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Strand Critical Current Determination by Fitting Voltage Current Measurement Values

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1) Introduction

Superconducting (SC) strand characterization is one of the major activities of the Short Sample Test Facility (SSTF) at Fermilab. The most important part of the characterization process is to measure the strands critical current values. Based on the current-voltage characteristics of superconducting strands the critical current value (I_c) and the n-value can be determined.

This note describes a procedure for analyzing current-voltage data systematically as well providing the relevant error analysis for the measurement to obtain error information for critical current and n-values. A stand alone program based on MatLab macros has been developed that allows performing the analysis automatically.

2) Voltage-Current Measurement of SC strands

The test facility is capable of measuring 75 cm long strand samples at maximum of 15*T* (4.2*K*) perpendicular magnetic field values. A typical short sample measurement of a strand is performed under constant magnetic field. While the current is ramped up (~20A/sec) the voltage is recorded about every second. In order to record sufficient amount of data points, right before the SC reaches its transition region, the current ramp rate is reduced to ~1 *A/sec*.

At the transition from superconducting to normal state the voltage measured between two voltage taps becomes nonzero. Increasing the current will gradually increases the resistance of the strand and consequently the voltage rise will be proportional to resistance and to the applied current value. The critical current value (at SSTF) is defined as the current value at which the average resistivity of the superconductor strand is $\rho = \rho_c = 10^{-14}\Omega \cdot m$. The quality factor n is calculated using the following power law parameterization:

$$\boxed{\rho(J) = \rho_c \left(\frac{j}{j_c} \right)^n} \qquad eq.1$$

Where:

- j_c is the critical current density;

3) Exponential law

The current-voltage characteristics of type II superconductors can be described by the following formula [1]:

$$\boxed{j(E) = j_c + j_1 \ln\left(E/E_0\right)} \qquad \text{for } E<E_f. \qquad eq.2$$

Where:

- E_f is the electric field at which the superconductor enters into the flux flow regime;
- E_0 is the electric field developed in the superconductor at $j = j_c$;
- j_1 is a constant that describes the current density increase as a function of the electric field.

In order to determine Jc and n-value based on the $\rho = \rho_c = 10^{-14}\Omega \cdot m$ criteria and (*eq.1*) power law parameterization *eq.2* has to be re-arranged:

$$E_0 = \rho_c \cdot j_c; \qquad j_1 = f(n)$$

From *eq. 2*

$$e^{j(E)-j_c} = \left(E/E_0\right)^{j_1} \qquad E = E_0 e^{\frac{j(E)-j_c}{j_1}} \qquad \boxed{E = \rho_c j_c e^{\frac{j(E)-j_c}{j_1}}} \qquad eq.3$$

Differentiating *eq.3* in *j:*

$$\frac{dE}{dj} = \rho_c j_c \frac{1}{j_1} e^{\frac{j(E)-j_c}{j_1}} \qquad \boxed{\left.\frac{dE}{dj}\right|_{j_c} = \rho_c \frac{j_c}{j_1}} \qquad eq.4$$

From *eq. 1*

$$E = j \cdot \rho_c \left(\frac{j}{j_c}\right)^n$$

Differentiating the above equation in *j* we obtain:

$$\frac{dE}{dj} = \frac{\rho_c}{j_c^n}(n+1)j^n \qquad \boxed{\left.\frac{dE}{dj}\right|_{j_c} = \rho_c(n+1)} \qquad eq.5$$

Equalizing *eq.4* and *5*:

$$\rho_c \frac{j_c}{j_1} = \rho_c (n+1) \qquad \boxed{j_1 = \frac{j_c}{(n+1)}} \qquad eq.6$$

Eq.3 can than be written as:

$$\boxed{E = \rho_c j_c e^{\frac{j(E)-j_c}{j_c}(n+1)}} \qquad eq.\ 7$$

The voltage signal during critical current measurement can then be parameterized as:

$$V = \frac{L}{A} \cdot \rho_c \cdot I_c \cdot e^{\frac{I-I_c}{I_c} \cdot (n+1)}$$

Where:

- V is the measured voltage (in volts) between the two voltage taps
- L is the length of the strand between the two voltage taps
- A is the average area of the strand cross-section

Considering that the voltage measurements can have an offset, the right parameterization is the following:

$$\boxed{V = V_{offset} + \frac{L}{A} \cdot \rho_c \cdot I_c \cdot e^{\frac{I-I_c}{I_c} \cdot (n+1)}} \qquad eq.\ 8$$

