.15	4.70
New	6.69	204	170	321	19.4	19.4	3.15	3.15	4.45

MC-MC: Quench In Main Coil. Dump Resistors around Main Coils



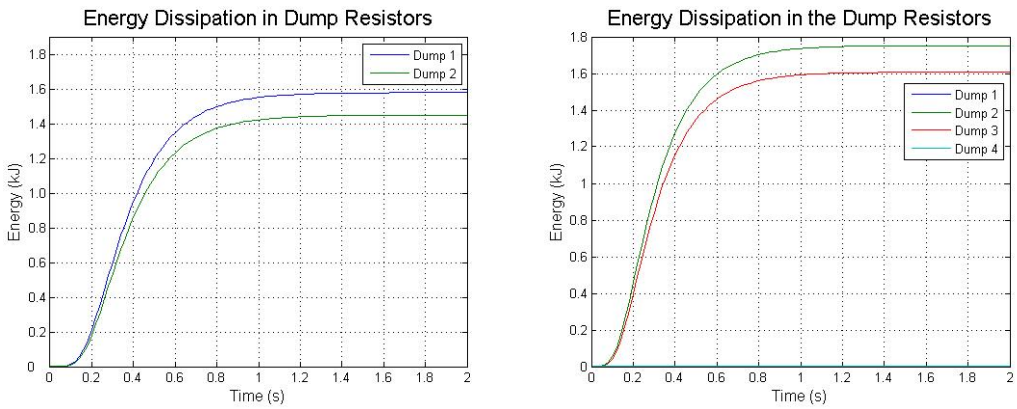
a) b)

Fig. 10. Currents in the circuit: a) old method; b) new method.



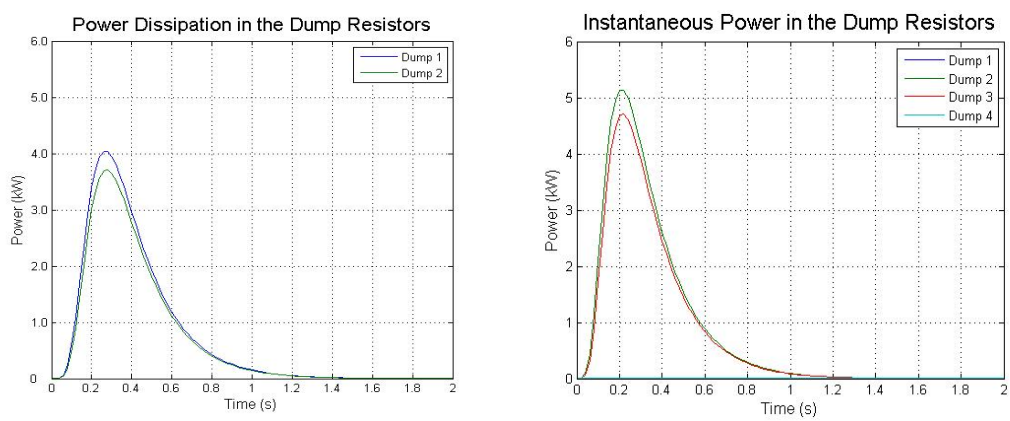
a) b)

Fig. 11. Voltages to ground in the circuit: a) old method; b) new method



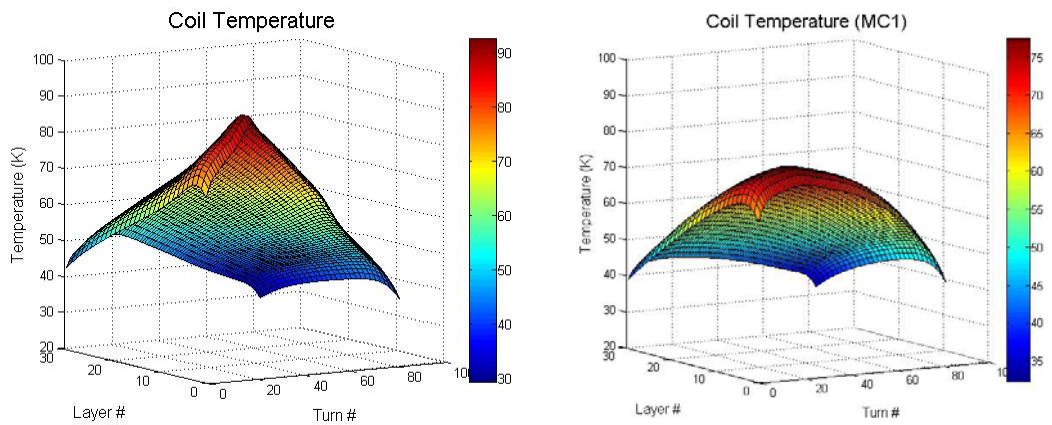
a) b)

Fig. 12. Energy dissipation in the dump resistors: a) old method; b) new method



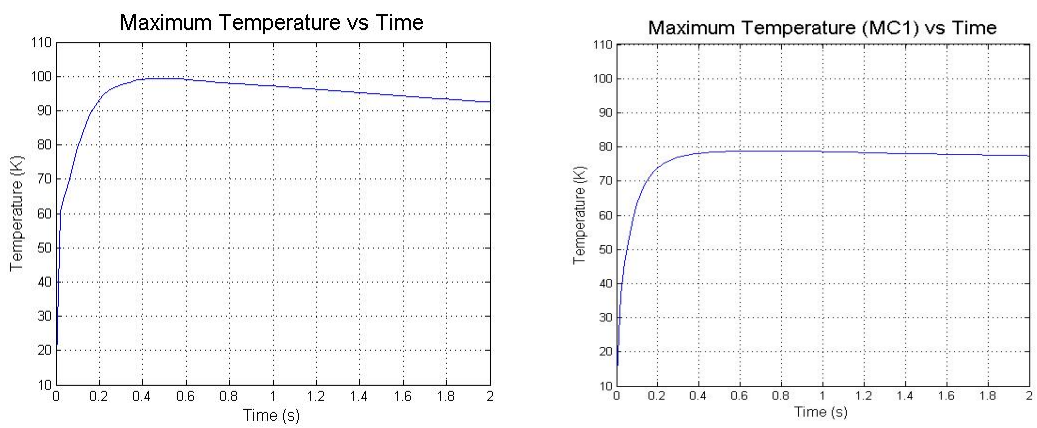
a) b)

