Estimation of the emittance dilution caused by the couplers in the ILC main linac.

TD-09-014

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1. The transverse force caused by RF kick and wake field in the linear accelerator has one component that is not zero on the accelerating beam axis. However, this transverse force depends on the longitudinal coordinate of the particle inside the bunch, that causes an emitance dilution.

Let's consider the simplest model of the emittance dilution produced by the RF kick and wake in the ILC main linac. The linac with "homogeneous" focusing and acceleration will be considered. The particle transverse dynamics is described by the following equation of motion:

$$\frac{dP_x}{dt} = F_{focus} + F_{kick},$$

where F_{focus} is focusing force, F_{kick} is a kick per unit length, s is the particle distance from the bunch center. The kick F_{kick} is equal to

$$F_{kick} = G\left(\mathrm{Re}\left(\frac{V_y}{V_z}e^{iks+i\varphi}\right)\right) + QW_\perp(0,0,s) \approx G\left(\mathrm{Re}\left(\frac{V_y}{V_z}e^{i\varphi}\right)\cos(ks) - \mathrm{Im}\left(\frac{V_y}{V_z}e^{i\varphi}\right)ks\right)$$

$$+QW_\perp(0,0,0) + QW_\perp^{'}s = \left[G\,\mathrm{Re}\left(\frac{V_y}{V_z}e^{i\varphi}\right)\cos(ks) + QW_\perp(0,0,0)\right] + \left[QW_\perp^{'} - Gk\,\mathrm{Im}\left(\frac{V_y}{V_z}e^{i\varphi}\right)\right]s \equiv$$

$$\equiv F_0 + sF',$$

where G is acceleration gradient, $\frac{V_y}{V_z}$ is RF kick, $k=2\pi/\lambda_{RF}$, λ_{RF} is RF wavelength, φ is RF phase, Q is the drive bunch charge, $W_\perp(0,0,s)$ is the transverse wake potential per unit length, s is the distance from the bunch center, $s<<\lambda_{RF}$. The first term is responsible for force that acts on the bunch particles the same way, and, thus, may be compensated using the beam alignment technique. The second term is responsible for the kick different for the different parts of the bunch and, thus, cannot be compensated. Thus, we will consider the second term only.

For ILC collider the average beta-function β is constant along the linac. In this case, the equation of the transverse motion is the following ($\gamma(z)$ is relativistic factor, z is longitudinal coordinate, U_0 is the initial energy in eV):

$$\frac{d}{dz}\left(\gamma(z)\frac{dx}{dz}\right)+\gamma(z)x/\beta^2=\frac{sF'\gamma(0)}{U_0},$$

or

$$\frac{\partial^2 x}{\partial z^2}+\frac{1}{\gamma(z)}\cdot\frac{\partial\gamma(z)}{\partial z}\cdot\frac{\partial x}{\partial z}+\frac{x}{\beta^2}=\frac{F's\gamma(0)}{U_0\gamma(z)}, \qquad (1)$$

In our case,

$$\frac{\partial\gamma(z)}{\partial z}=\gamma^{'}=const.$$

and, thus,

$$\gamma(z)=\gamma_0+\gamma^{'}z.$$

General solution of (1) may be expressed the following way:

$$x(z)=C_1x_1(z)+C_2x_2(z)+\int_0^z G(z,z^{'})f(z^{'})dz^{'}, \qquad (2)$$

where C_1 and C_2 are the constants determined by initial conditions, $x_1(z)$ and $x_2(z)$ are basic solutions of homogeneous equation,

$$\frac{\partial^2 x}{\partial z^2}+\frac{\gamma^{'}}{\gamma_0+\gamma^{'}z}\cdot\frac{\partial x}{\partial z}+\frac{x}{\beta^2}=0, \qquad (3)$$

$f(z)$ is a right-hand,

$$f(z)=\frac{sF'\gamma_0}{U_0\gamma(z)}\equiv\frac{F}{\gamma(z)},$$

and $G(z,z')$ is the Green function:

$$G(z,z')=-\frac{x_1(z)x_2(z^{'})-x_2(z)x_1(z^{'})}{W(z^{'})},$$

and $W(z')$ is Wronskian:

$$W(z^{'})=x_1(z^{'})x_2^{'}(z^{'})-x_2(z^{'})x_1^{'}(z^{'})$$

For $x(0)=0$ and $x'(0)=0$ $C_1=C_2=0$.

Note that (3) is Bessel equation and

$$x_1(z)=J_0\left(\frac{\gamma_0/\gamma' + z}{\beta}\right)=J_0\left(\gamma(z)/\beta\gamma'\right),$$

$$x_2(z)=Y_0\left(\frac{\gamma_0/\gamma' + z}{\beta}\right)=Y_0\left(\gamma(z)/\beta\gamma'\right),$$

$$W(z')=J_0(\gamma(z')/\beta\gamma')Y_0'(\gamma(z')/\beta\gamma')-Y_0(\gamma(z')/\beta\gamma')J_0'(\gamma(z')/\beta\gamma')=\frac{2\gamma'}{\pi\gamma(z)},$$

$$G(z,z')=-\frac{\pi\gamma(z)}{2\gamma'}\left(J_0(\gamma(z)/\beta\gamma')Y_0(\gamma(z')/\beta\gamma')-Y_0(\gamma(z)/\beta\gamma')J_0(\gamma(z')/\beta\gamma')\right).$$

and

$$\int_0^z G(z,z')f(z')dz'=-\frac{\pi F}{2\gamma'}\int_0^z\left[J_0(\gamma(z)/\beta\gamma')Y_0(\gamma(z')/\beta\gamma')-Y_0(\gamma(z)/\beta\gamma')J_0(\gamma(z')/\beta\gamma')\right]dz'.$$