4) Linear fit

Introducing $z \equiv V - \overline{V}_{offset}$, where $\overline{V}_{offset}$ is the average voltage offset measured by fitting a straight line to the voltage signal prior to the transition region, and taking the logarithm of eq.8 we can use a linear approach to fit the data:

$$\ln z = \ln\left(\frac{L}{A} \cdot \rho_c \cdot I_c \cdot e^{\frac{I-I_c}{I_c} \cdot (n+1)} \right)$$

$$y \equiv \ln(z) \qquad \boxed{y = a_0 + a_1 \cdot I} \qquad eq.\ 9$$

$$\boxed{a_0 = \ln\left(\frac{L}{A} \cdot \rho_c \cdot I_c \right) - (n+1)} \qquad eq.\ 10$$

$$\boxed{a_1 = \frac{n+1}{I_c}} \qquad \textit{eq. 11}$$

$$x \equiv I \qquad \boxed{y = a_0 + a_1 \cdot x} \qquad \textit{eq. 12}$$

There is an analytical solution to find a_0 and a_1 using the least squares method:

$$\chi^2 \equiv \sum \left[\frac{y_i - y(x_i)}{\sigma_{y_i}} \right]^2 = \sum \left[\frac{y_i - a_0 - a_1 x_i}{\sigma_{y_i}} \right]^2$$

Minimizing χ^2:

$$\boxed{a_0 = \frac{1}{\Delta} \left(\sum \frac{x_i^2}{\sigma_{y_i}^2} \sum \frac{y_i^2}{\sigma_{y_i}^2} - \sum \frac{x_i}{\sigma_{y_i}^2} \sum \frac{x_i y_i}{\sigma_{y_i}^2} \right)} \qquad \textit{eq. 13}$$

$$\boxed{a_1 = \frac{1}{\Delta} \left(\sum \frac{1}{\sigma_{y_i}^2} \sum \frac{x_i y_i}{\sigma_{y_i}^2} - \sum \frac{x_i}{\sigma_{y_i}^2} \sum \frac{y_i}{\sigma_{y_i}^2} \right)} \qquad \textit{eq. 14}$$

$$\Delta = \sum \frac{1}{\sigma_{y_i}^2} \sum \frac{x_i^2}{\sigma_{y_i}^2} - \left(\sum \frac{x_i}{\sigma_{y_i}^2} \right)^2$$

For proper evaluation of the coefficients we need to estimate the error of y for each data point ($\sigma_{yi.}$). It is assumed that the measurement error is uniform as a function of current so $\sigma_{zi}=\sigma_z$ equation is valid for any i. Using this assumption and the error propagation method it follows:

$$\sigma_{y_i} = \frac{dy}{dz} \sigma_{z_i} = \frac{\sigma_{z_i}}{z_i} = \frac{\sigma_z}{z_i} \qquad \boxed{\sigma_{y_i} = \frac{\sigma_z}{z_i}} \qquad \textit{eq. 15}$$

Substituting in *eq. 13* and *14* $\sigma_{yi.}$ with σ_z (using *eq. 15*) one can easily see that coefficients a_0 and a_1 will only depend on the magnitude of z_i but they will not depend on σ_z. We can then calculate the parameters and obtain the fitting curve:

$$\boxed{z = e^{a_0 + a_1 x}} \qquad \textit{eq. 16}$$

The best estimation of the variance of the data sample is then:

$$\sigma_z^2 \approx s_z^2 \equiv \frac{1}{N-m} \sum (z_i - z)^2 \qquad \textit{eq. 17}$$

where *N-m* is the number of degrees of freedom and it is equal to the number of measurements minus the number of parameters used in the fit. Thus using this technique, we can obtain the common error in our measurements of z from the fit although at the expense of any information about the quality of the fit [2]. The goodness of the fit using the exponential law has been proven by many experiments [1].

Once the parameters of the fit are obtained, the I_c can be calculated by applying electrical resistivity of $10^{-14}\Omega m$ and solving the following equation:

$$\boxed{b_1 I_c = e^{a_0 + a_1 I_c}} \qquad \textit{eq. 18}$$

$$b_1 = \frac{L}{A} \cdot \rho_c$$

The n-value can be obtained by solving eq. 11.