Fig. 13. Instantaneous power in the dump resistors: a) old method; b) new method



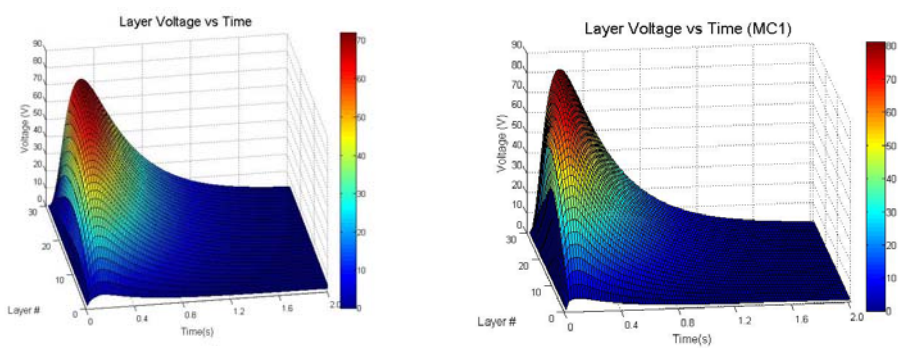
a) b)

Fig. 14. Temperature distribution in quenching coil, at 2 sec.: a) old method; b) new method



a) b)

Fig. 15. Maximum temperature in the quenching coil: a) old method; b) new method



a) b)

Fig. 16. Voltage in the layers of the quenching coil relative to the inner layer: a) old method; b) new method

Table 2. Comparison of the results of the two methods for the MC-MC Case.

Method	Max. Coil Resistance (Ohm)	Max. Coil Temperature (K)	Max. Voltage to Ground (V)	Max. Voltage across Coil (V)	Max. Power in Dump 2 (kW)	Max. Power in Dump 3 (kW)	Energy Dissipation in Dump 2 (kJ)	Energy Dissipation in Dump 3 (kJ)	Energy Dissipation in Coil (kJ)
Old	1.88	99.4	72.5	71.9	4.04	3.71	1.58	1.45	7.98
New	2.33	78.9	81.7	81.08	5.14	4.71	1.75	1.61	7.56

For the BC-MC case, the results of the two programs are almost identical for the following categories: currents, voltages, energy dissipation, and instantaneous power. However, the distribution of temperature in the quenching coil is somewhat different. Also, the maximum temperature and the maximum resistance developed in the quenching coil are slightly higher for the new program. The predictions of the two programs differ by less than five percent.

For the MC-MC case, all of the variables have somewhat different instantaneous values. Results obtained by the two programs differ by a maximum of about twenty-five percent. The disparities in the results of the two programs could are due to following reasons. The most important one is that the magnetic field calculations for the newer version of the program takes into account the ratios of currents in two or three coils, while the older version calculates the magnetic field based on ratios of currents in only one or two coils, which results in different magnetic field on turns, turn temperature, and resistance. Besides, the calculations made using the general approach use dump resistors with resistances of 100 Ohms, as opposed to the old approach with an absence of resistors connected in parallel to some coils.

IV. Summary

A new quench protection technique for use with the SS2 focusing lens of the HINS linac was made possible by developing a tool for modeling it. The technique employs dump resistors permanently connected to each coil in the system. Generalized approach used in the developed code allows analysis of more wide class of systems with inductively coupled superconducting coils. Comparison of results obtained by using the new code with those obtained previously in [2] demonstrates satisfactory accuracy of the QP modeling for the SS2 lens system using the "old" protection technique and the "old" modeling tool. More elaborate study of the protection system using the new protection technique and the new modeling approach will follow.

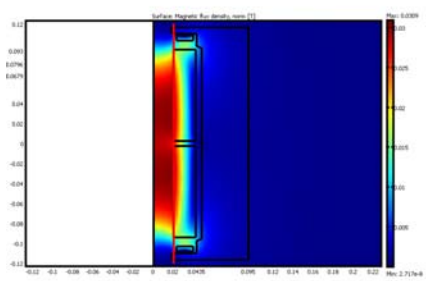
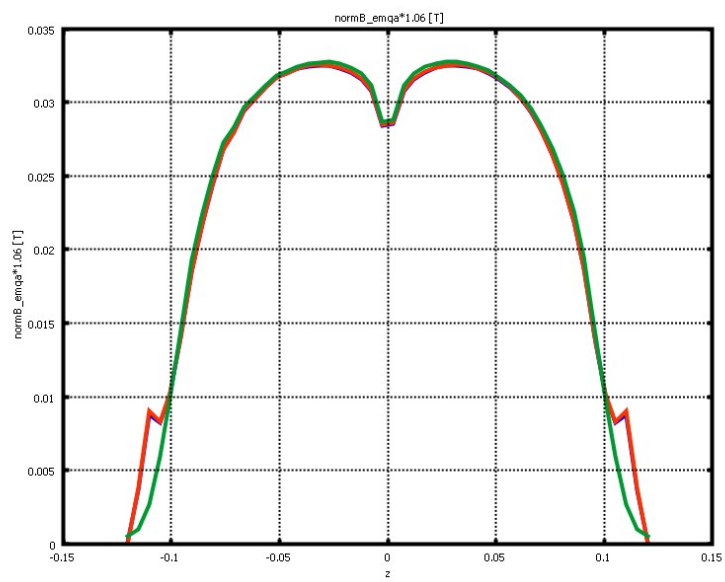
References:

1. M. Tartaglia, et. al., "Solenoid Focusing Lenses for the R&D Proton Linac at Fermilab", PAC-09.
2. E. Khabiboulline, I Terechkine, "SSR2 Focusing Lens Quench Protection Study", FNAL TD note TD-11-006, April 2011.