In our case, when

$$\frac{\gamma(z)}{\gamma'\beta}>>1,$$

it is possible to use the Bessel functions approximation for large argument:

$$J_0\left(\gamma(z)/\beta\gamma'\right)=\sqrt{\frac{2\gamma'\beta}{\pi\gamma(z)}}\cos\left(\gamma(z)/\gamma'\beta-\pi/4\right),$$
$$Y_0\left(\gamma(z)/\beta\gamma'\right)=\sqrt{\frac{2\gamma'\beta}{\pi\gamma(z)}}\sin\left(\gamma(z)/\gamma'\beta-\pi/4\right),$$

and

$$\int_0^z G(z,z')f(z')dz'=-\frac{F\beta}{\sqrt{\gamma(z)}}\int_0^z\frac{\sin\left[(z'-z)/\beta\right]}{\sqrt{\gamma(z')}}dz'.$$

Introducing a new variable

$$\chi = \frac{\gamma(z)}{\gamma' \beta} = \frac{\gamma_0 + \gamma' z}{\gamma' \beta},$$

one has

$$\int_0^z G(z,z') f(z') dz' = -\frac{F\beta}{\gamma' \sqrt{\chi}} \left[\cos(\chi) \int_{\chi_0}^{\chi} \frac{\sin(\chi')}{\sqrt{\chi'}} d\chi' - \sin(\chi) \int_{\chi_0}^{\chi} \frac{\cos(\chi')}{\sqrt{\chi'}} d\chi' \right], \qquad (4)$$

where

$$\chi_0 = \frac{\gamma(0)}{\gamma' \beta} = \frac{\gamma_0}{\gamma' \beta}.$$

The integrals in (4) are Fresnel integrals:

$$\int_{\chi_0}^{\chi} \frac{\sin(\chi')}{\sqrt{\chi'}} d\chi' = \sqrt{2\pi} \left[S(\sqrt{2\chi/\pi}) - S(\sqrt{2\chi_0/\pi}) \right],$$

$$\int_{\chi_0}^{\chi} \frac{\cos(\chi')}{\sqrt{\chi'}} d\chi' = \sqrt{2\pi} \left[C(\sqrt{2\chi/\pi}) - C(\sqrt{2\chi_0/\pi}) \right]$$

For $2\chi/\pi >> 1$ one has

$$S(\sqrt{2\chi/\pi}) \approx \frac{1}{2} - \frac{\cos(\chi)}{\sqrt{2\pi\chi}},$$

$$C(\sqrt{2\chi/\pi}) \approx \frac{1}{2} + \frac{\sin(\chi)}{\sqrt{2\pi\chi}},$$

and

$$\int_0^z G(z,z') f(z') dz' \approx -\frac{F\beta}{\gamma'} \left[\frac{\cos(\chi - \chi_0)}{\sqrt{\chi\chi_0}} - \frac{1}{\chi} \right] = -F\beta^2 \left[\frac{\cos(z/\beta)}{\sqrt{\gamma(z)\gamma_0}} - \frac{1}{\gamma(z)} \right]. \qquad (5)$$

Thus, from (2) and (5) one can find

$$x(z) \approx x_0 \sqrt{\frac{\gamma_0}{\gamma(z)}} \cos(z/\beta) + x_0' \beta \sqrt{\frac{\gamma_0}{\gamma(z)}} \sin(z/\beta) - \frac{sF'\beta^2}{U_0} \sqrt{\frac{\gamma_0}{\gamma(z)}} \times \left[\cos(z/\beta) - \sqrt{\frac{\gamma_0}{\gamma(z)}} \right],$$

$$x'(z) \approx -\frac{x_0}{\beta} \sqrt{\frac{\gamma_0}{\gamma(z)}} \sin(z/\beta) + x_0' \sqrt{\frac{\gamma_0}{\gamma(z)}} \cos(z/\beta) + \frac{sF'\beta}{U_0} \sqrt{\frac{\gamma_0}{\gamma(z)}} \sin(z/\beta), \qquad (6)$$

or

$$x(z) \approx \left(x_0 - \frac{sF'\beta^2}{U_0} \right) \sqrt{\frac{\gamma_0}{\gamma(z)}} \cos(z/\beta) + x_0^{'}\beta \sqrt{\frac{\gamma_0}{\gamma(z)}} \sin(z/\beta) + \frac{sF'\beta^2}{U_0} \left(\frac{\gamma_0}{\gamma(z)} \right),$$

$$x'(z) \approx -\frac{1}{\beta} \left(x_0 - \frac{sF'\beta^2}{U_0} \right) \sqrt{\frac{\gamma_0}{\gamma(z)}} \sin(z/\beta) + x_0^{'} \sqrt{\frac{\gamma_0}{\gamma(z)}} \cos(z/\beta).$$

One can see that the rf kick simply shifts the center of betatron oscillations off the axis. Because this shift depends on the longitudinal particle coordinate inside the bunch, the r.m.s. emittance averaged over the bunch increases.

2. For ILC Main Linac one has

$\mathrm{Im}\left(\frac{V_y}{V_z} \right)$=1.17e-5 (HFSS calculations, RF phase φ is -5.3°, [1])

G=23.93 MeV/m (31.5 MeV/m in the structures) ;

σ=0.3 mm;