4.1 Error estimation

In order to estimate the error of I_c we also have to include in *eq. 17* the error of the offset:

$$b_1 I_c + V_{offset} = e^{a_0 + a_1 I_c} + \overline{V}_{offset}$$

$$\boxed{b_1 I_c = e^{a_0 + a_1 I_c} + \overline{V}_{offset} - V_{offset}} \qquad \textit{eq. 19}$$

Since a_0, a_1, and V_{off} are correlated proper error propagation calculation is quite tedious. We used a different approach. For every VI measurements the following procedure is applied to obtain the error:

1. Fit the data using the method described in section 4.
2. Calculate the rms value of the data spread using *eq.17*.
3. Use smearing technique (randomization based on this rms value) to introduce many new data sets by randomizing the curve.
4. Make fits to every newly created data sets and find a graphical solution for I_c.
5. The rms values of the spread of I_c and n-values will be representing the errors.

Generating 200 data samples were enough to be able to estimate the rms value within 10% accuracy.

Examples of this fitting technique used on a 1mm Modified Jelly Roll strand is shown in fig. 1-2-3.

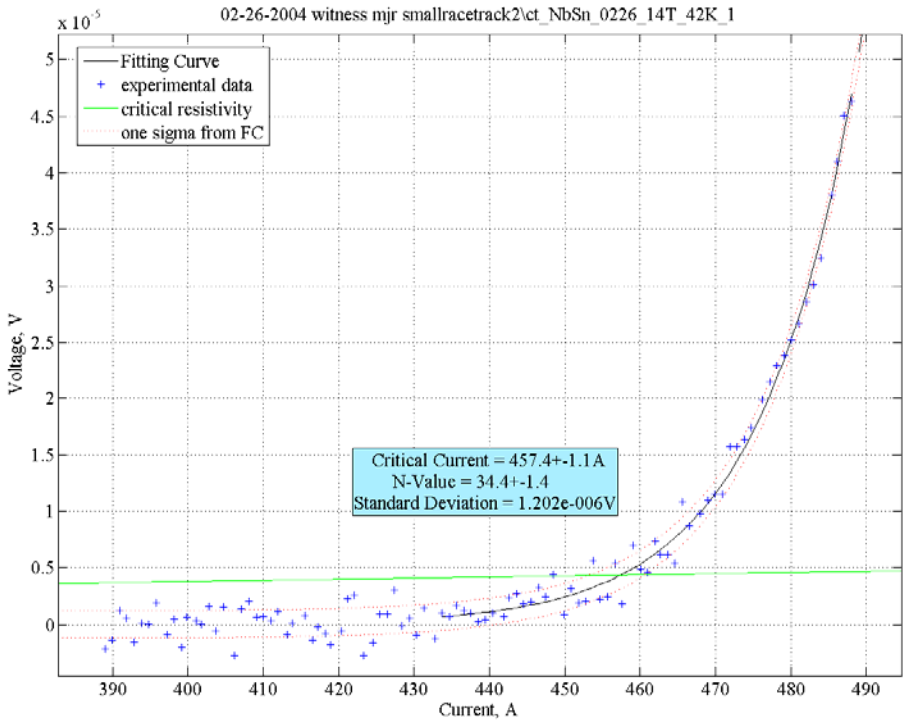
02-26-2004 witness mjr smallracetrack2\ct_NbSn_0226_14T_42K_1
x 10^-5
Fitting Curve
experimental data
critical resistivity
one sigma from FC
Critical Current = 457.4+-1.1A
N-Value = 34.4+-1.4
Standard Deviation = 1.202e-006V
Voltage, V
Current, A