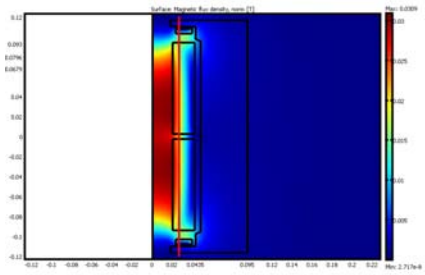
Appendix, #1: Comsol Modeling of Magnetic Field in Coils

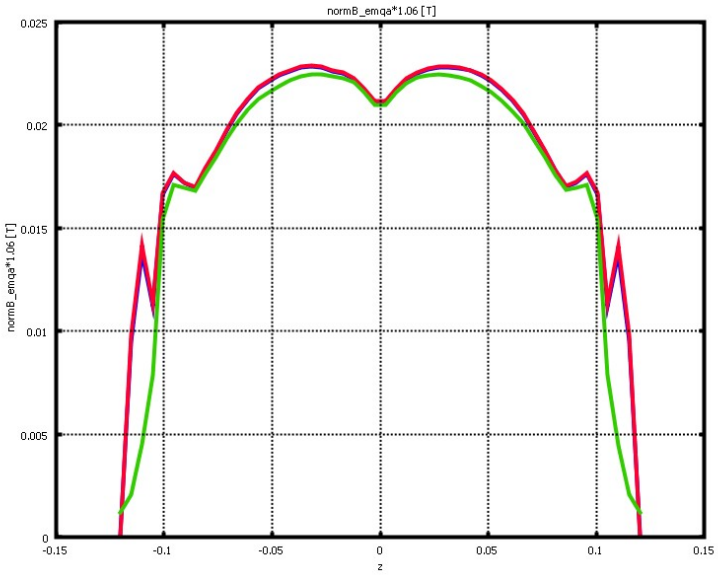
It may be possible to use an analytical approach to calculate the magnetic field in the coils, instead of interpolation of Comsol-generated data which takes into account the iron surrounding the coils. This note investigates the influence of the nonlinearity of the magnetic permeability of the iron and the presence of the iron on the magnetic field inside the coils.

The modeling was done with all current ratios set equal to 1, and with the total current equal to 1A. The permeability of the iron was set to 100, MUR (a nonlinear function defined for soft iron), or 1 (representing no iron). When generating the graph for permeability of 1, the magnetic field strength was multiplied by a constant value of 1.06. This multiplication represents the possible scaling that could be done in an analytical approach for this system of coils.

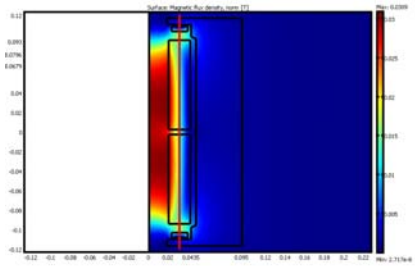
R Position (m)	Relative permeability Blue: μ = 100 Red: μ = MUR(B) (Soft Iron) Green: μ = 1 (multiplied by factor of 1.06)
$20*10^{-3}$ 	

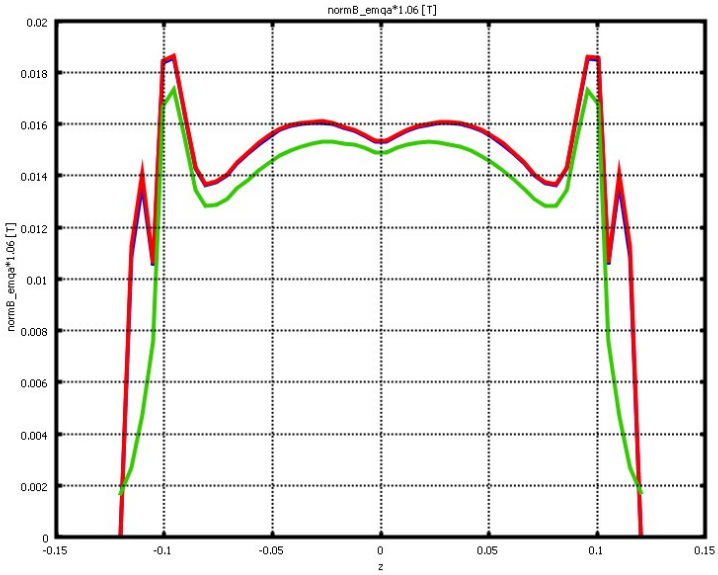
$26.5*10^{-3}$

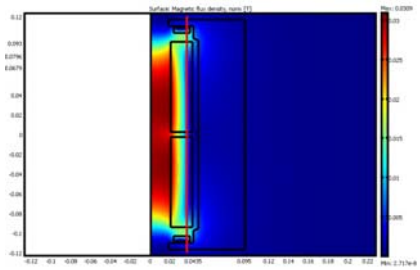
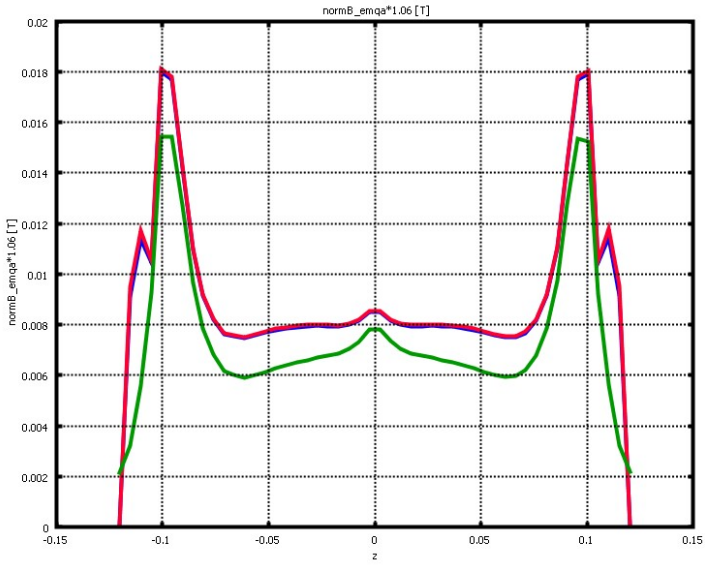
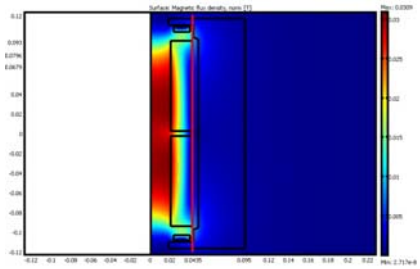
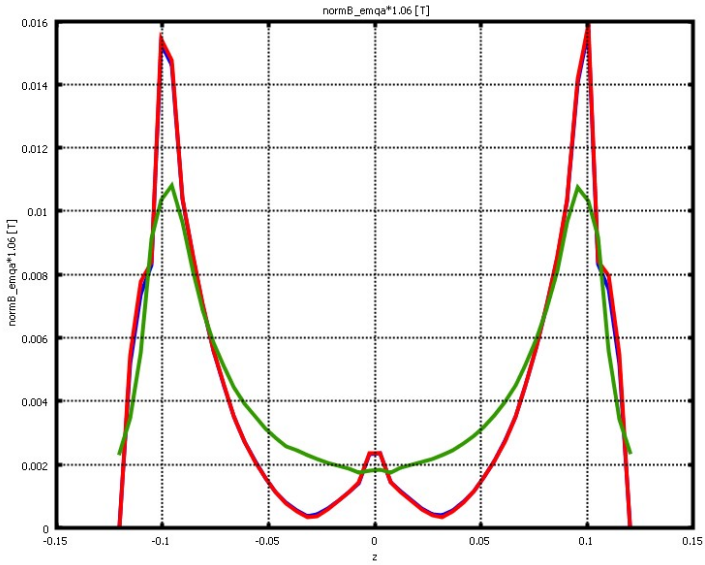