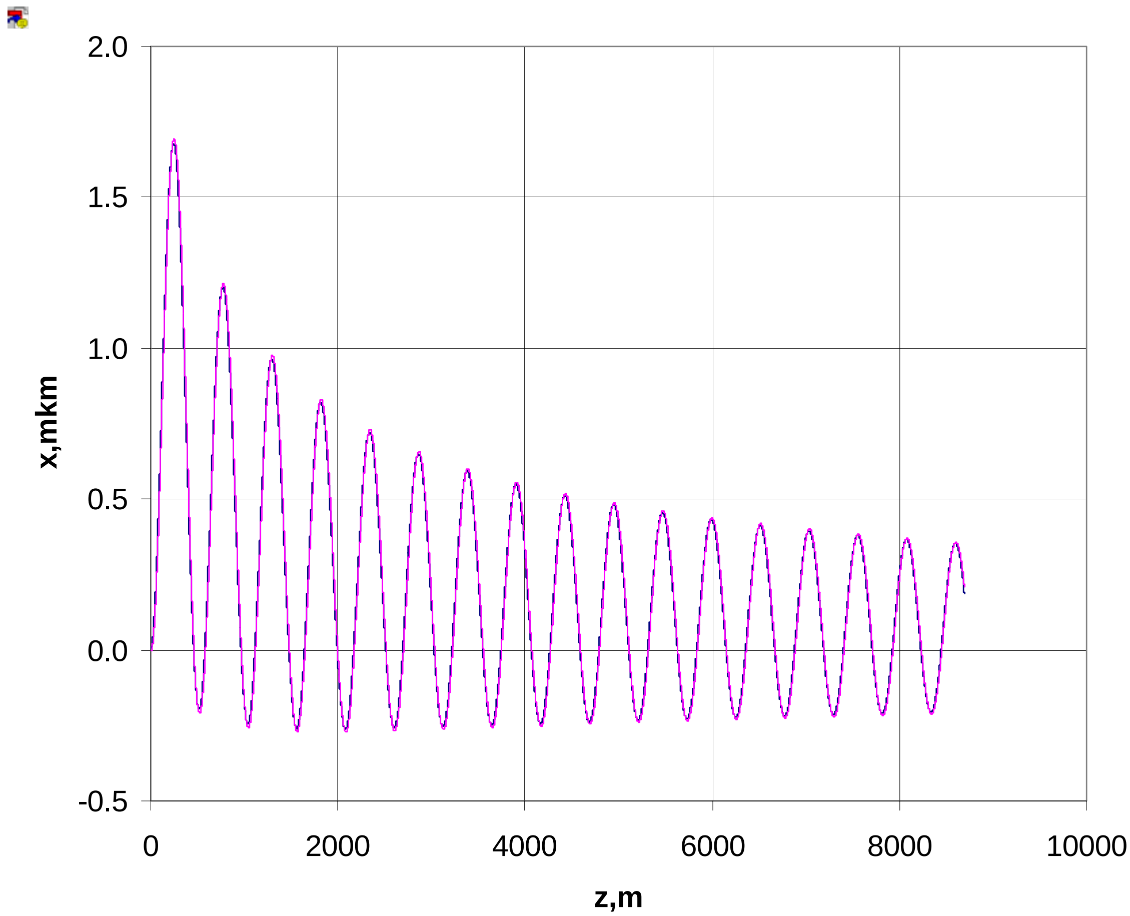
U_0=15 GeV;

λ_{RF}=0.23 m;

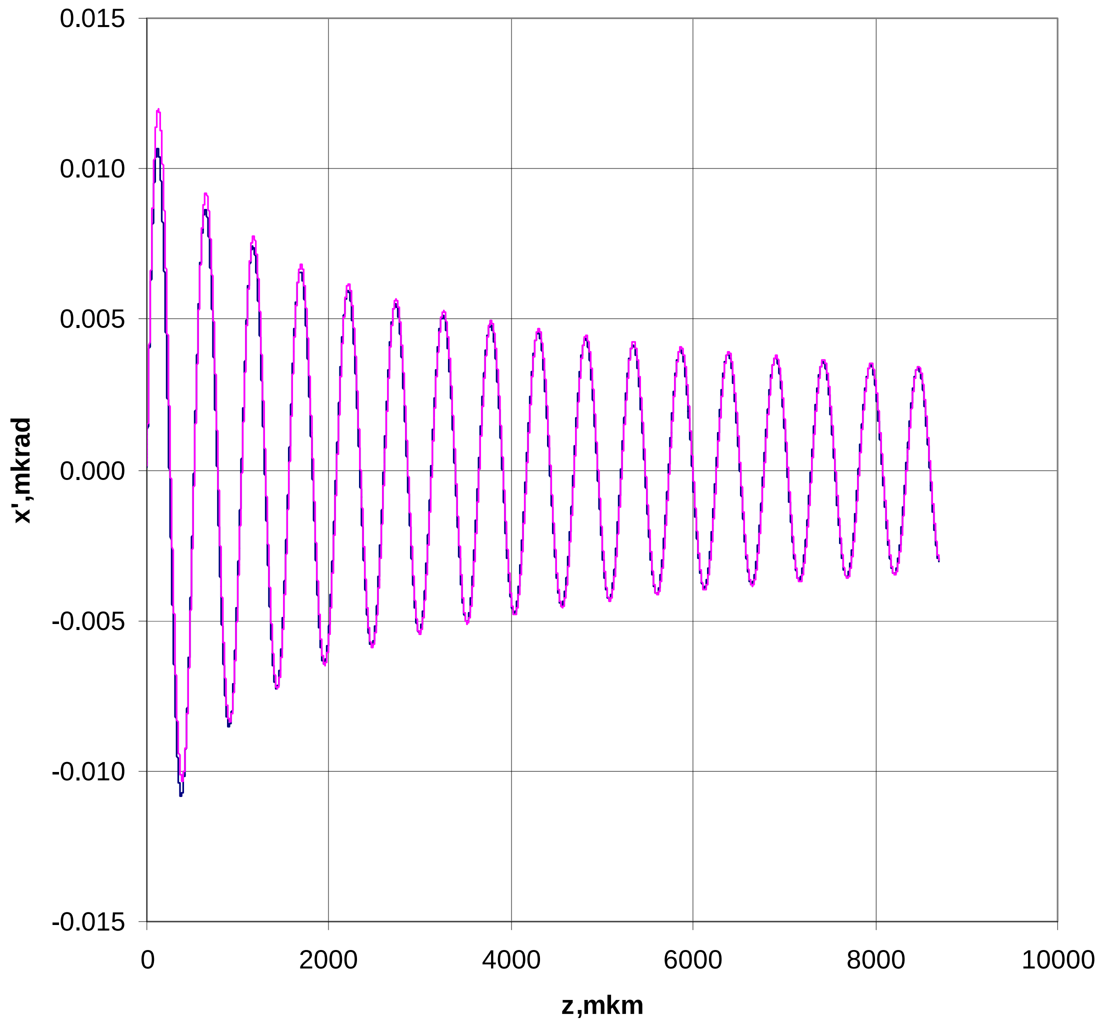
γ_0=3e4 (15 GeV);