Fig. 1

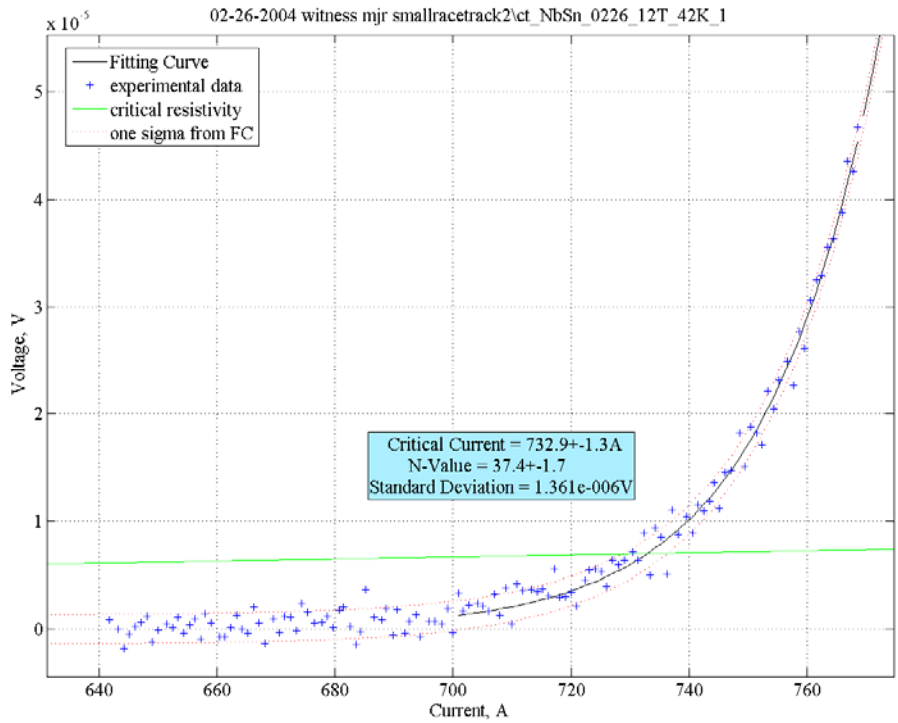
02-26-2004 witness mjr smallracetrack2\ct_NbSn_0226_12T_42K_1
x 10^-5
Fitting Curve
experimental data
critical resistivity
one sigma from FC
Critical Current = 732.9+-1.3A
N-Value = 37.4+-1.7
Standard Deviation = 1.361e-006V
Voltage, V
Current, A

Fig. 2

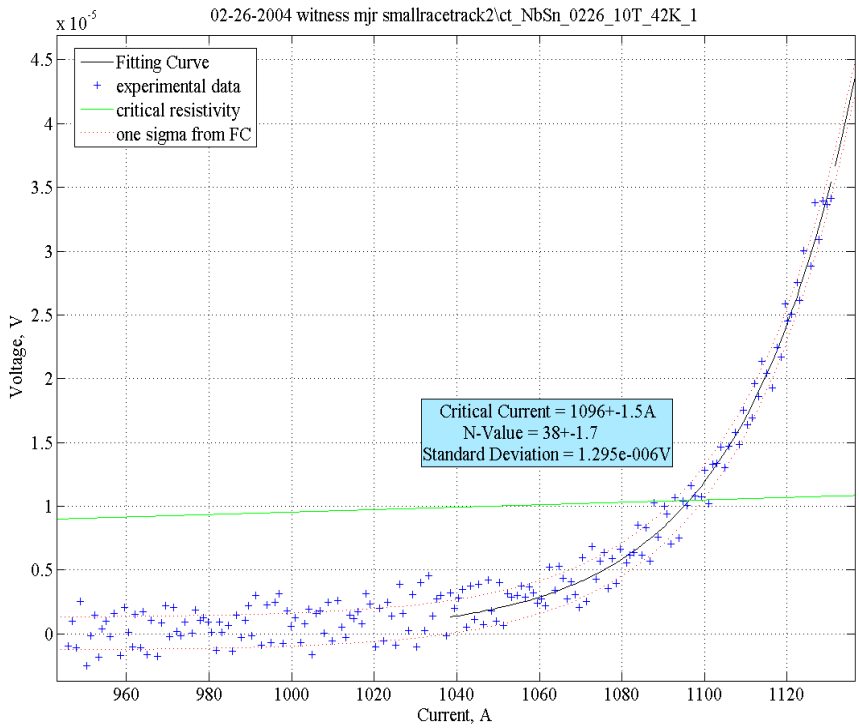

Fig. 3

5 Non-linear fit using 'uifit' subroutine

In order to use the linear method for fitting the data we had to take the logarithm of the voltages. This transformation only works when the voltage is positive so for the fit we are able to use only the last part of the data set. In order to get more accurate results non-linear fitting technique had to be introduced. This technique is based on minimization method. In our program we implemented a freely available subroutine (www.mathworks.com/matlabcentral/fileexchange), 'uifit', which is capable of performing non-linear fitting using the matlab core function 'fminsearch'. This technique allows us to estimate the I_c, the n-value and the associated errors even when we have only few data points at the transition region. The error analysis is performed the same way as it was described in section 4.1 because this subroutine does not provide the error matrix of the parameters [2].