$31.05*10^{-3}$





$36.5*10^{-3}$ 	
$42.1*10^{-3}$ 	

The curves representing magnetic permeability of 100 (blue) and MUR (red) are almost the same, which means that the nonlinearity of the permeability of the iron is not significant.

The graphs with magnetic permeability of MUR (red) and 1 (green) are almost equal closer to the center of the magnet system (low R values). In the graphs, the main coils are positioned along Z at [-0.094, -0.0025] and [0.0025, 0.094]. Observations of these sections indicate that the magnetic fields calculated for the main coils appear to be similar for both methods. The bucking coils are positioned along Z at [-0.1084, -0.103] and [0.103, 0.1084]. The graphs show that there exists a substantial disparity between the magnetic fields calculated using the two methods.

Closer to the outside of the coils (high R values), the graphs show more discrepancy. The graph with R=42.1e-3 is the worst case, and this figure shows a maximum difference of about 50% (for the main coil, since the cross section does not include any part of the bucking coil). The two versions of the magnetic field in the bucking coils are substantially different for high values of R as well as low values.

Since initial quenching is most likely to happen in the areas where the magnetic field is greatest, the inaccuracies caused by using an analytical approach for the main coils are not significant. Most of the subsequent quenching after the initial quench is determined by heat propagation, partly because the quench current is high at low magnetic fields. Therefore, the quench propagation model should not be considerably affected by inaccuracies in the magnetic field. For these reasons, calculation of the field in the main coils can either be completed from interpolation of Comsol-generated results taking into account the surrounding iron or through an analytical approach, which may generate insignificant inaccuracies (Appendix, #2). It may be beneficial to compare the two approaches because of potential simplification of the code.

However, modeling of the magnetic field in the bucking coils cannot be done through an analytical approach, because the surrounding iron exerts a strong influence on the magnetic field in each bucking coil. Instead, interpolation of Comsol-generated data, taking into account the surrounding iron, must be implemented in the program.

Appendix, #2: Analytical Modeling of Magnetic Field in Coils

In modeling quench propagation, it may be necessary to analytically determine the distribution of magnetic field in the coils. This note describes one method which makes use of the Biot-Savart law. Other methods may exist, such as those which utilize vector potentials.

To begin, the magnetic field generated by current in a circular loop in the XY plane is calculated at point P in the YZ plane (see Fig. 1).

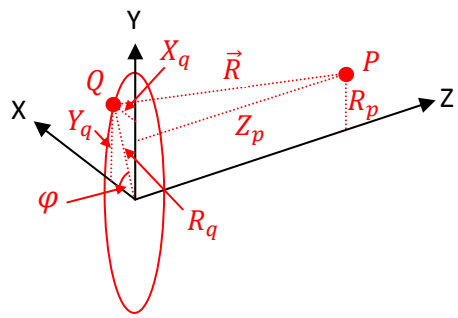


Fig.1 Diagram of circular loop of current and point P in a coordinate plane.

The Biot-Savart law defines $dH = \frac{I}{4\pi} * \frac{\vec{dl} \times \hat{r}}{|r|^2} = \frac{I}{4\pi} * \frac{\vec{dl} \times \vec{R}}{|R|^3}$. In the diagram above, the vector $\vec{R}$ from point Q to point P can be represented as $\vec{R} = \hat{x}(x_p - x_q) + \hat{y}(y_p - y_q) + \hat{z}(z_p - z_q)$. If φ is the angle between the x-axis and the line from the origin to Q, then

$$(x_q, y_q, z_q) = (R_q \cos\varphi, R_q \sin\varphi, z_q).$$

In our case, $(x_p, y_p, z_p) = (0, R_p, z_p)$.

Then, $\vec{R} = \hat{x}(-R_q \cos\varphi) + \hat{y}(R_p - R_q \sin\varphi) + \hat{z}(z_p - z_q)$.

Using trigonometry, φ can be eliminated, so

$$\vec{R} = \hat{x}\left(-R_q \frac{x_q}{\sqrt{{x_q}^2 + {y_q}^2}}\right) + \hat{y}\left(R_p - R_q \frac{y_q}{\sqrt{{x_q}^2 + {y_q}^2}}\right) + \hat{z}(z_p - z_q).$$

On the other hand,

$$\vec{dl} = \hat{x} * dx_q + \hat{y} * dy_q + \hat{z} * dz_q.$$

Because the loop of current is always located at point 0 on the z-axis with no tilt over the z-axis, $dz_q = 0$. Since the loop is a circle, ${x_q}^2 + {y_q}^2 = {R_q}^2$, so $x_q dx_q + y_q dy_q = 0$. Then, $dy_q = -\frac{x_q}{y_q} dx_q$ and substitution gives

$$\vec{dl} = \hat{x} dx_q - \hat{y} \frac{x_q}{y_q} dx_q + \hat{z} * 0.$$

The cross-product is now determined:

$$\vec{dl} \times \vec{R} = dx\left[\left(-\frac{x_q}{y_q}(z_p - z_q)\right) * \hat{x} - (z_p - z_q) * \hat{y} + \left(R_p - \frac{{R_q}^2}{y_q}\right) * \hat{z}\right].$$

Next, the contour integral must be taken. The distance between points Q and P is

$$r = \left[\left(x_p - x_q\right)^2 + \left(y_p - y_q\right)^2 + \left(z_p - z_q\right)^2\right]^{\frac{1}{2}}.$$