$\gamma(z_{max})$=4.36e5 (223 GeV)

the $x(z)$ and $x'(z)$ for $s = -\sigma$ are shown calculated by direct integration of the Eq. (1) and according to the formula (6) for zero initial condition for both x and x'. Calculations were made for β= 83 m. One can see that the analytical solution works pretty well.



a)



b)

Figure 1. (a) *x(z)* and (b) *x'(z)* calculated by direct integration (purple) and analytically (blue).

In Figure 2 vertical coordinate *y(z)* dependence is shown calculated numerically by PLACET [2] (blue) and LUCRETIA [3] (black) for realistic TESLA lattice structure (but no misalignments and no wakes) and analytically (red) according to formula (6) for zero initial conditions and for s = 0.3 mm. In order to fit the betatron phase the average β-function in (6) was taken 83 m. Results of PLACET and LUCRETIA coincide with the precision of about ±2 nm.

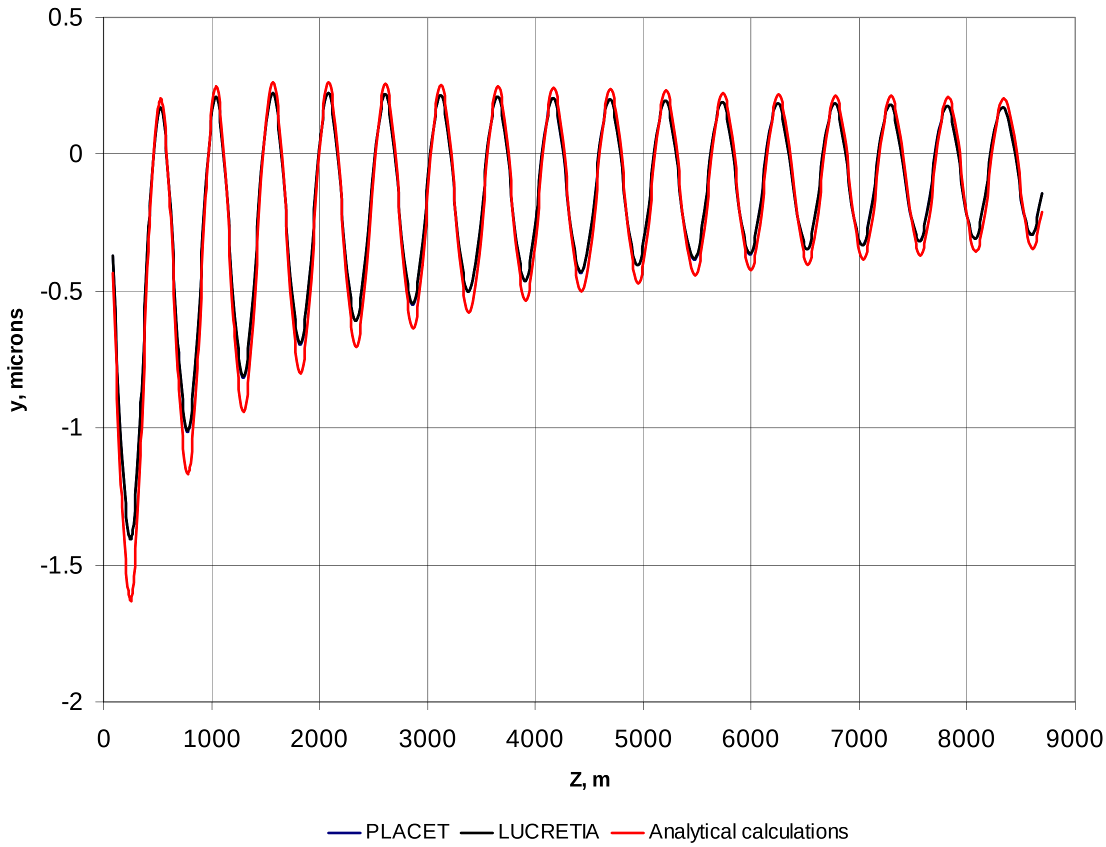


Figure 2. The electron trajectory calculated by PLACET (blue), LUCRETIA (black) and analytically, based on the smooth approximation for the focusing system (red). Results of PLACET and LUCRETIA coincide.