The disadvantage of the non-linear fitting method is that the subroutine needs quite accurate initial values to start the minimization process and the computation takes significantly longer time to obtain the results. If there is sufficient amount of data points at the transition region it is more practical to use the linear method. Fig.4 shows a typical data set when the non-linear fit is the more appropriate one to be used.

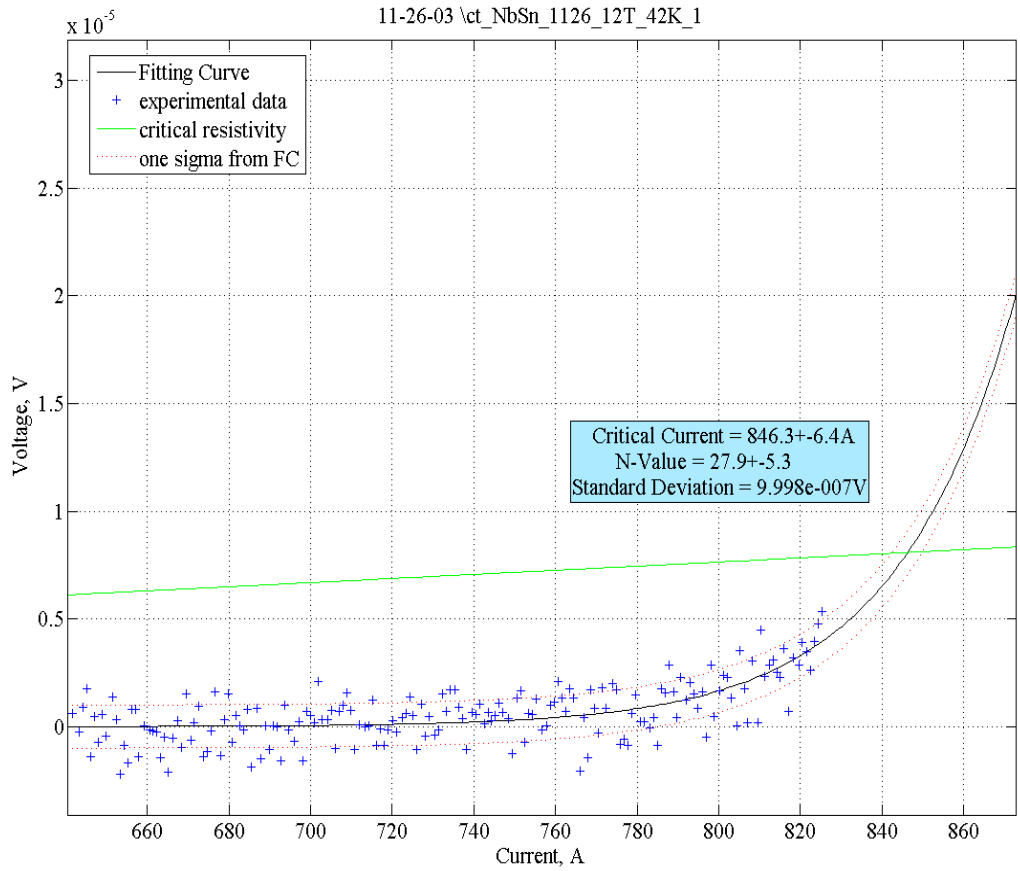

Fig. 4

6 Non-linear fit using the 'fit' subroutine

In order to reduce the computing time significantly we examined the capabilities of Matlab fitting tool box that became available recently. Incorporating the 'fit' function in to the program we achieved the desired results including the error estimate hundreds times faster than using the method described in section 5. In order to calculate the parameters errors, the 'fit' function uses the 'error matrix' so it is not necessary to calculate hundreds of time the fitting.

The 'fit' function allows choosing between different solving algorithms, we used the *trust region* algorithm that represents an improvement over the popular *Levenberg-Marquardt* algorithm [2].

We also compared the results and error estimates with those obtained in section 5 and we found a reasonable agreement. Fig. 5 shows the results obtained by the new version of our program with the same set of data used to get fig. 4.

The running time of the newly released version of the program is very short, it takes about the same amount of time as the linear fit so there were no reasons to keep the linear method any more consequently it was removed from the program.