The overall integral is $B = \mu_0 * H = \frac{\mu_0 * I}{4\pi} * \int \frac{\vec{dl} \times \vec{R}}{|R|^3} dx_q$. H can be separated into three components: H_x, H_y, H_z:

$$H_x = \frac{I}{4\pi} * \int \frac{-\frac{x_q}{y_q}(z_p - z_q)}{\left[(x_p - x_q)^2 + (y_p - y_q)^2 + (z_p - z_q)^2\right]^{\frac{3}{2}}} dx_q,$$

$$H_y = \frac{I}{4\pi} * \int \frac{-(z_p - z_q)}{\left[(x_p - x_q)^2 + (y_p - y_q)^2 + (z_p - z_q)^2\right]^{\frac{3}{2}}} dx_q,$$

$$H_z = \frac{I}{4\pi} * \int \frac{\left(R_p - \frac{{R_q}^2}{y_q}\right)}{\left[(x_p - x_q)^2 + (y_p - y_q)^2 + (z_p - z_q)^2\right]^{\frac{3}{2}}} dx_q.$$

The integrals for the three components written above can be split into parts that have limits of integration that together comprise integration around the full circle of the source loop: $[R_q, 0]$, $[0, -R_q]$, $[-R_q, 0]$, and $[0, R_q]$.

The magnetic field distribution due to the loop of current is azimuthally symmetrical, so in this case the field will be evaluated at the point where $x_p = 0$ and $y_p = R_p$. Furthermore, ${x_q}^2 + {y_q}^2 = {R_q}^2$, so $y_q = \pm\sqrt{{R_q}^2 - {x_q}^2}$. To eliminate the uncertainty in the sign of y_q, slightly different expressions must be written for the different parts of the contour integral. For example:

$$H_x = \frac{I}{4\pi} * \left(\int_{R_q}^{-R_q} \frac{-\frac{x_q}{\sqrt{{R_q}^2 - {x_q}^2}}(z_p - z_q) dx_q}{\left[{x_q}^2 + \left(R_p - \sqrt{{R_q}^2 - {x_q}^2}\right)^2 + (z_p - z_q)^2\right]^{\frac{3}{2}}} + \int_{-R_q}^{R_q} \frac{\frac{x_q}{\sqrt{{R_q}^2 - {x_q}^2}}(z_p - z_q) dx_q}{\left[{x_q}^2 + \left(R_p - (-\sqrt{{R_q}^2 - {x_q}^2})\right)^2 + (z_p - z_q)^2\right]^{\frac{3}{2}}} \right)$$

$$H_y = \frac{I}{4\pi} * \left(\int_{R_q}^{-R_q} \frac{-(z_p - z_q) dx_q}{\left[{x_q}^2 + \left(R_p - \sqrt{{R_q}^2 - {x_q}^2}\right)^2 + (z_p - z_q)^2\right]^{\frac{3}{2}}} + \int_{-R_q}^{R_q} \frac{-(z_p - z_q) dx_q}{\left[{x_q}^2 + \left(R_p - (-\sqrt{{R_q}^2 - {x_q}^2})\right)^2 + (z_p - z_q)^2\right]^{\frac{3}{2}}} \right)$$

$$H_z = \frac{I}{4\pi} * \left(\int_{R_q}^{-R_q} \frac{\left(R_p - \frac{{R_q}^2}{\sqrt{{R_q}^2 - {x_q}^2}}\right) dx_q}{\left[{x_q}^2 + \left(R_p - \sqrt{{R_q}^2 - {x_q}^2}\right)^2 + (z_p - z_q)^2\right]^{\frac{3}{2}}} + \int_{-R_q}^{R_q} \frac{\left(R_p + \frac{{R_q}^2}{\sqrt{{R_q}^2 - {x_q}^2}}\right) dx_q}{\left[{x_q}^2 + \left(R_p - (-\sqrt{{R_q}^2 - {x_q}^2})\right)^2 + (z_p - z_q)^2\right]^{\frac{3}{2}}} \right)$$

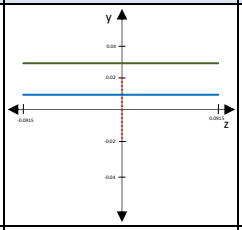
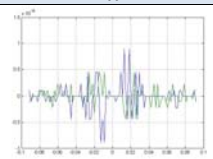
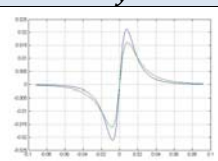
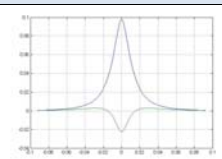
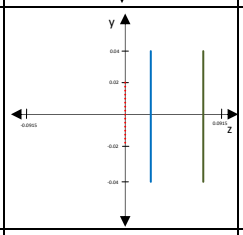
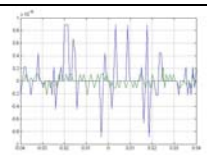
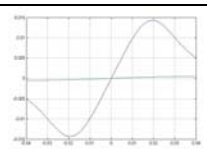
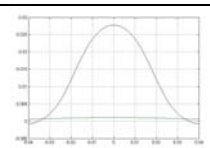
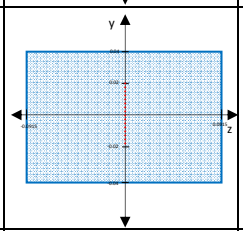
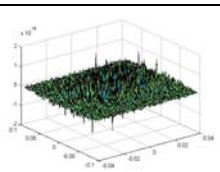
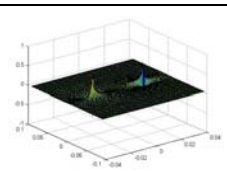
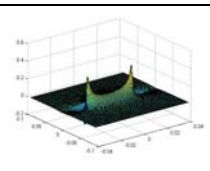
In the above integrals, R_q is the radius of the source loop, x_q represents the position of a source current on the loop, z_p is the distance along the z-axis from the source loop to the point, z_q is the z-component of the coordinates of the source loop. As mentioned earlier, for this case

$z_q = 0$ since the source loop is centered over the x and y axes with no tilt over the z-axis. R_p is the distance from point P to the z-axis.