3. Note that for the bunch matched to the focusing system having β=*const*

$$x_0 = x_{00}\cos\psi,$$
$$x_0^{'} = -\frac{x_{00}}{\beta}\sin\psi, \tag{7}$$

where ψ is the particle betatron phase.
The r.m.s. emittance of the bunch is equal to

$$\varepsilon = \gamma(z)\left[<(x)^2><(x^{'})^2> - <xx'>^2\right]^{1/2}, \tag{8}$$

Using (6) and (7) and averaging over transverse and longitudinal coordinates one has

$$\gamma(z)<(x)^2> = \varepsilon_0\beta + \frac{<s^2>F'^2\beta^4\gamma_0}{U_0^2}\left(\cos(z/\beta) - \sqrt{\frac{\gamma_0}{\gamma(z)}}\right)^2,$$
$$\gamma(z)<(x')^2> = \frac{\varepsilon_0}{\beta} + \frac{<s^2>F'^2\beta^2\gamma_0}{U_0^2}\sin^2(z/\beta),$$

$$\gamma(z) < xx' >= -\frac{< s^2 > F'^2 \beta^3 \gamma_0}{U_0^2}\left(\cos(z/\beta) - \sqrt{\frac{\gamma_0}{\gamma(z)}}\right)\sin(z/\beta),$$

and, thus,

$$\varepsilon = \left[\varepsilon_0^2 + \varepsilon_0 \frac{< s^2 > F'^2 \beta^3 \gamma_0}{U_0^2}\left(1 - 2\sqrt{\frac{\gamma_0}{\gamma(z)}}\cos(z/\beta) + \frac{\gamma_0}{\gamma(z)}\right)\right]^{1/2}$$

where ε_0 is initial r.m.s. emittance and $<s^2>$ is r.m.s. bunch length. For Gaussian longitudinal bunch profile one has

$$\varepsilon = \left[\varepsilon_0^2 + \varepsilon_0 \frac{\sigma^2 F'^2 \beta^3 \gamma_0}{U_0^2}\left(1 - 2\sqrt{\frac{\gamma_0}{\gamma(z)}}\cos(z/\beta) + \frac{\gamma_0}{\gamma(z)}\right)\right]^{1/2}$$

If the r.m.s. emittance dilution is small compared to the initial emittance, i.e., if $\varepsilon_0 << \frac{\sigma^2 F'^2 \beta^3 \gamma_0}{2U_0^2}$,

$$\varepsilon \approx \varepsilon_0 + \frac{(F')^2 \sigma^2 \beta^3 \gamma_0}{2U_0^2}\left(1 - 2\sqrt{\frac{\gamma_0}{\gamma(z)}}\cos(z/\beta) + \frac{\gamma_0}{\gamma(z)}\right). \qquad (9)$$

If the focusing system is design such a way, that $z_{max}/\beta = 2\pi n$, then

$$\varepsilon \approx \varepsilon_0 + \frac{(F')^2 \sigma^2 \beta^3 \gamma_0}{2U_0^2}\left(1 - \sqrt{\frac{\gamma_0}{\gamma(z)}}\right)^2, \qquad (10)$$

and r.m.s. emittance dilution is

$$\Delta\varepsilon \approx \frac{(F')^2 \sigma^2 \beta^3 \gamma_0}{2U_0^2}\left(1 - \sqrt{\frac{\gamma_0}{\gamma(z)}}\right)^2.$$

If there is no acceleration, the minimal r.m.s. emittance dilution is zero.

4. One can see from the Formula (9), that the r.m.s. emittance oscillates with the period equal to the betatron wavelength $2\pi\beta$. It may be illustrated in the simplest case, when there is no acceleration. In Figure 3 the phase diagram is shown for three transverse bunch cross sections that correspond to head, center and tail of the bunch. Initially the areas in phase space occupied by the particles that belong to these cross sections coincide. But the rf kick is a permanent force that has different sign for head and tail of the bunch, moving the potential well, where the particles oscillate, off the $(x,x'\beta)$-coordinate center in different directions for tail and head. Thus, the center of rotation in phase space is different for different cross sections (head, center and tail of the bunch), and areas in the phase space occupied by the particles that belong to different cross section are not coincide, when the betatron phase is not equal to $2\pi n$. Thus, minimal r.m.s. emittance corresponds to the betratron phase of $2\pi n$ and is equal to the initial r.m.s.

emittance as illustrated in Figure 3. Maximal r.m.s. emittance corresponds to the betatron phase of *(2n+1)π*. In longitudinal geometrical space cross section the bunch tilt oscillates as shown in Firgure 4.

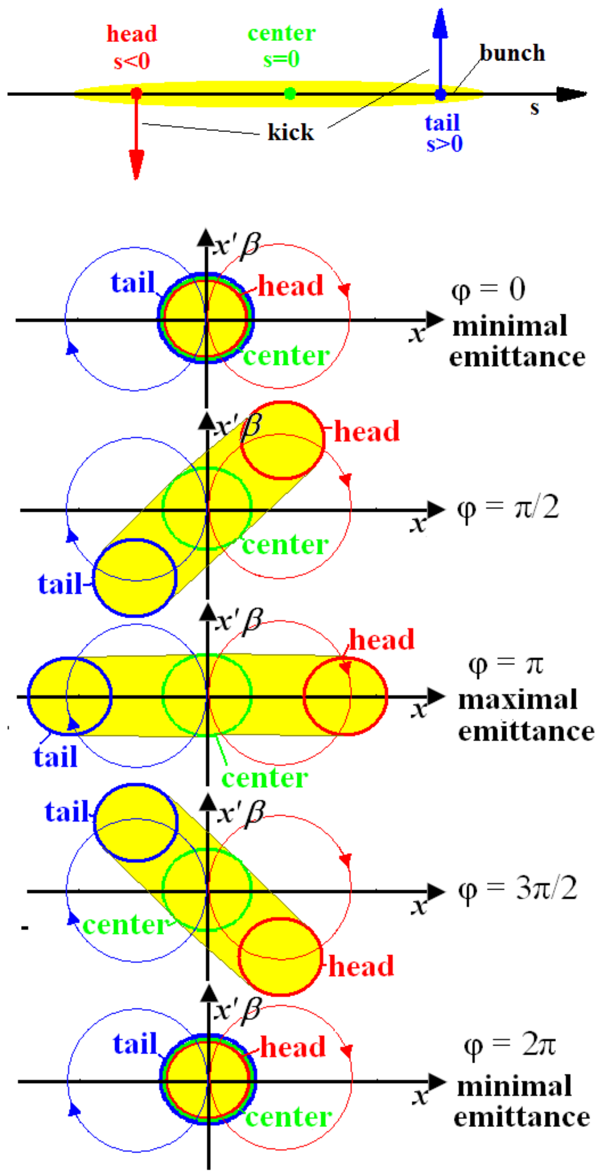


Figure 3. The phase diagrams for different betatron phase for particle that belongs to different transverse bunch cross section: head (red), center (green), and tail (blue). No acceleration. The emittance of the entire bunch is shown in yellow.