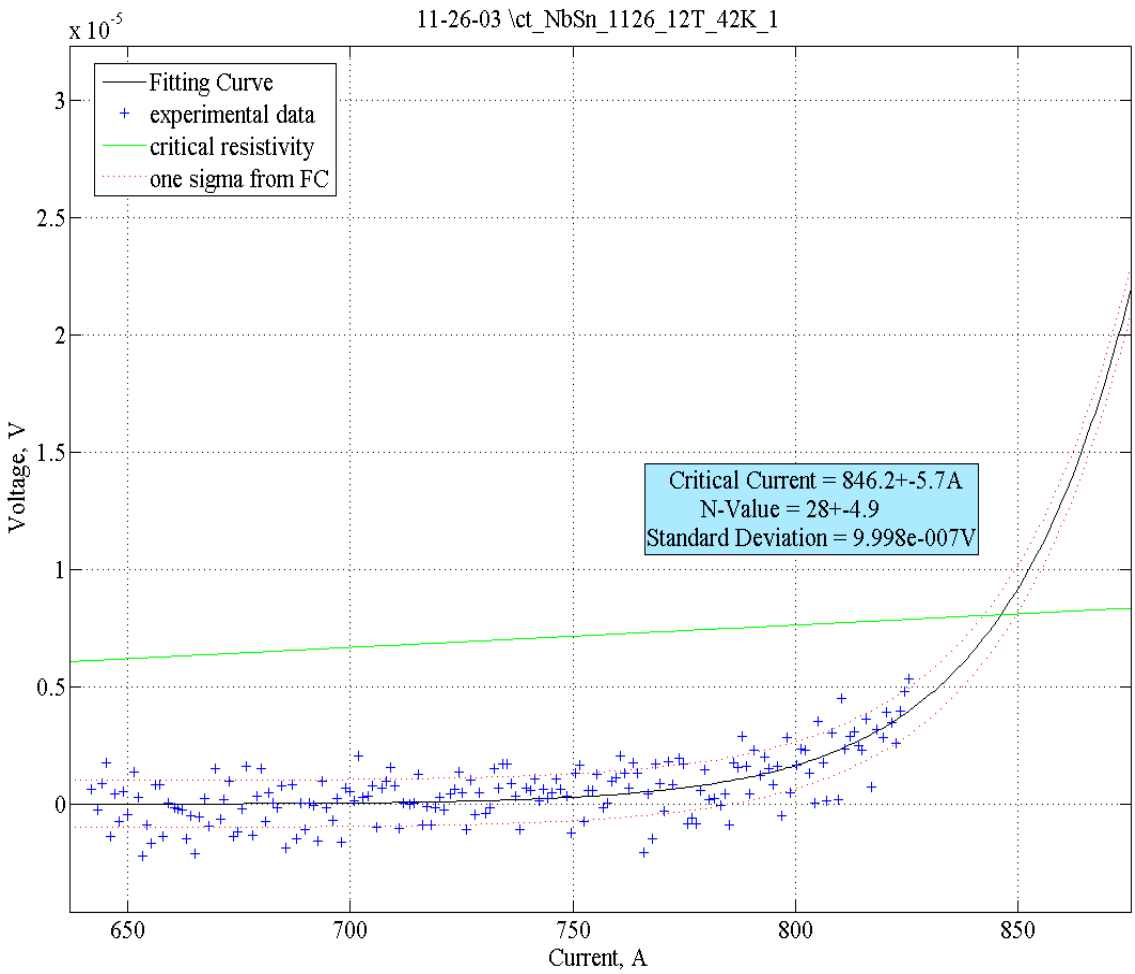

Fig. 5

7 Conclusions

A new data-analysis tool has been developed in order to measure systematically the I_c and n-value of SC strands and their errors.

This new tool utilizes a non linear fitting method to obtain I_c and n-value as parameters of the fit of the current voltage measurements.

Matlab software packages were used to develop a stand alone program which allows the user performing the data analysis automatically. Initially the user was able to choose to analyze data using either a linear or non-linear parameter fit. While the linear method was computationally much faster, the non-linear method was more accurate especially when only few data points were available at the transition region of the SC.

Since the running time of the new released version of the program is very short, the linear method was removed.

The program is based on the hypothesis that the uncertainty on the voltage measurements is independent from the voltage and current values.

Reference

1 A.V. Gurevich, R.G. Mints, A.L. Rakhmanov, "The Physics of Composite Superconductors", begel house, inc New York 1997

2 P.R. Bevington, D.K. Robinson, "Data Reduction and Error Analysis for the Physical Sciences", WCB/McGraw-Hill, Boston 1992