After integration, the magnetic field at point P can be evaluated as:

$$B = \sqrt{{B_x}^2 + {B_y}^2 + {B_z}^2} = \mu_0 * \sqrt{{H_x}^2 + {H_y}^2 + {H_z}^2}$$

The equations derived above were tested through a program written in the MATLAB language. To analyze the results, R_p and Z_p could be changed while the other parameters of the system remained constant. In generating the graphs below, the following parameters were used: $I = 250\ A$, $R_q = 0.020\ m$, $z_q = 0$. Zp changed in the interval $-0.0915 \leq Z_p \leq 0.0915$, and Rp changed in the interval $-0.040 \leq R_p \leq 0.040$. The figures below compare well with data obtained from an equivalent model generated using Comsol MultiPhysics.

Parameters	Layout	B_x	B_y	B_z
$R_p = 0.010$ $R_p = 0.030$ $-0.0915 \leq Z_p \leq 0.0915$ $I = 250$ $R_q = 0.020$ $z_q = 0$				
$-0.040 \leq R_p \leq 0.040$ $Z_p = 0.020$ $Z_p = 0.080$ $I = 250$ $R_q = 0.020$ $z_q = 0$				
$-0.040 \leq R_p \leq 0.040$ $-0.0915 \leq Z_p \leq 0.0915$ $I = 250$ $R_q = 0.020$ $z_q = 0$				

The next step was to expand the single source loop into a solenoid. This was achieved through integration along R_q to form an annulus, and then integration along Z_q to form the solenoid. The following expressions involving triple integrals were used:

$$H_x = \frac{J}{4\pi} * \left(\int_{-\frac{L}{2}}^{\frac{L}{2}} \int_{R_i}^{R_o} \int_{R_q}^{-R_q} \frac{-\frac{x_q}{\sqrt{{R_q}^2 - {x_q}^2}}(z_p - z_q)dx_q dR_q dz_q}{\left[{x_q}^2 + \left(R_p - \sqrt{{R_q}^2 - {x_q}^2}\right)^2 + (z_p - z_q)^2\right]^{\frac{3}{2}}} + \int_{-\frac{L}{2}}^{\frac{L}{2}} \int_{R_i}^{R_o} \int_{-R_q}^{R_q} \frac{\frac{x_q}{\sqrt{{R_q}^2 - {x_q}^2}}(z_p - z_q)dx_q dR_q dz_q}{\left[{x_q}^2 + \left(R_p - (-\sqrt{{R_q}^2 - {x_q}^2})\right)^2 + (z_p - z_q)^2\right]^{\frac{3}{2}}} \right)$$

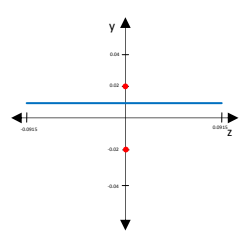
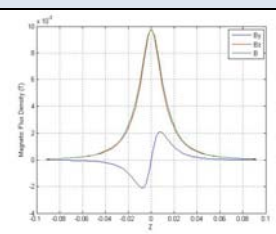
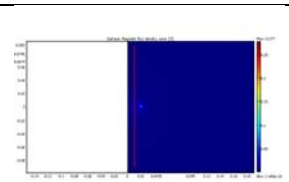
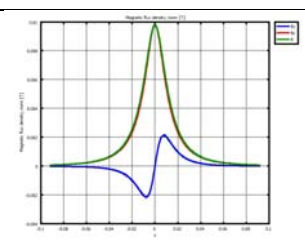
$$H_y = \frac{J}{4\pi} * \left(\int_{-\frac{L}{2}}^{\frac{L}{2}} \int_{R_i}^{R_o} \int_{R_q}^{-R_q} \frac{-(z_p - z_q)dx_q dR_q dz_q}{\left[{x_q}^2 + \left(R_p - \sqrt{{R_q}^2 - {x_q}^2}\right)^2 + (z_p - z_q)^2\right]^{\frac{3}{2}}} + \int_{-\frac{L}{2}}^{\frac{L}{2}} \int_{R_i}^{R_o} \int_{-R_q}^{R_q} \frac{-(z_p - z_q)dx_q dR_q dz_q}{\left[{x_q}^2 + \left(R_p - (-\sqrt{{R_q}^2 - {x_q}^2})\right)^2 + (z_p - z_q)^2\right]^{\frac{3}{2}}} \right)$$

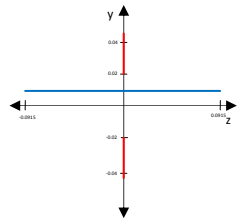
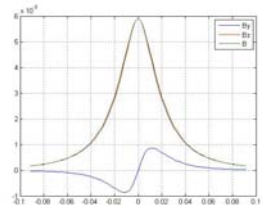
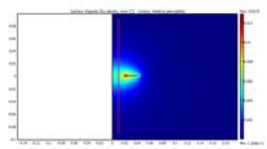
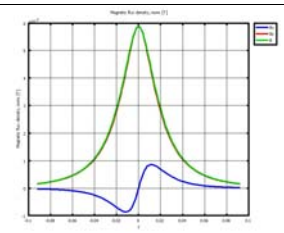
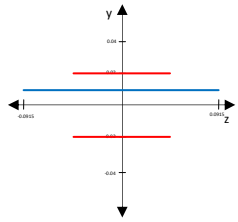
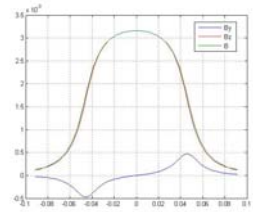
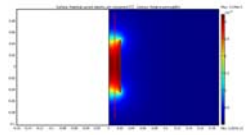
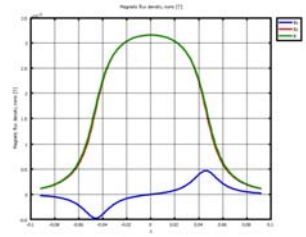
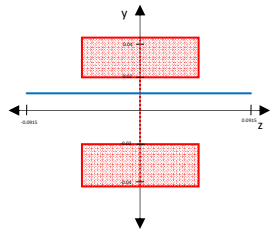
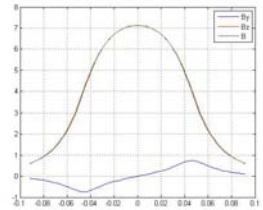
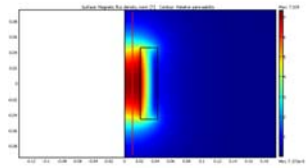
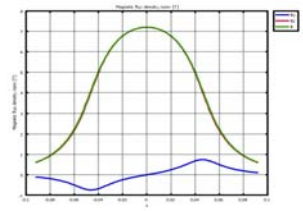
$$H_z = \frac{J}{4\pi} * \left(\int_{-\frac{L}{2}}^{\frac{L}{2}} \int_{R_i}^{R_o} \int_{R_q}^{-R_q} \frac{\left(R_p - \frac{{R_q}^2}{\sqrt{{R_q}^2 - {x_q}^2}} \right) dx_q dR_q dz_q}{\left[{x_q}^2 + \left(R_p - \sqrt{{R_q}^2 - {x_q}^2} \right)^2 + (z_p - z_q)^2 \right]^{\frac{3}{2}}} + \int_{-\frac{L}{2}}^{\frac{L}{2}} \int_{R_i}^{R_o} \int_{-R_q}^{R_q} \frac{\left(R_p + \frac{{R_q}^2}{\sqrt{{R_q}^2 - {x_q}^2}} \right) dx_q dR_q dz_q}{\left[{x_q}^2 + \left(R_p - (-\sqrt{{R_q}^2 - {x_q}^2}) \right)^2 + (z_p - z_q)^2 \right]^{\frac{3}{2}}} \right)$$