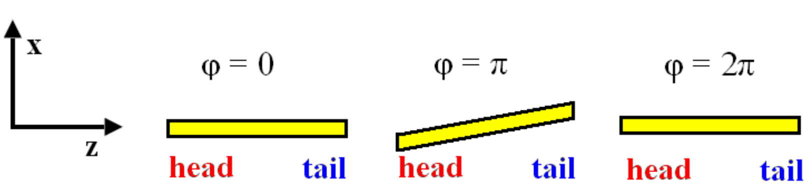


Figure 4. The bunch behavior in *(x,z)* coordinates without acceleration. On can see that the bunch is tilted versus the z-axis, but is parallel to z-axis, when the betatron phase is equal to 2πn.

In presence of acceleration the circles of the rotation do not touch the coordinate center $x=0$ and $x'=0$ because the circle center x_c shift (or potential well shift) is proportional $1/\gamma$, but the circle radius R is proportional to $1/\gamma^{1/2}$ due to adiabatic decay (see Figure 5), and thus, the areas do not coincide when the betatron phase is $2\pi n$, and the minimal r.m.s. emittance is greater than the initial one. In Figure 6 the phase diagram is shown for three transverse bunch cross sections after acceleration.

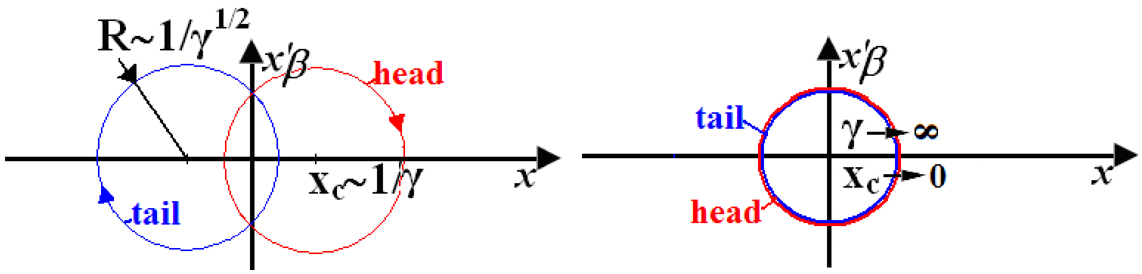


Figure 5. Center of rotation and radius for the particle with $x(0)=0$ and $x'(0)=0$ for the the bunch head (red) and tail (blue) after acceleration.

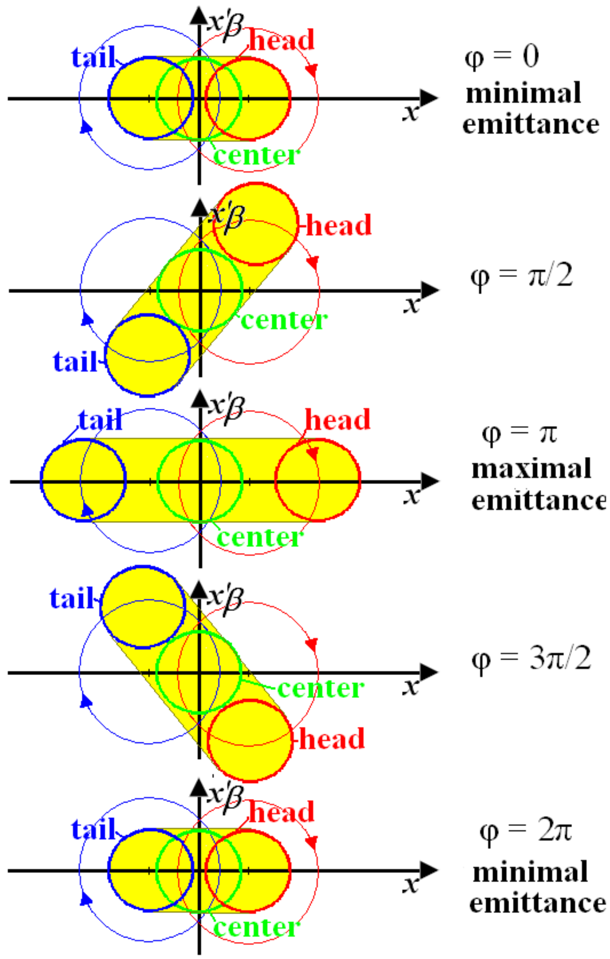


Figure 6. The phase diagrams for different betatron phase for particle that belongs to different transverse bunch cross section: head (red), center (green), and tail (blue). in presence of acceleration. The emittance of the entire bunch is shown in yellow. In longitudinal geometrical space cross section the bunch tilt oscillates as shown in Firgure 7.

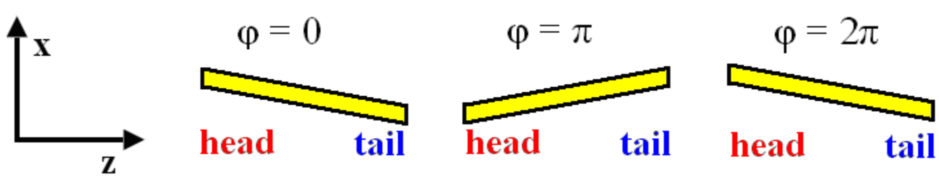


Figure 7. The bunch behavior in *(x,z)* coordinates with acceleration.