In the integrals, R_i is the inner radius of the solenoid while R_o is the outer radius. L is the total length of the solenoid. Since the expressions represent a solenoid instead of a loop of current, I (current) is replaced with J (current density), which equals $\frac{(N_t * N_l) * I}{L * (R_o - R_i)}$, where N_t is the number of turns in the solenoid and N_l is the number of layers.

The limits of integration of the innermost integral depend on the value of the variable integrated in the second integral. MATLAB's standard integration feature does not ordinarily allow this situation. One possible workaround is to set the innermost limits of integration equal to the extreme values specified by the second integral (in this case, R_q in the innermost limits of integration would be replaced with R_o). Then, a MATLAB function could use a logic operation to define the integrand equal to the above expressions when $x_q \leq R_q$. When $x_q > R_q$, the integrand is set to 0. This way, changing the innermost limits of integration to R_o provides the proper results, since adding 0's does not change the calculated answer and values of x_q greater than R_q are effectively ignored. Essentially, the integrand equals zero outside of the desired region of integration and therefore does not contribute to the integral. Although this method provides a suitable workaround, it lengthens computational time. MATLAB's Symbolic Toolbox has functions that could achieve the desired result without a workaround.

In order to further verify the equations and the accompanying MATLAB program, calculations were done to find the magnetic field produced by several different geometrical structures: loop, annulus, thin solenoid, and full solenoid. The results were then compared to data collected from equivalent models created in Comsol MultiPhysics. The figures below represent the magnetic field found at points throughout a line inside the solenoid, at $R_p = 0.010$ and Z_p ranging from -0.0915 to 0.0915.

Program	Structure	Parameters	Layout	B_y, B_z, B
MATLAB	Loop	$R_p = 0.010$ $-0.0915 \leq Z_p \leq 0.0915$ $I = 250$ $R_q = 0.020$ $z_q = 0$		
Comsol	Loop	$R_p = 0.010$ $-0.0915 \leq Z_p \leq 0.0915$ $I = 250$ $R_q = 0.020$ $z_q = 0$		

MATLAB	Annulus	$R_p = 0.010$ $-0.0915 \le Z_p \le 0.0915$ $I = 250$ $R_i = 0.02$ $R_o = 0.0421$ $z_q = 0$		
Comsol	Annulus	$R_p = 0.010$ $-0.0915 \le Z_p \le 0.0915$ $I = 250$ $R_i = 0.02$ $R_o = 0.0421$ $z_q = 0$		
MATLAB	Thin Solenoid	$R_p = 0.010$ $-0.0915 \le Z_p \le 0.0915$ $I = 250$ $R_q = 0.020$ $L = 0.0915$		
Comsol	Thin Solenoid	$R_p = 0.010$ $-0.0915 \le Z_p \le 0.0915$ $I = 250$ $R_q = 0.020$ $L = 0.0915$		
MATLAB	Full Solenoid	$R_p = 0.01$ $-0.0915 \le Z_p \le 0.0915$ $I = 250$ $R_i = 0.02$ $R_o = 0.0421$ $L = 0.0915$		
Comsol	Full Solenoid	$R_p = 0.01$ $-0.0915 \le Z_p \le 0.0915$ $I = 250$ $R_i = 0.02$ $R_o = 0.0421$ $L = 0.0915$		

Besides the data samples shown in the above figures, the magnetic field data was verified for the entire region that experienced significant magnetic field produced by the sample structures. Since the results of the MATLAB and Comsol programs compare well, the MATLAB program and the accompanying equations seem to be correct. Furthermore, accuracy in the MATLAB

program could be improved by lowering the tolerance for error of integration, although computational time would then increase.

An analytical approach based on these equations could be implemented in the quench propagation program as long as there are no significant external influences (e.g. surrounding iron) affecting the magnetic fields produced by the coils.

Appendix, #3: Calculating Currents for General Quench Protection Scheme

One way to analyze the magnet system (with dump resistors connected in parallel to every coil) is to come out with an system of circuit equations and solve them having in mind certain initial conditions. As before [A1] we will assume that the induction matrix of the circuit is known and can be presented as a square matrix:

$$\{M\} = \begin{Bmatrix} M11 & M12 & M13 & M14 \\ M21 & M22 & M23 & M24 \\ M31 & M32 & M33 & M34 \\ M41 & M42 & M43 & M44 \end{Bmatrix}$$

We will also use vector of current derivatives:

$$\{dIdt\} = \begin{Bmatrix} dI1/dt \\ dI2/dt \\ dI3/dt \\ dI4/dt \end{Bmatrix},$$

vector of coil currents:

$$\{CCur\} = \begin{Bmatrix} I1 \\ I2 \\ I3 \\ I4 \end{Bmatrix},$$

vector of dump resistor currents:

$$\{DCur\} = \begin{Bmatrix} Id1 \\ Id2 \\ Id3 \\ Id4 \end{Bmatrix},$$

vector of dump resistor resistances:

$$\{Rd\} = \begin{Bmatrix} Rd1 \\ Rd2 \\ Rd3 \\ Rd4 \end{Bmatrix},$$

and vector of coil resistances:

$$\{Rc\} = \begin{Bmatrix} Rc1 \\ Rc2 \\ Rc3 \\ Rc4 \end{Bmatrix}.$$

Before the initiation of the quench, we know the initial conditions in the system, so we can complete {CCur} and {DCur}. Furthermore, the resistances developed in the quenching coils can be calculated from the initial conditions, so we know {Rc}. {Rd} is a vector of constant values.

To find the four coil current derivatives in {dIdt},we can use the four equations which relate the voltage across dump resistors to the equivalent voltage across the coils:

$$Vd1 = Rc(1) * CCur(1) + M(1,1) * dIdt(1) + M(1,2) * dIdt(2) + M(1,3) * dIdt(3) + M(1,4) * dIdt(4)$$
$$Vd2 = Rc(2) * CCur(2) + M(2,1) * dIdt(1) + M(2,2) * dIdt(2) + M(2,3) * dIdt(3) + M(2,4) * dIdt(4)$$
$$Vd3 = Rc(3) * CCur(3) + M(3,1) * dIdt(1) + M(3,2) * dIdt(2) + M(3,3) * dIdt(3) + M(3,4) * dIdt(4)$$
$$Vd4 = Rc(4) * CCur(4) + M(4,1) * dIdt(1) + M(4,2) * dIdt(2) + M(4,3) * dIdt(3) + M(4,4) * dIdt(4)$$

The above statements can be represented in one equation: *Vd = Rc .* CCur + M * dIdt*. In this statement, "**.*** "symbolizes the multiplication of corresponding matrix elements, while "*****" means matrix multiplication.

This single equation can be rewritten as *Vd - Rc .* CCur = M * dIdt*, which is represented in four equations as:

$$DCur(1) * Rd(1) - Rc(1) * CCur(1) = M(1,1) * dIdt(1) + M(1,2) * dIdt(2) + M(1,3) * dIdt(3) + M(1,4) * dIdt(4)$$
$$DCur(2) * Rd(2) - Rc(2) * CCur(2) = M(2,1) * dIdt(1) + M(2,2) * dIdt(2) + M(2,3) * dIdt(3) + M(2,4) * dIdt(4)$$
$$DCur(3) * Rd(3) - Rc(3) * CCur(3) = M(3,1) * dIdt(1) + M(3,2) * dIdt(2) + M(3,3) * dIdt(3) + M(3,4) * dIdt(4)$$
$$DCur(4) * Rd(4) - Rc(4) * CCur(4) = M(4,1) * dIdt(1) + M(4,2) * dIdt(2) + M(4,3) * dIdt(3) + M(4,4) * dIdt(4)$$

We can solve for the current derivatives {dIdt} by conducting matrix operations with the matrix of mutual inductances {M} and a new matrix {U}, which is created based on the equations above. Essentially, matrices are used to solve a system of linear equations.

$$\{U\} = \begin{Bmatrix} DCur(1) * Rd(1) - Rc(1) * CCur(1) \\ DCur(2) * Rd(2) - Rc(2) * CCur(2) \\ DCur(3) * Rd(3) - Rc(3) * CCur(3) \\ DCur(4) * Rd(4) - Rc(4) * CCur(4) \end{Bmatrix}$$

$$\{dIdt\} = inv(M) * U = M \backslash U$$

By using relevant equations, we can find a number of other values (such as voltages and temperatures) describing the system during the present time step.

Before the beginning of the next time step, the new values for the currents, {CCur} and {DCur}, have to be calculated. To calculate {CCur} in the next time step, we can use an equation relating the current to its derivative:

$$\{CCur\}_{n+1} = \{CCur\}_n + \{dIdt\} * stepT$$

In this equation, $\{CCur\}_{n+1}$ describes the currents in the coils in the next time step, $\{CCur\}_n$ describes the currents in the coils in the present time step, {dIdt} is the current derivatives in the coils in the present time step, and stepT is the length of the program's time step.

In the initial time step, {DCur} = 0, but we need to find {DCur} for the next step. To begin, we set up four equations that contain the unknown values of dump resistor currents in the next time step. The first three equations state that the sum of the currents in each partial circuit (consisting of dump resistor connected in parallel to coil) must be the same. The fourth equation states that the sum of voltage drops across all elements in the total discharge circuit (Fig. 1) is equal to zero.

$$I1 + Id1 = I2 + Id2$$
$$I1 + Id1 = I3 + Id3$$
$$I1 + Id1 = I4 + Id4$$
$$0 = Id1 * Rd(1) + Id2 * Rd(2) + Id3 * Rd(3) + Id4 * Rd(4)$$

These equations can be rewritten as:

$$I1 - I2 = -Id1 + Id2 + 0 * Id3 + 0 * Id4$$
$$I1 - I3 = -Id1 + 0 * Id2 + Id3 + 0 * Id4$$
$$I1 - I4 = -Id1 + 0 * Id2 + 0 * Id3 + Id4$$
$$0 = Id1 * Rd(1) + Id2 * Rd(2) + Id3 * Rd(3) + Id4 * Rd(4)$$

In the above equations, I1, I2, etc. describe the coil currents in the next time step. Similarly, Id1, Id2, etc. describe the dump resistor currents in the next time step. We can set up matrices to solve for these dump resistor currents in the system of linear equations. This process is similar to the above procedure used to calculate the current derivatives.

$$\{D\} = \begin{Bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ Rd(1) & Rd(2) & Rd(3) & Rd(4) \end{Bmatrix}$$

$$\{F\} = \begin{Bmatrix} I1 - I2 \\ I1 - I3 \\ I1 - I4 \\ 0 \end{Bmatrix}$$

$$\{E\} = inv\{D\} * \{F\} = D \backslash F = \begin{Bmatrix} Id1 \\ Id2 \\ Id3 \\ Id4 \end{Bmatrix}$$

Knowing both {CCur} and {DCur} in the next time step, we can calculate {Rc} in the next iteration of the program loop. Since we now know all the necessary conditions for the next time step, we can calculate any other values needed to describe the system. The program can proceed to model quench propagation through a series of time steps.

It should be indicated that there exists an uncertainty in the initial steps of the program modeling when {Rd} is too high or the size of the time step is too large. The result is that the program terminates because of an error. To ensure that the program functions properly, the following condition should roughly be followed:

$$\{Rd\} * stepT \leq 10^{-2}$$

In any case, decreasing the step size should result in more accurate modeling.

References:

[A1] E. Khabiboulline, I Terechkine, “SSR2 Focusing Lens Quench Protection Study”, FNAL TD note TD-11-006, April 2011.