5. For the initial vertical r.m.s emittance of 20 nm the r.m.s. emittance dilution versus *z* is shown in Figure 8, calculated numerically by PLACET (blue) and LUCRETIA (black) for the same TESLA lattice structure (no misalignments and no wakes as well) and analytically (red) according to formula (9). In order to fit the betatron phase the average β-function in (9) was taken 83 m. One can see that the minimal values of the

vertical r.m.s. emittance dilution calculated by PLACET and LUCLETIA are close to estimation by formula (9).

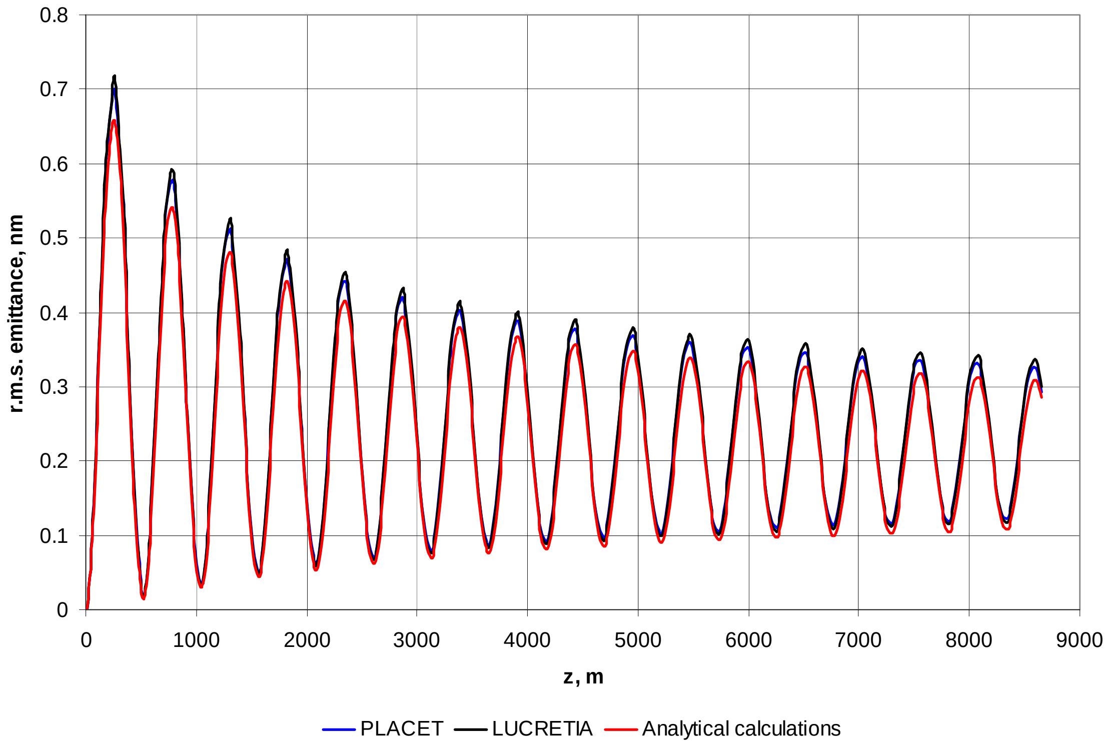


Figure 8. Vertical emittance change vs. z calculated numerically using PLACET (blue), LUCRETIA (black) and analytically (red) using the formula (9).

6. In general case, when the average rf-kick and wake caused by couplers is compensated by the beam alignment correction, the emittance dilution can be expressed by the formula

$$\Delta\varepsilon \approx \frac{\sigma_F^2 \beta^3 \gamma_0}{2U_0^2}\left(1 - 2\sqrt{\frac{\gamma_0}{\gamma(z)}}\cos(z/\beta) + \frac{\gamma_0}{\gamma(z)}\right),$$

where

$$\sigma_F^2 = \left< \left(F_{kick}(s) - \overline{F}_{kick}\right)^2 \right>,$$
$$F_{kick}(s) = G\left(\mathrm{Re}\left(\frac{V_y}{V_z}e^{iks+i\varphi}\right)\right) + QW_\perp(0,0,s), \qquad (11)$$

and

$$\overline{F}_{kick} = \left< F_{kick}(s) \right> = \left< \left(G\left(\mathrm{Re}\left(\frac{V_y}{V_z}e^{iks+i\varphi}\right)\right) + QW_\perp(0,0,s) \right) \right>.$$

For the ILC cavity one has

$\dfrac{V_y}{V_z}$=(-7.3+11.1i)×10^{-6},

φ=-5.3°,

G=23.93 MeV/m (31.5 MeV/m in the structures) and wake $W_\perp(0,0,s)$ is shown in Figure 9 (vertical and horizontal components) for the bunch r.m.s. length of 0.3 mm [4]:

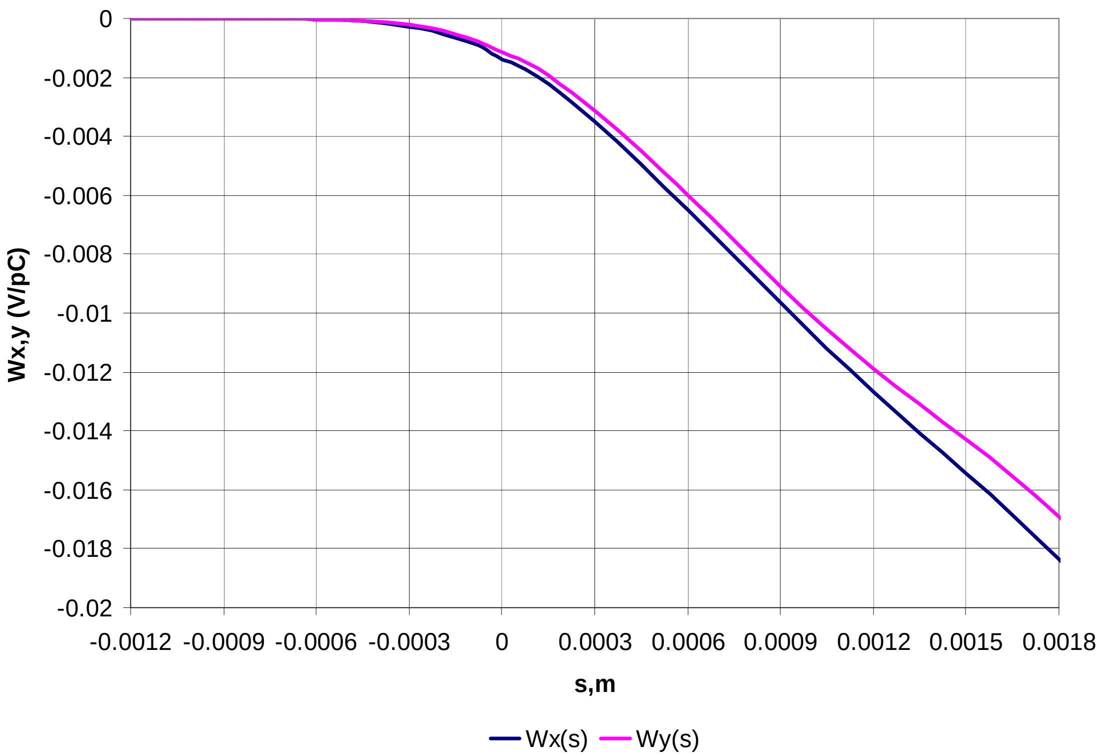


Figure 9. Transverse wake potential caused by the couplers (per cavity) for the bunch r.m.s. length of 0.3 mm.

Dependence of the $F_{kick}(s)$ versus s is shown in Figure 10 in red. In blue the acceleration RF field versus s is shown. The total vertical wake is shown in pink. The bunch profile is shown also in black in a.u. One can see that rf field and wake has the same slope and do not compensate each other.

For these parameters one has for σ_F from (11) and for the minimal vertical emittance dilution

$$\Delta\varepsilon \approx \frac{\sigma_F^2\beta^3\gamma_0}{2U_0^2}\left(1-\sqrt{\frac{\gamma_0}{\gamma(z)}}\right)^2 \qquad (12)$$

the following results for the main linac with the TESLA lattice (β=83 m):

Table I

	σ_F ,V/m	$\Delta\varepsilon$, nm formula (12)	$\Delta\varepsilon$, nm PLACET (1-to-1 correction and dispersion removal)
rf-kick only	2.3	0.11	0.16
wake only	3.9	0.31	0.32
rf-kick + wake	6.1	0.74	0.85

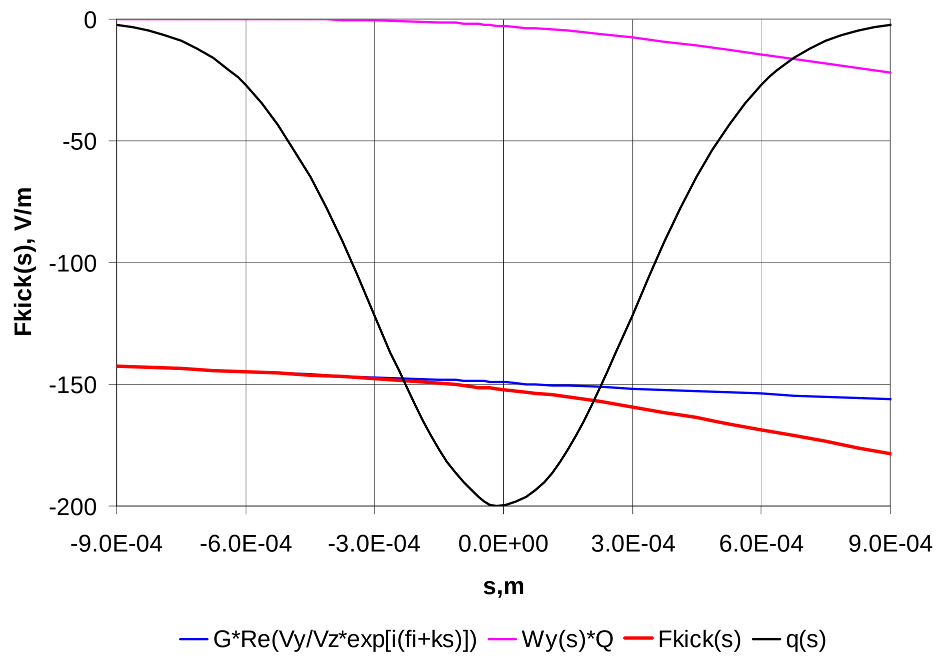


Figure 10. $F_{kick}(s)$ dependence (red). In blue the RF gradient versus s and in pink wake are shown. The bunch profile is shown also in black in a.u.

One can see that Formula (12) gives good estimation for the emittance dilution caused by the rf-kick and wave from the couplers in presence of 1-to-1 correction and dispersion removal.

References:

[1] V. Yakovlev, I. Gonin, A. Lunin, and N. Solyak, "Coupler RF kick simulations," SLAC Wakefest2007, December 11, 2007, http://ilcagenda.linearcollider.org/conferenceDisplay.py?confId=2378

[2] Tracking code PLACET, https://savannah.cern.ch/pro jects /placet/

[3] The LUCRETIA Project, http://www.slac.stanford.edu/accel/ilc/codes/Lucretia-orig/web/overview.html

[4] N. Solyak, A. Lunin, and V. Yakovlev, "Transverse Wake Field Simulations for the ILC Acceleration Structure," EPAC2008, Genoa, June 23-27, 2008, p. 640.